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On the number of Perfect Matchings of Tubular Fullerene Graphs

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Abstract: The perfect matchings counting problem of graphs has important applications in combinatorial optimization, statistical physics, quantum chemistry and other fields. A perfect matching of a graph G is a set of non-adjacent edges that covers all vertices of G. The number of perfect matchings of a graph is closely related to its number of vertices. A fullerene graph F is a 3-connected cubic planar graphs all of whose faces are pentagons and hexagons. Došlić obtained that a fullerene graph with p vertices has at least $\frac{p}{2}+1$ perfect matchings, Zhang et al. proved a better lower bound $\left\lceil \frac{3(p+2)}{4} \right\rceil$ of the number of perfect matchings of a fullerene graph. We have known that the fullerene graph has a nontrivial cyclic 5-edge-cut if and only if it is isomorphic to the graph T_n for some integer $n \geq 1$, where T_n is the tubular fullerene graph comprised of two caps formed of six pentagons joined by n concentric layers of hexagons. In this paper, the perfect matchings of the graph T_n is classified by matching a certain vertex, and recursive relations of a set of perfect matching numbers are obtained. Then the calculation formula of the number of perfect matchings of the graph T_n is given by recursive relationships. Finally, we get the number of perfect matchings of T_n with p vertices.

Index Terms: Fullerene Graph, Tubular Fullerene, Perfect Matching, Cyclic Edge-cut

1. Introduction

The enumeration problem of perfect matchings of graphs is an important research topic in graph theory. Perfect matchings of graphs are called dimer configurations in statistical physics [1] and Kekulé structures in quantum chemistry [2, 3]. As a very important topological index, the number of perfect matchings has been applied in many fields, such as estimating resonance energy and π-electron energy, calculating pauling bond orders, etc [2-4]. So far, the problem of counting the perfect matchings of graphs has attracted extensive attention of many mathematicians, physicists and chemists [1,5-8].

However, Valiant [9] proved that the counting of perfect matchings in a graph (even a bipartite graph) is NP-hard, so it is difficult to find a polynomial algorithm. Physicist Kasteleyn [10] first proposed to calculate the number of perfect matchings of graphs by the Pfaffian method when he used matching theory to study the Ising model of magnetism. After that, the number of perfect matchings of various grid graphs also entered the scope of research by many scholars. Lu and Wu [11,12] gave explicit expressions for the number of perfect matchings of quadrilateral grid graphs on the Möbius strip, Klein bottle, and cylindrical surface. In recent years, Tang et al. [13,14] have been committed to solving the perfect matchings counting problem of some special graph classes.

Since the discovery of the first fullerene molecule in 1985 [15], the fullerenes have been objects of interest to scientists all over the world. Mathematical tools and results can be used to study many properties of fullerene molecules. A *fullerene graph* F is a 3-regular and 3-connected plane graph whose faces are of length 5 or 6. It follows from Euler's formula that the number of pentagonal faces is 12. For all even $p \geq 20$ vertices or carbons, fullerene F_p can be constructed except for $p = 22$.

A *matching* in a graph G is a set M of edges of G such that no two edges in M have a vertex in common. A matching M of G is *perfect* if any vertex of G is incident with an edge of M. We also say that every vertex of G is *covered* or *saturated* by an edge from M. Let G be a graph with perfect matchings, and we say that M_1 and M_2 are two different perfect matchings of G if the two perfect matchings M_1 and M_2 of the graph G have one edge that differs. In 1891, Petersen [16] concluded that every 3-regular graph with no more than two cut-edges has a perfect matching, which means that the fullerene graph has a perfect matching. An equivalent proposition of the famous Four-Color theorem states that fullerene graphs have at least three perfect matchings [17]. Došlić [18] pointed out that every edge of a fullerene graph is in a perfect matching and gave a lower bound $\frac{p}{2}+1$ of the number of perfect matchings of fullerene graphs with p vertices in 1998. Later, Zhang [19] gave a better lower bound $\left\lceil \frac{3(p+2)}{4} \right\rceil$ for fullerene graphs with p vertices. In 2009 [20], Kardoš et al. proved that the exponential lower bound is $2^{\frac{p-380}{61}}$ for the number of perfect matchings of fullerene graphs. However, the exact expression for the number of perfect matchings of a general fullerene graph has not been given.

Let T_n be a tubular fullerene graph that consists of $n(n \geq 1)$ concentric layers of hexagons and capped on each end by a cap formed of six pentagons [21, 22]. For example, see Figure 1 for T_3. In this paper, we give the formula for calculating the number of perfect matchings of T_n.

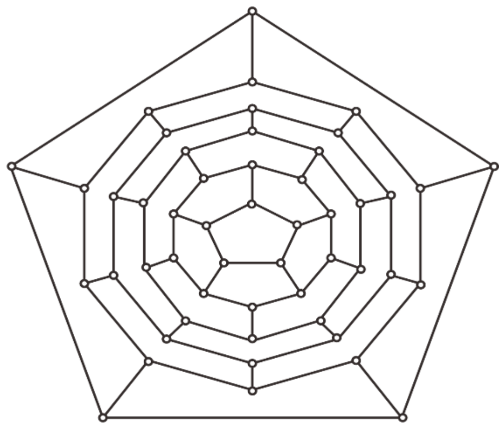

Fig. 1. Illustration for a tubular fullerene T_3.

2. Research Method

In the next section, we define basic concepts and present some preliminary results. Inspired by articles of Tang et al.[13,14], we use the methods of division, sum, and then recursion to get the number of perfect matchings of a tubular fullerene graph T_n. Before calculating the number of perfect matchings of T_n, we first give the definitions of the graph G_n, the graph Z_n and the graph H_n, and get recursive relational expressions for the numbers of perfect matchings of these graphs. In section 4, we classify the perfect matchings of T_n, find the recursive formula for the number of perfect matchings in each class, and then use these recursive relations to obtain the calculation formula of perfect matchings of T_n with p vertices.

3. Definitions and Preliminary Results

Let G be a connected plane graph with vertex-set $V(G)$ and edge-set $E(G)$. For a graph *H*, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we say that H is a *subgraph* of G, written as $H \subseteq G$. For $S \subseteq V(G)$, $G-S$ is the subgraph of G obtained by deleting all the vertices in S and their incident edges. S is called a *vertex cut* of G if $G-S$ is disconnected. G is called *k-connected* if $k < |V(G)|$ and $G-S$ is connected for every $S \subseteq V(G)$ with $|S| < k$. The greatest integer k for which G is *k*-connected is defined as the *connectivity* of *G*. For $S \subseteq V(G)$, let $\bar{S} = V(G) - S$, we define $E(S, \bar{S})$ as the set of edges that each has one end-vertex in S and the other in $\bar{S}$. An *edge-cut* of a connected graph G is a set of edges $C \subseteq E(G)$ such that $G-C$ is disconnected. An *k-edge-cut* is an edge-cut with k edges. A graph G is *k-edge-connected* if G cannot be separated into two components by removing less than *k* edges. An edge-cut C of a graph G is *cyclic* if each component of $G-C$ contains a cycle and it is called a *trivial cyclic-k-edge-cut* if at least one of the resulting two components induces a single *k*-cycle, otherwise, it is *nontrivial*. A graph G is *cyclically k-edge-connected* if G cannot be separated into at least two components, each containing a cycle, by removing less than *k* edges. Let T_n be a tubular fullerene graph as defined above. The graph T_n is the only fullerene graph with nontrivial cyclic 5-edge-cuts and has at least one perfect matching [16,22].

Theorem 3.1 ([21,22]) A fullerene graph has a nontrivial cyclic 5-edge-cut if and only if it is isomorphic to the graph T_n for some integer $n \geq 1$.

Let the planar embedding of T_n be shown in Figure 2. It is easy to know that T_n consists of $n+3$ concentric rings, with 5 vertices each on the innermost and outermost rings, and 10 vertices on each of the remaining rings. From the inside to the outside, we write that these concentric rings are the $0th, 1st, 2nd, 3rd, \cdots$, $(n+2)th$ rings, respectively. The vertices on the $0th$ ring are numbered $v_{10}, v_{11}, v_{12}, v_{13}, v_{14}$ clockwise and the vertices on the $(n+2)th$ ring v_{20}, v_{21}, v_{22}, v_{23}, v_{24} clockwise. Let us denote the vertices on the *k*th concentric ring by $u_{k0}, u_{k1}, u_{k2}, \cdots, u_{k9}$ in clockwise order $(k=1,2,3,\cdots,n+1)$ so that $u_{1a}, u_{2a}, u_{3a}, \cdots, u_{n+1,a}$ are on the same straight line, where $a=0,1,2,\cdots,9$ (see the labeling of T_n in Figure 2). The edge that is not on the concentric ring is called a *radial edge*.

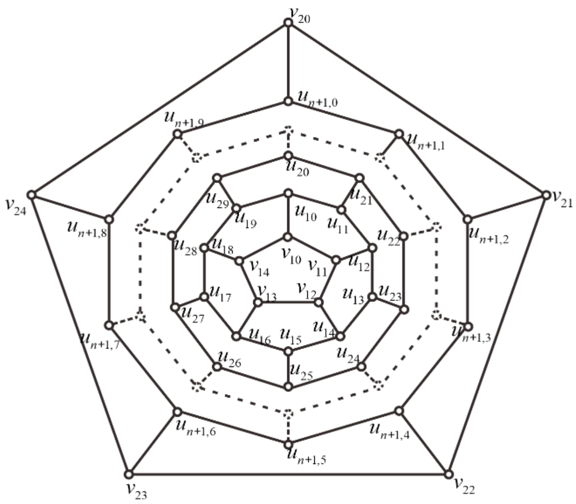

Fig. 2. The labeling of T_n.

Let $S \subseteq V(T_n)$ and $S=\{v_{10}, u_{10}, v_{11}, u_{12}, v_{12}, u_{14}, v_{13}, v_{14}\}$. Denoted by $G_{n-1}=T_n-S$ (see Figure 3). We can know that the graph G_{n-1} has $n-1$ concentric layers of hexagons with one cap as the same as T_n. Then we define the graph G_n with one more hexagonal concentric layer than G_{n-1}, as shown in Figure 4.

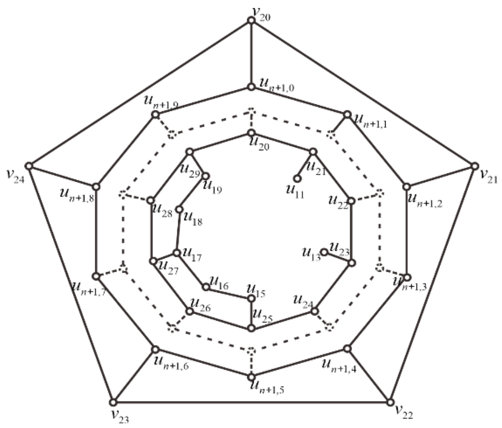

Fig. 3. The graph G_{n-1}.

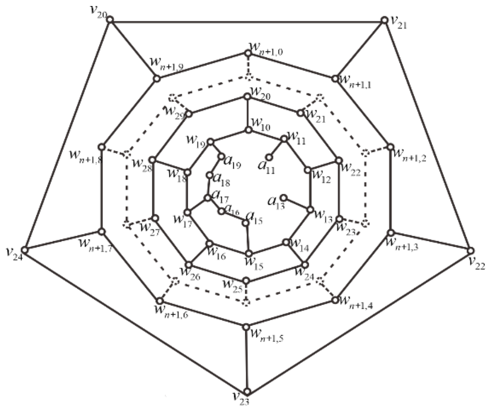

Fig. 4. The graph G_n .

The above are some basic concepts, terms and symbols that need to be used in this article. The following describes the calculation method for solving the recursive relation of the number of perfect matchings of graph G_n . Let's first give a definition of the linear constant coefficient homogeneous recurrence relation.

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k} = 0, \tag{1}$$

$$a_0 = d_0, a_1 = d_1, \cdots a_{k-1} = d_{k-1}, \tag{2}$$

If $c_1, c_2, \cdots, c_k, d_0, d_1, \cdots d_{k-1}$ are constants, (1) is called t*he linear constant coefficient homogeneous recursion relation of order k.*

Corresponding to the recursive relation to (1) is

$$C(x) = x^k + c_1 x^{k-1} + \cdots c_{k-1} x + c_k, \tag{3}$$

which is called *the characteristic polynomial* of (1).

Lemma 3.2 The number of perfect matchings in the graph G_n is denoted by $g(n)$, then

$$g(n) = \left(\frac{5+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{5-\sqrt{5}}{2}\right)^{n+1}. \tag{4}$$

Proof: It is easy to know that the graph G_n exists perfect matchings. The labeling of the vertices of a graph G_n is shown in Figure 4. Let $\beta(G_n)$ be the set of perfect matchings of G_n . The vertices of degree 1 are a_{11}, a_{13} , then $a_{11}w_{11}, a_{13}w_{13} \in M$ for every $M \in \beta(G_n)$. Without loss of generality, take a vertex a_{15}, let the perfect matchings set with edge $a_{15}w_{15}$ be β_1, and the sets of perfect matchings with edges $a_{15}a_{16}, a_{17}w_{17}$ and $a_{15}a_{16}, a_{19}w_{19}$ be β_2 and β_3 , respectively. Then $\beta_i \cap \beta_j = \varnothing (1 \le i < j \le 3)$. Hence

$$\beta(G_n) = \beta_1 \cup \beta_2 \cup \beta_3, \tag{5}$$

$$g(n) = |\beta(G_n)| = |\beta_1| + |\beta_2| + |\beta_3|. \tag{6}$$

We have the following claim:

Claim 1: $|\beta_1| = g(n-1)$.

Proof: β_1 must contain the edges $a_{11}w_{11}, a_{13}w_{13}$ and $a_{15}w_{15}$, then $a_{16}a_{17}, a_{18}a_{19} \in M$ for every $M \in \beta_1$. For $S \subseteq V(G_n)$, let $S = \{a_{11}, w_{11}, a_{13}, w_{13}, a_{15}, w_{15}, a_{16}, a_{17}, a_{18}, a_{19}\}$, then $G_n - S$ does not affect the number of perfect matchings of β_1 and $G_n - S \cong G_{n-1}$. That is, the number of perfect matchings of G_{n-1} is $|\beta_1|$. From the definition of $g(n)$, we have

$$|\beta_1| = g(n-1). \tag{7}$$

Next we consider $|\beta_2|$. We first define a graph Z_n. Let $S \subseteq V(G_n)$, $S = \{a_{11}, w_{11}, a_{13}, w_{13}, a_{15}, a_{16}, a_{17}, w_{17}, a_{18}, a_{19}\}$, the graph obtained by $G_n - S$ is denoted as Z_{n-1}. The graph Z_{n-1} has $n-1$ concentric hexagonal layers (see Figure 5). Z_n is shown in Figure 6 (one more concentric hexagonal layer than Z_{n-1}). We can see that the graph Z_n consists of $n(n \geq 1)$ concentric hexagonal layers each of five hexagons, capped by six pentagons in outer end, and the inner end is composed of two hexagons and a vertex of degree 1. It is obvious that Z_n has $n+2$ concentric rings. The six 2-degree vertices on the inner cap are labeled $b_{14}, b_{15}, b_{16}, b_{18}, b_{19}, b_{10}$ along the clockwise direction, and the 1-degree vertex is labeled b_{12} (see Figure 6). For the graph Z_n, we define the $1st, 2nd, 3rd, \cdots, (n+2)th$ rings from the inside to the outside. Let us denote the vertices on the kth concentric ring by $t_{k0}, t_{k1}, t_{k2}, \cdots, t_{k9}$ in clockwise order $(k = 1, 2, 3, \cdots, n+1)$ so that $t_{1a}, t_{2a}, t_{3a}, \cdots, t_{n+1,a}$ are on the same straight line, where $a = 0, 1, 2, \cdots, 9$, and the five vertices on the ($n+2$)th ring are labeled $v_{20}, v_{21}, v_{22}, v_{23}, v_{24}$ (The vertices of Z_n are labeled as shown in Figure 6.)

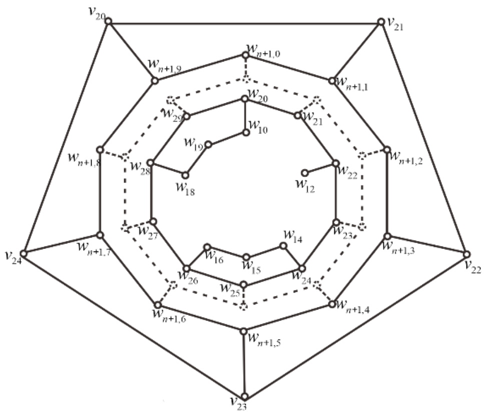

Fig. 5. The graph Z_{n-1}.

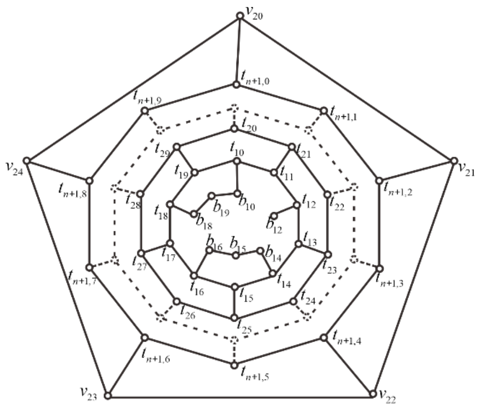

Fig. 6. The graph Z_n.

Claim 2: Let the number of perfect matchings of Z_n be $z(n)$, then

$$z(n) = g(n-1) + 3z(n-1). \tag{8}$$

Proof: We know that the graph Z_n has perfect matchings. The perfect matchings set of Z_n is denoted as $\mathcal{B}(Z_n)$. b_{12} is a 1-degree vertex, thus $b_{12}t_{12} \in M$ for every $M \in \mathcal{B}(Z_n)$.

(i) The set of perfect matchings containing edges $b_{12}t_{12}, b_{10}t_{10}, b_{14}t_{14}$ is denoted by $\mathcal{B}_1$. Then $b_{15}b_{16}, b_{18}b_{19} \in M$ for any $M \in \mathcal{B}_1$. Let $S \subseteq V(Z_n)$, $S = \{b_{12}, t_{12}, b_{10}, t_{10}, b_{14}, t_{14}, b_{15}, b_{16}, b_{18}, b_{19}\}$, then $Z_n - S \cong G_{n-1}$. So the perfect matchings number of G_{n-1} is $|\mathcal{B}_1|$. That is

$$|\mathcal{B}_1| = g(n-1). \tag{9}$$

(ii) Let $\mathcal{B}_2$ denote the set of perfect matchings containing edges $b_{12}t_{12}, b_{10}t_{10}, b_{16}t_{16}$. Obviously, $b_{14}b_{15}, b_{18}b_{19} \in M$ for any $M \in \mathcal{B}_2$. Let $S = \{b_{12}, t_{12}, b_{10}, t_{10}, b_{16}, t_{16}, b_{14}, b_{15}, b_{18}, b_{19}\}$ for $S \subseteq V(Z_n)$, then $Z_n - S \cong Z_{n-1}$. It is easy to know that the number of perfect matchings of graph Z_{n-1} is $|\mathcal{B}_2|$. Thus

$$|\mathcal{B}_2| = z(n-1). \tag{10}$$

(iii) We denote the set of perfect matchings containing edges $b_{12}t_{12}, b_{18}t_{18}, b_{14}t_{14}$ as $\mathcal{B}_3$. Then for any $M \in \mathcal{B}_3$, M must also have edges $b_{10}b_{19}$ and $b_{15}b_{16}$. For $S = \{b_{12}, t_{12}, b_{14}, t_{14}, b_{18}, t_{18}, b_{10}, b_{19}, b_{15}, b_{16}\}$, $Z_n - S \cong Z_{n-1}$. Thus

$$|\mathcal{B}_3| = z(n-1). \tag{11}$$

(iv) The perfect matchings set which contains edges $b_{12}t_{12}, b_{18}t_{18}, b_{16}t_{16}$ is denoted by $\mathcal{B}_4$. In the same way, we have

$$|\mathcal{B}_4| = z(n-1). \tag{12}$$

The above four types of perfect matchings in $\mathcal{B}(Z_n)$ do not contain each other, do not intersect with each other, and exhaust all perfect matchings in $\mathcal{B}(Z_n)$. That is $\mathcal{B}_i \cap \mathcal{B}_j = \varnothing$ $(1 \le i < j \le 4)$, $\mathcal{B}(Z_n) = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$. We have

$$z(n) = |\mathcal{B}(Z_n)| = |\mathcal{B}_1| + |\mathcal{B}_2| + |\mathcal{B}_3| + |\mathcal{B}_4|. \tag{13}$$

Thus

$$z(n) = g(n-1) + 3z(n-1). \tag{14}$$

Claim 3: $|\beta_2| = z(n-1)$.

Proof: We can know that $a_{11}w_{11}, a_{13}w_{13}, a_{15}a_{16}, a_{17}w_{17} \in M$ for any $M \in \beta_2$, then $a_{18}a_{19} \in M$. For $S \subseteq V(G_n)$, $S = \{a_{11}, w_{11}, a_{13}, w_{13}, a_{15}, a_{16}, a_{17}, w_{17}, a_{18}, a_{19}\}$. We can know that $G_n - S \cong Z_{n-1}$. Hence

$$|\beta_2| = z(n-1). \tag{15}$$

Using a similar analytical method to find $|\beta_3|$, we have the following result.

$$|\beta_3| = g(n-1). \tag{16}$$

To sum up, we have

$$g(n) = |\beta_1| + |\beta_2| + |\beta_3| = 2g(n-1) + z(n-1). \tag{17}$$

Combining equations (14) and (17), we have that

$$g(n) = 5g(n-1) - 5g(n-2). \tag{18}$$

The characteristic equation of linear recurrence (18) is

$$x^2 - 5x + 5 = 0. \tag{19}$$

We can get that the characteristic roots of (19) are $x=\frac{5\pm\sqrt{5}}{2}$. Therefore, its general solution is

$$g(n)=b_1\left(\frac{5+\sqrt{5}}{2}\right)^n+b_2\left(\frac{5-\sqrt{5}}{2}\right)^n, \tag{20}$$

It is easy to verify that $g(1)=15$ and $g(2)=50$. So

$$\begin{cases} g(1)=b_1\left(\frac{5+\sqrt{5}}{2}\right)+b_2\left(\frac{5-\sqrt{5}}{2}\right)=15 \\ g(2)=b_1\left(\frac{5+\sqrt{5}}{2}\right)^2+b_2\left(\frac{5-\sqrt{5}}{2}\right)^2=50 \end{cases}, \tag{21}$$

we can get

$$\begin{cases} b_1=\frac{5+\sqrt{5}}{2} \\ b_2=\frac{5-\sqrt{5}}{2} \end{cases}, \tag{22}$$

Hence the solution of equation (18) is

$$g(n)=\left(\frac{5+\sqrt{5}}{2}\right)^{n+1}+\left(\frac{5-\sqrt{5}}{2}\right)^{n+1}. \tag{23}$$

4. Main Results

Theorem 4.1 The number of perfect matchings of the tubular fullerene graph T_n is denoted by $f(n)$, then

$$f(n)=1+5\times\left(\frac{5+\sqrt{5}}{2}\right)^n+5\times\left(\frac{5-\sqrt{5}}{2}\right)^n+5^{n+2}. \tag{24}$$

Proof: Suppose the set of perfect matchings of T_n is $\mathcal{H}(T_n)$. Since every vertex of T_n is matched by a perfect matching M of T_n, without loss of generality, say the matched vertex by M be v_{10} (see Figure 2 the labeling of T_n), then the perfect matchings of T_n is classified by v_{10}. We denote the set of perfect matchings with edges $v_{10}u_{10}, v_{10}v_{11}$, and $v_{10}v_{14}$ in the graph T_n by $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$, respectively. Since $\mathcal{H}_i\cap\mathcal{H}_j=\varnothing$ $(1\le i<j\le 3)$, we have

$$\mathcal{H}(T_n)=\mathcal{H}_1\cup\mathcal{H}_2\cup\mathcal{H}_3, \tag{25}$$

$$f(n)=\left|\mathcal{H}(T_n)\right|=\left|\mathcal{H}_1\right|+\left|\mathcal{H}_2\right|+\left|\mathcal{H}_3\right|. \tag{26}$$

By symmetry, $\left|\mathcal{H}_2\right|=\left|\mathcal{H}_3\right|$.

Now we consider $\left|\mathcal{H}_1\right|$. In every perfect matching of T_n, all vertices of T_n are matched vertices. We know that $v_{10}u_{10}\in M$ for every $M\in\mathcal{H}_1$. Then the remaining four vertices $v_{11}, v_{12}, v_{13}, v_{14}$ on the 0th ring must be covered by four radial edges or two radial edges or 0 radial edges. So the perfect matchings of $\mathcal{H}_1$ can be divided into the following five cases.

Case 1: We denote the perfect matchings set with edges $v_{10}u_{10}, v_{11}u_{12}, v_{12}u_{14}, v_{13}u_{16}$ and $v_{14}u_{18}$ by $\mathcal{H}_{11}$. Then the remaining vertices are covered by radial edges. So

$$\mathcal{H}_{11} = 1. \tag{27}$$

Case 2: Suppose that the set of perfect matchings with edges $v_{10}u_{10}, v_{11}u_{12}, v_{12}u_{14}$ and $v_{13}v_{14}$ is $\mathcal{H}_{12}$. Let $S \subseteq V(T_n)$ and $S = \{v_{10}, u_{10}, v_{11}, u_{12}, v_{12}, u_{14}, v_{13}, v_{14}\}$. Then $T_n - S \cong G_{n-1}$. So we have

$$|\mathcal{H}_{12}| = g(n-1). \tag{28}$$

Case 3: The set of perfect matchings containing edges $v_{10}u_{10}, v_{11}u_{12}, v_{12}v_{13}$ and $v_{14}u_{18}$ is denoted by $\mathcal{H}_{13}$. Similar to the analysis for Case 2, we get

$$|\mathcal{H}_{13}| = g(n-1). \tag{29}$$

Case 4: The set of perfect matchings is denoted by $\mathcal{H}_{14}$ if it contains edges $v_{10}u_{10}, v_{11}v_{12}, v_{13}u_{16}$ and $v_{14}u_{18}$. From the above analysis, it is easy to verify that

$$|\mathcal{H}_{14}| = g(n-1). \tag{30}$$

Case 5: Let the set of perfect matchings containing edges $v_{10}u_{10}, v_{11}v_{12}, v_{13}v_{14}$ be $\mathcal{H}_{15}$. For $S \subseteq V(T_n)$, let $S = \{v_{10}, u_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$, we define the graph $T_n - S$ as H_{n-1}. Then the number of perfect matchings of the graph H_{n-1} is $|\mathcal{H}_{15}|$. It is clear that H_{n-1} has $n-1$ hexagon layers (see Figure 7). We write that H_n has one more concentric hexagonal layer than H_{n-1}. The vertex labels of the graph H_n are shown in Figure 8.

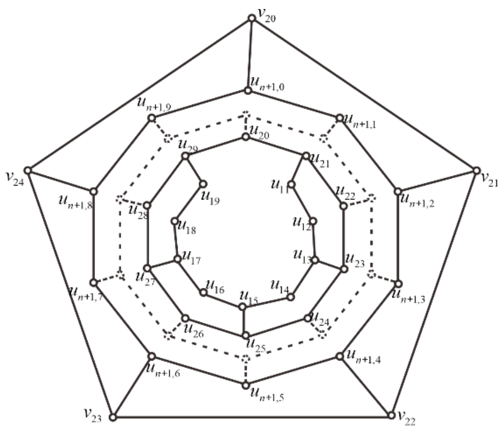

Fig. 7. The graph H_{n-1}.

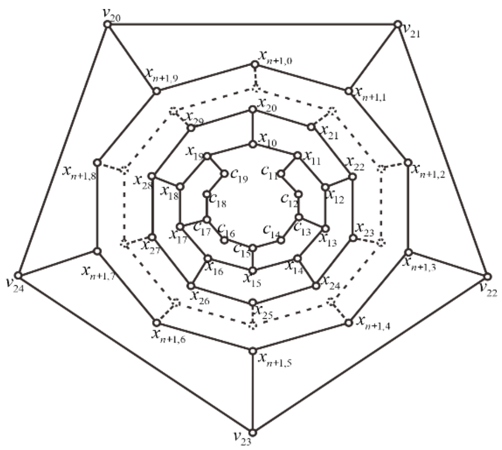

Fig. 8. The graph H_n.

Claim 4: Let the number of perfect matchings of the graph H_n be $h(n)$. Then

$$h(n)=5^{n+2}. \tag{31}$$

Proof: The set of perfect matchings of the graph H_n is denoted by $\mathcal{D}(H_n)$. Let the set of perfect matchings containing edges $c_{11}x_{11}, c_{13}x_{13}, c_{15}x_{15}, c_{17}x_{17}$ and $c_{19}x_{19}$ be $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ and $\mathcal{D}_5$, respectively. Then we have $\mathcal{D}_i \cap \mathcal{D}_j = \varnothing (1 \le i < j \le 5)$, $\mathcal{D}(H_n) = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5$. Hence

$$h(n)=\left|\mathcal{D}(H_n)\right|=\left|\mathcal{D}_1\right|+\left|\mathcal{D}_2\right|+\left|\mathcal{D}_3\right|+\left|\mathcal{D}_4\right|+\left|\mathcal{D}_5\right|. \tag{32}$$

From the structural properties of H_n, $|\mathcal{D}_1|$, $|\mathcal{D}_2|$, $|\mathcal{D}_3|$, $|\mathcal{D}_4|$ and $|\mathcal{D}_5|$ are equivalent. Therefore

$$h(n)=\left|\mathcal{D}(H_n)\right|=5\left|\mathcal{D}_1\right|. \tag{33}$$

Since $c_{11}x_{11} \in M$ for any $M \in \mathcal{D}_1$, we have $c_{12}c_{13}, c_{14}c_{15}, c_{16}c_{17}, c_{18}c_{19} \in M$. Let $S = \{c_{11}, x_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{17}, c_{18}, c_{19}\}$ for $S \subseteq V(H_n)$, then $H_n - S \cong H_{n-1}$. From the definition of $h(n)$, we have

$$\left|\mathcal{D}_1\right|=h(n-1). \tag{34}$$

So we get the following result.

$$h(n)=\left|\mathcal{D}(H_n)\right|=5\left|\mathcal{D}_1\right|=5h(n-1). \tag{35}$$

We can easily verify that $h(1)=125$. Thus

$$h(n)=5^{n+2}. \tag{36}$$

To sum up,

$$\left|\mathcal{H}_{15}\right|=h(n-1)=5^{n+1}. \tag{37}$$

Then

$$\left|\mathcal{H}_1\right|=\left|\mathcal{H}_{11}\right|+\left|\mathcal{H}_{12}\right|+\left|\mathcal{H}_{13}\right|+\left|\mathcal{H}_{14}\right|+\left|\mathcal{H}_{15}\right|=1+3g(n-1)+h(n-1). \tag{38}$$

Next we consider the $|\mathcal{H}_2|$. We use a similar analysis to find $\mathcal{H}_2$. The set of perfect matchings of $|\mathcal{H}_2|$ can be classified into three types. The perfect matchings set containing edges $v_{10}v_{11}, v_{12}v_{13}$ and $v_{14}u_{18}$ is denoted as $\mathcal{H}_{21}$. Including the edges $v_{10}v_{11}, v_{12}u_{14}$ and $v_{13}u_{16}$ is denoted as $\mathcal{H}_{22}$. The third type is a perfect matchings set containing edges $v_{10}v_{11}, v_{12}u_{14}$ and $v_{13}v_{14}$, which we define as $\mathcal{H}_{23}$. The number of perfect matchings of $\mathcal{H}_{21}$ is $h(n-1)$. It can be seen that the number of perfect matchings of $\mathcal{H}_{22}$ is $g(n-1)$. And the number of perfect matchings of $\mathcal{H}_{23}$ is $h(n-1)$. The above three types of perfect matchings in $\mathcal{H}_2$ are exclusive of each other, disjoint each other, and contain all perfect matchings of $\mathcal{H}_2$. Hence we have

$$\left|\mathcal{H}_2\right|=\left|\mathcal{H}_{21}\right|+\left|\mathcal{H}_{22}\right|+\left|\mathcal{H}_{23}\right|=g(n-1)+2h(n-1), \tag{39}$$

$$\left|\mathcal{H}_3\right|=\left|\mathcal{H}_2\right|=g(n-1)+2h(n-1). \tag{40}$$

In summary, we have the following conclusion.

$$f(n)=\left|\mathcal{H}(T_n)\right|=\left|\mathcal{H}_1\right|+\left|\mathcal{H}_2\right|+\left|\mathcal{H}_3\right|=1+5g(n-1)+5h(n-1). \quad (41)$$

From equations (4) and (31), we obtain

$$f(n)=1+5\times\left(\frac{5+\sqrt{5}}{2}\right)^n+5\times\left(\frac{5-\sqrt{5}}{2}\right)^n+5^{n+2}. \quad (42)$$

We have completed the proof of Theorem 4.1.

If T_n is a tubular fullerene graph with p vertices, it can be seen from the previous analysis that $p=20+10n$.

From the above proofing process, and combining with Theorem 4.1, we can get the following direct result.

Corollary 4.2 Let the number of perfect matchings of a tubular fullerene graph T_n with p $(p\geq 20)$ vertices is $N(p)$, then

$$N(p)=1+5\times\left(\frac{5+\sqrt{5}}{2}\right)^{\frac{p-20}{10}}+5\times\left(\frac{5-\sqrt{5}}{2}\right)^{\frac{p-20}{10}}+5^{\frac{p}{10}}. \quad (43)$$

Let's do a simple verification. When $n=1$, then $p=30$, From (42) (43) , $f(1)=151$, $N(30)=151$. We can verify the number of perfect matchings of a tubular fullerene graph T_1 is also 151, that is to say, it is consistent with the results we get. In particular, $n=0$, i.e. $p=20$, we get the number of perfect matchings of the dodecahedron F_{20} also conforms to the above formulas.

5. Conclusions

In this paper, we study the formula for calculating the number of perfect matchings of a tubular fullerene graph T_n. We know that it is very difficult to get the counting formula for perfect matchings of a graph. But for some special graph classes, using the recursive method to find the explicit expression of the number of perfect matchings is an effective method. The method in this article has reference value. In later work, we can use this idea to find the perfect matchings counting formula for certain graph classes.

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