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Concepts of Bezier Polynomials and its Application in Odd Higher Order Non-linear Boundary Value Problems by Galerkin WRM

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Abstract: Many different methods are applied and used in an attempt to solve higher order nonlinear boundary value problems (BVPs). Galerkin weighted residual method (GWRM) are widely used to solve BVPs. The main aim of this paper is to find the approximate solutions of fifth, seventh and ninth order nonlinear boundary value problems using GWRM. A trial function namely, Bezier Polynomials is assumed which is made to satisfy the given essential boundary conditions. Investigate the effectiveness of the current method; some numerical examples were considered. The results are depicted both graphically and numerically. The numerical solutions are in good agreement with the exact result and get a higher accuracy in the solutions. The present method is quit efficient and yields better results when compared with the existing methods. All problems are performed using the software MATLAB R2017a.

Index Terms: Higher order non-linear differential equations, Numerical solutions, Galerkin method, Bezier polynomials.

1. Introduction

The BVPs of higher order have been examined due to their mathematical importance and applications in diversified applied sciences. Higher order non-linear BVPs occur in the study of fluid dynamics, astrophysics, hydrodynamic, hydro magnetic stability, astronomy, beam and long wave theory, induction motors, engineering and applied physics.. Many phenomena in applied Mathematics and other sciences can be described very successfully by models using mathematical tools from ordinary differential equations. On the other hand, solving nonlinear ordinary differential equations analytically may guide Mathematicians to know how to describe some physical process deeply and sometimes lead to know some facts which are not simply understood through common observations.

Fifth order BVPs arise in some branches of applied mathematics, engineering and many other fields of advanced physical sciences especially in the mathematical modeling of viscoelastic flows. Wazwaz [1] developed decomposition technique for computing fifth order linear and non-linear BVPs while Erturk [2] solving non-linear problems by using differential transformation technique. These techniques have also been proven to be eighth order convergent. The application of seventh order BVPs available in engineering sciences. These problems arise in Mathematical modeling of induction motors with two rotor circuits. The performance of the induction motor behavior is modeled by a seventh order ordinary differential equation. This model is constructed with two stator state variables, one shaft speed and two rotor state variables. Presently, the literature on the numerical solutions of seventh order BVPs is not too much available. Siddiqi and Iftkhar [3] determined the approximate solutions of seventh, eighth, and tenth-order BVPs using homotopy analysis method (HAM). On the other hand, reproducing kernel space method was used by Ghazala *et al* [4]. The existence and uniqueness theorem of solutions of BVPs was presented in a book written by Agarwal [5] which does not contain any numerical examples. Recently, Siddiqi and Muzammal [6] developed variation of parameters method to solve seventh order BVPs.

The BVPs of ninth order have been presented due to their mathematical importance and the potential for applications in hydrodynamic and hydro magnetic stability. Few techniques including finite element, finite-difference, polynomial and non-polynomial spline, homotopy perturbation, decomposition etc. have been used to solve ninth order non-linear boundary value problems [7,8,9,10,11]. Most of these techniques used so far are well known that they provide the solution only at grid points. Investigation of haar wavelet Collocation method to solve ninth order boundary value problems was developed by Reddy *et al* [12]. Kasi Viswanadham, and Reddy [13] determined the numerical solution of ninth order boundary value problems by Petrov-Galerkin method with Quintic B-splines as basis functions and Septic B-splines as weight functions. The modified variational iteration and homotopy perturbation methods have been applied for solving tenth and ninth order BVPs and twelfth order BVPs by Mohy-ud-Din and Yildirim [15] and Mohamed Othman *et al* [16] respectively. Nadjafi and Zahmatkesh [17] also investigated the homotopy perturbation method for solving higher order BVPs.

The rest of the paper is organized in four sections. In section two, we provide a short discussion on Bezier polynomials. The analysis of Galerkin weighted residual method is presented in section three. In section four, considers certain numerical problems and their results to show the accuracy of the present method. Finally, we draw a conclusion.

2. Bezier Polynomials

Bezier polynomials are popular because their mathematical descriptions are compact, intuitive, and elegant. They are easy to compute and easy to use for solving higher order linear and non-linear BVPs.

The Bezier polynomials $B_{m,n}(x)$ of order n are defined as follows

$$B_{m,n}(x) = \sum_{m=0}^{n} \binom{n}{m} x^m (1-x)^{n-m} P_m, 0 \le x \le 1$$

Some Bezier polynomials are given below:

$B_0(x) = (1-x)^{19}$

$B_1(x) = 19(1-x)^{18} x$

$B_2(x) = 171(1-x)^{17} x^2$

$B_3(x) = 969(1-x)^{16} x^3$

$B_4(x) = 3876(1-x)^{15} x^4$

$B_5(x) = 11628(1-x)^{14} x^5$

$B_6(x) = 27132(1-x)^{13} x^6$

$B_7(x) = 50388(1-x)^{12} x^7$

$B_8(x) = 75582(1-x)^{11} x^8$

$B_9(x) = 92378(1-x)^{10} x^9$

$B_{10}(x) = 92378(1-x)^{9} x^{10}$

$B_{11}(x) = 3876(1-x)^{8} x^{11}$

$B_{12}(x) = 75582(1-x)^{7} x^{12}$

$B_{13}(x) = 27132(1-x)^{6} x^{13}$

$B_{14}(x) = 11628(1-x)^{5} x^{14}$

$B_{15}(x) = 3876(1-x)^{4} x^{15}$

$B_{16}(x) = 969(1-x)^{3} x^{16}$

$B_{17}(x) = 171(1-x)^{2} x^{17}$

$B_{18}(x) = 19(1-x) x^{18}$

$B_{19}(x) = x^{19}$

Since Bezier polynomials have special properties at $x = 0 : B_{m,n}(0) = 0$ and $x = 1 : B_{m,n}(1) = 0$, $m = 1, 2, 3, \ldots, n-1$ respectively, so that they can be used as set of basis function to satisfy the corresponding homogeneous form of the essential boundary conditions in the Galerkin weighted residual method to solve a BVP over the interval $[0,1]$. In this research paper we used Bezier polynomials as basis functions to solve higher order non-linear BVPs.

3. Methodology

In the present section, we formulate Galerkin WRM for the computed solutions of fifth-order non-linear BVPs and then we extend our idea for solving seventh and ninth order nonlinear BVPs. This article considers the following non-linear boundary value problem

$$a_5 \frac{d^5 y}{dx^5} + a_4 \frac{d^4 y}{dx^4} + a_3 \frac{d^3 y}{dx^3} + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = q, c < x < d \tag{1}$$

with the given boundary conditions

$$y(\mathrm{c}) = \mathrm{C}_0, y(d) = \mathrm{D}_0, y'(c) = \mathrm{C}_1, y'(d) = \mathrm{D}_1, y''(c) = \mathrm{C}_2 \tag{2}$$

where $\mathrm{C}_m, m = 0,1,2$ and $\mathrm{D}_n, n = 0,1$ are finite real constants and $\mathrm{a}_i, i = 0,1,2,3,4,5$ and q are all continuous and differentiable functions of x defined on the interval $[c, d]$.

We approximate $\tilde{y}(x)$ as follows

$$\tilde{y}(x) = \theta_0(x) + \sum_{i=1}^{n-1} \gamma_i B_i(x) \; , n \ge 2 \tag{3}$$

Here $\theta_0(x)$ is satisfied the essential boundary conditions and $B_i(\mathrm{c}) = B_i(d) = 0$ for each $i = 1, 2, 3, \ldots, n-1$.

Putting equation (3) into equation (1), the Galerkin weighted residual equations are

$$\int_{\mathrm{c}}^{d} \left[a_5 \frac{d^5 \tilde{y}}{dx^5} + a_4 \frac{d^4 \tilde{y}}{dx^4} + a_3 \frac{d^3 \tilde{y}}{dx^3} + a_2 \frac{d^2 \tilde{y}}{dx^2} + a_1 \frac{d\tilde{y}}{dx} + a_0 \tilde{y} - q \right] B_j(x)\, dx = 0, \; j = 1, 2, .., n-1 \tag{4}$$

Integrating up to second derivative on the LHS of equation (4), we have

$$\begin{aligned}
\int_{\mathrm{c}}^{d} \left[a_5 \frac{d^5 \tilde{y}}{dx^5} \right] B_j(x)\, dx &= \left[a_5 \frac{d^4 \tilde{y}}{dx^4} B_j(x) \right]_{\mathrm{c}}^{d} - \int_{\mathrm{c}}^{d} \frac{d}{dx}\left(a_5 B_j(x)\right) \frac{d^4 \tilde{y}}{dx^4} dx \\
&= -\left[\frac{d}{dx}\left[a_5 B_j(x)\right] \frac{d^3 \tilde{y}}{dx^3} \right]_{\mathrm{c}}^{d} + \int_{\mathrm{c}}^{d} \frac{d^2}{dx^2}\left(a_5 B_j(x)\right) \frac{d^3 \tilde{y}}{dx^3} dx \quad [\because \; B_i(\mathrm{c}) = B_i(d) = 0 \\
&= -\left[\frac{d}{dx}\left[a_5 B_j(x)\right] \frac{d^3 \tilde{y}}{dx^3} \right]_{\mathrm{c}}^{d} + \left[\frac{d^2}{dx^2}\left[a_5 B_j(x)\right] \frac{d^2 \tilde{y}}{dx^2} \right]_{\mathrm{c}}^{d} - \int_{\mathrm{c}}^{d} \frac{d^3}{dx^3}\left(a_5 B_j(x)\right) \frac{d^2 \tilde{y}}{dx^2} dx \\
&= -\left[\frac{d}{dx}\left[a_5 B_j(x)\right] \frac{d^3 \tilde{y}}{dx^3} \right]_{\mathrm{c}}^{d} + \left[\frac{d^2}{dx^2}\left[a_5 B_j(x)\right] \frac{d^2 \tilde{y}}{dx^2} \right]_{\mathrm{c}}^{d} - \left[\frac{d^3}{dx^3}\left[a_5 B_j(x)\right] \frac{d\tilde{y}}{dx} \right]_{\mathrm{c}}^{d} + \int_{\mathrm{c}}^{d} \frac{d^4}{dx^4}\left[a_5 B_j(x)\right] \frac{d\tilde{y}}{dx} dx
\end{aligned} \tag{5}$$

$$\int_{\mathrm{c}}^{d} \left[a_4 \frac{d^4 \tilde{y}}{dx^4} \right] B_j(x)\, dx = -\left[\frac{d}{dx}\left[a_4 B_j(x)\right)\right] \frac{d^2 \tilde{y}}{dx^2} \right]_{\mathrm{c}}^{d} + \left[\frac{d^2}{dx^2}\left[a_4 B_j(x)\right] \frac{d\tilde{y}}{dx} \right]_{\mathrm{c}}^{d} - \int_{\mathrm{c}}^{d} \frac{d^3}{dx^3}\left[a_3 B_j(x)\right] \frac{d\tilde{y}}{dx} dx \tag{6}$$

$$\int_{\mathrm{c}}^{d} \left[a_3 \frac{d^3 \tilde{y}}{dx^3} \right] B_j(x)\, dx = -\left[\frac{d}{dx}\left[a_3 B_j(x)\right] \frac{d\tilde{y}}{dx} \right]_{\mathrm{c}}^{d} + \int_{\mathrm{c}}^{d} \frac{d^2}{dx^2}\left[a_3 B_j(x)\right] \frac{d\tilde{y}}{dx} dx \tag{7}$$

$$\begin{aligned}
\int_{\mathrm{c}}^{d} \left[a_2 \frac{d^2 \tilde{y}}{dx^2} \right] B_j(x)\, dx &= \left[a_2 \frac{d\tilde{y}}{dx} B_j(x) \right]_{\mathrm{c}}^{d} - \int_{\mathrm{c}}^{d} \frac{d}{dx}\left[a_2 B_j(x)\right] \frac{d\tilde{y}}{dx} dx \\
&= -\int_{\mathrm{c}}^{d} \frac{d}{dx}\left[a_2 B_j(x)\right] \frac{d\tilde{y}}{dx} dx \quad [\because \; B_i(\mathrm{c}) = B_i(d) = 0 \,]
\end{aligned} \tag{8}$$

Substituting equations (5) to equation (8) into equation (4) and applying approximate solution $\tilde{y}(x)$ which is given in equation (3) and imposing the above boundary conditions we developed a system of equations in matrix form as follows

$$\sum_{i=1}^{n-1} K_{i,j}\,\gamma_i = L_j\ ,\ j = 1,2,\dots,n-1 \tag{9}$$

where

$$\begin{aligned} K_{i,j} = \int_c^d \left[\left\{ -\frac{d^3}{dx^3}\left(a_4 B_j(x)\right) + \frac{d^2}{dx^2}\left(a_3 B_j(x)\right) - \frac{d}{dx}\left(a_2 B_j(x)\right) + a_1 B_j(x) \right\} \frac{d}{dx}\left(B_i(x)\right) + a_0 B_i(x) B_j(x) \right] dx \\ - \left[\frac{d}{dx}\left[a_4 B_j(x)\right] \frac{d^2}{dx^2}[B_i(x)] \right]_{x=d} + \left[\frac{d}{dx}\left[a_4 B_j(x)\right] \frac{d^2}{dx^2}[B_i(x)] \right]_{x=c} \end{aligned} \tag{10}$$

$$\begin{aligned} L_j = \int_c^d \left\{ q(x) B_j(x) + \left[\frac{d^3}{dx^3}\left(a_4 B_j(x)\right) - \frac{d^2}{dx^2}\left(a_3 B_j(x)\right) + \frac{d}{dx}\left(a_2 B_j(x)\right) - a_1 B_j(x) \right] \frac{d\theta_0}{dx} - a_0 \theta_0 B_j(x) \right\} dx \\ - \left[\frac{d^2}{dx^2}\left[a_4 B_j(x)\right] \right]_{x=c} \times \mathrm{C}_2 + \left[\frac{d}{dx}\left[a_3 B_j(x)\right] \right]_{x=d} \times \mathrm{D}_1 - \left[\frac{d}{dx}\left[a_3 B_j(x)\right] \right]_{x=c} \times \mathrm{C}_1,\ j = 1,2,\dots,n-1 \end{aligned} \tag{11}$$

We solve the systems (9) by neglecting non-linear terms. Then by applying the most common Newton's iterative technique we get the computed solutions for the desired non-linear BVP. This formulation is described through the numerical examples in the next section.

4. Results & Discussion

To implement the present method, three non-linear problems of various orders are considered. For all the examples, the solutions obtained by the proposed method are compared with the exact solutions. All problems are performed using the software MATLAB R2017a.

4.1 Investigation by First Example

Consider the following fifth-order nonlinear differential equation [1,2]

$$\frac{d^5 y}{dx^5} = y^2 e^{-x},\ x \in [0,1] \tag{12}$$

Boundary conditions:

$$y(0) = 1,\ y(1) = e,\ y'(0) = 1,\ y'(1) = e,\ y''(0) = 1 \tag{13}$$

Analytic solution: $y(x) = e^x$

Solution

Using the method discussed in section three, we approximate $y(x)$ in the following form

$$\tilde{y}(x) = \theta_0(x) + \sum_{i=1}^{n-1} \gamma_i B_i(x)\ ,n \ge 2 \tag{14}$$

Here $\theta_0(x) = 1 - x(1-e)$ is satisfied by the essential boundary conditions of equation (9b).

Now the unknown parameters $\gamma_i, (i = 1,2,\dots,n-1)$ specify the given system of non-linear equations

$$(K + M)\,N = L \tag{15}$$

where the elements of K, M, N, L are $k_{i,j}, m_{i,j}, \gamma_i, l_j$ respectively, followed by

$$k_{i,j} = \int_0^1 \left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d}{dx}\left[B_i(x)\right] - 2\theta_0 e^{-x} B_i(x) B_j(x) \right] dx - \left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^3}{dx^3}\left[B_i(x)\right] \right]_{x=1} + \left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^3}{dx^3}\left[B_i(x)\right] \right]_{x=0} + \left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^2}{dx^2}\left[B_i(x)\right] \right]_{x=1} \tag{16}$$

$$m_{i,j} = -\sum_{k=1}^{n-1} \gamma_k \int_0^1 \left[B_i(x) B_k(x) B_j(x) \right] e^{-x} dx \tag{17}$$

$$l_j = \int_0^1 \left[-\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d\theta_0}{dx} + \theta_0^2 e^{-x} B_j(x) \right] dx + \left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=1} - \left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=0} - \left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^2\theta_0}{dx^2} \right]_{x=1} + \left[\frac{d^2}{dx^2}\left[B_j(x)\right] \right]_{x=0} + \left[\frac{d^3}{dx^3}\left[B_j(x)\right] \right]_{x=1} \times (e) - \left[\frac{d^3}{dx^3}\left[B_j(x)\right] \right]_{x=0} \tag{18}$$

Solving the systems (15) by neglecting non-linear terms using Galerkin WRM we obtained the parameter γ_i. That is, to find initial coefficients we solve the following system of linear equations.

$$\mathrm{K}N = L \tag{19}$$

where the corresponding elements are

$$k_{i,j} = \int_0^1 \left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d}{dx}\left[B_i(x)\right] \right] dx - \left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^3}{dx^3}\left[B_i(x)\right] \right]_{x=1} + \left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^3}{dx^3}\left[B_i(x)\right] \right]_{x=0} + \left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^2}{dx^2}\left[B_i(x)\right] \right]_{x=1} \tag{20}$$

$$l_j = \int_0^1 \left[-\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d\theta_0}{dx} \right] dx + \left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=1} - \left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=0} - \left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^2\theta_0}{dx^2} \right]_{x=1} + \left[\frac{d^2}{dx^2}\left[B_j(x)\right] \right]_{x=0} + \left[\frac{d^3}{dx^3}\left[B_j(x)\right] \right]_{x=1} \times (e) - \left[\frac{d^3}{dx^3}\left[B_j(x)\right] \right]_{x=0} \tag{21}$$

If the initial values of the unknown parameters γ_i are obtained from equation (19), they are substituted into equation (15) to obtain new estimates for the values of unknown parameters γ_i.

This iteration process continues until the desired values of the unknown parameters are obtained. Substituting the final values of the parameters into equation (14), we obtain an approximate solution of the non-linear BVP (12).

Table 1. Comparing computed value with analytic value for the first example in y_i using 8 iterations

x	Exact values	14, Bezier polynomials			Reference[1,2]	
		Approximate	Relative Error	Maximum relative error	Maximum relative error by Wazwaz	Maximum relative error by Erturk
0.0	1.0000000000	1.0000000000	0.000000000			
0.1	1.1051709181	1.1051709181	1.91×10^{-14}			
0.2	1.2214027582	1.2214027582	3.10×10^{-13}			
0.3	1.3498588076	1.3498588076	4.65×10^{-13}			
0.4	1.4918246980	1.4918246980	1.43×10^{-13}			
0.5	1.6487212707	1.6487212707	1.81×10^{-13}	4.65×10^{-13}	4.10×10^{-8}	1.52×10^{-10}
0.6	1.8221188004	1.8221188004	5.43×10^{-14}			
0.7	2.0137527075	2.0137527075	1.11×10^{-14}			
0.8	2.2255409285	2.2255409285	2.91×10^{-13}			
0.9	2.4596031112	2.4596031112	1.32×10^{-13}			
1.0	2.7182818285	2.7182818285	0.000000000			

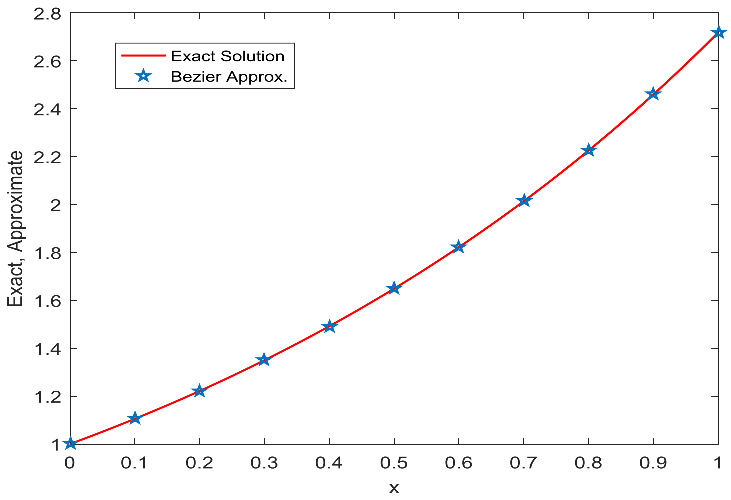

Fig.1. Computed vs analytical solutions for first example

The analytic solution and the approximate solution obtained are depicted in **Fig.1**. There is a very good agreement and relationship between the approximate solutions obtained by 8th iterations using Galerkin weighted residual method and the exact solutions which are shown in **Table 1**.

4.2 Investigation by Second Example

Consider the following seventh-order nonlinear differential equation [3,13]

$$\frac{d^7y}{dx^7} = y^2e^{-x}, x \in [0,1] \tag{22}$$

Boundary conditions:

$$y(0)=1, y(1)=e, y'(0)=1, y'(1)=e, y''(0)=1, y''(1)=e, y'''(0)=1$$

Analytic solution:

$$y(x) = e^x$$

Table 2. Comparing computed value with analytic value for the second example in y_i using 6 iterations

x	Exact values	15, Bezier polynomials			Reference[3,13]	
		Approximate	Relative Error	Maximum relative error	Maximum relative error by Siddiqi et al.	Maximum relative error by Bellal and Shafiq
0.0	1.0000000000	1.0000000000	0.000000000	7.17×10^{-14}	7.59×10^{-10}	7.15×10^{-12}
0.1	1.1051709181	1.1051709181	1.21×10^{-16}			
0.2	1.2214027582	1.2214027582	4.12×10^{-14}			
0.3	1.3498588076	1.3498588076	1.63×10^{-15}			
0.4	1.4918246980	1.4918246980	4.20×10^{-15}			
0.5	1.6487212707	1.6487212707	7.17×10^{-14}			
0.6	1.8221188004	1.8221188004	7.68×10^{-15}			
0.7	2.0137527075	2.0137527075	2.18×10^{-15}			
0.8	2.2255409285	2.2255409285	1.23×10^{-16}			
0.9	2.4596031112	2.4596031112	1.23×10^{-14}			
1.0	2.7182818285	2.7182818285	0.000000000			

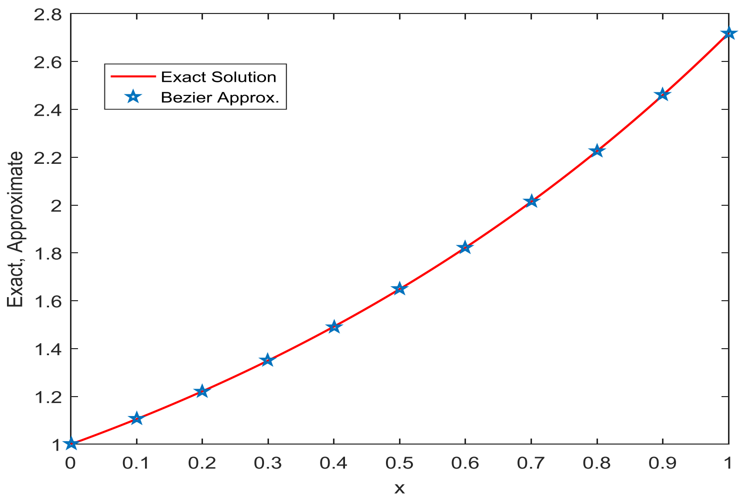

Fig.2. Computed vs analytical solutions for second example

The analytic solution and the approximate solution obtained are depicted in **Fig.2**. There is a very good agreement and relationship between the approximate solutions obtained by 6th iterations using Galerkin weighted residual method and the exact solutions which are shown in **Table 2**.

4.3 Investigation by Third Example

Consider the following ninth-order nonlinear differential equation [12,14]

$$\frac{d^9y}{dx^9} = y^2\frac{dy}{dx} + cos^3(x), x \in [0,1] \tag{23}$$

Boundary conditions:

$$y(0)=1, y(1)=\sin(1), y'(0)=1, y'(1)=\cos(1), y''(0)=0, y''(1)=-\sin(1), y'''(0)=-1,$$
$$y'''(1)=-\cos(1), y^{(iv)}(0)=0$$

Analytic solution: $y(x)=\sin x$

Table 3. Comparing computed value with analytic value for the third example in y_i using 8 iterations

x	Exact values	15, Bezier polynomials			Reference[12,14]	
		Approximate	Relative Error	Maximum relative error	Maximum relative error by Reddy et al.	Maximum relative error by Kasi et al.
0.0	1.0000000000	1.0000000000	0.000000000	3.65×10^{-14}	3.62×10^{-10}	5.57×10^{-6}
0.1	0.09983341665	0.09983341665	1.41×10^{-14}			
0.2	0.19866933080	0.19866933080	1.72×10^{-15}			
0.3	0.29552020666	0.29552020666	2.69×10^{-14}			
0.4	0.38941834231	0.38941834231	1.01×10^{-15}			
0.5	0.47942553860	0.47942553860	3.65×10^{-14}			
0.6	0.56464247340	0.56464247340	1.22×10^{-14}			
0.7	0.64421768724	0.64421768724	2.66×10^{-14}			
0.8	0.71735609090	0.71735609090	1.88×10^{-14}			
0.9	0.78332690963	0.78332690963	1.45×10^{-15}			
1.0	0.8414709848	0.8414709848	0.000000000			

The analytic solution and the approximate solution obtained are depicted in **Fig.3**. There is a very good agreement and relationship between the approximate solutions obtained by 8th iterations using Galerkin weighted residual method and the exact solutions which are shown in **Table 3**.

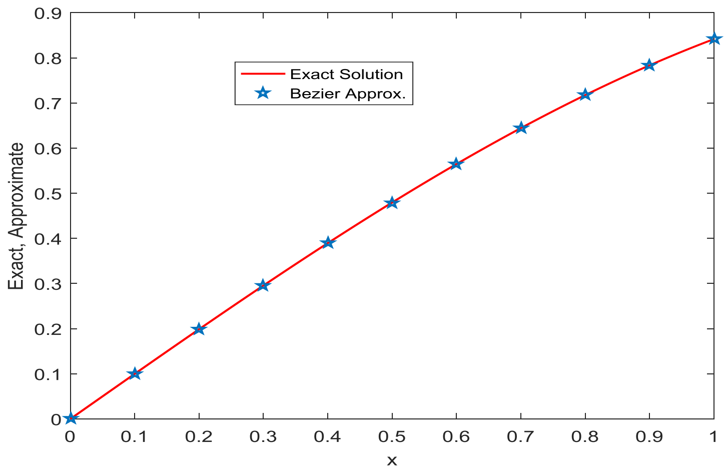

Fig.3. Computed vs analytical solutions for third example

5. Conclusions

The main objective of our work is to determine better numerical approximation for higher order non-linear boundary value problems. Therefore, we apply Galerkin weighted residual method to solve boundary value problems of various orders and compare the result with their analytic solution. The results are presented in a data structured table and sketching graphically. By observing all those figures and tables, it is clear that the presented outcome exhibits the higher estimated order of convergence of this method. So, we can conclude that GWRM is an efficient, unconditionally stable, highly modular and easily expandable method that can be applied for the solution of more complicated mathematical, physical and engineering problems.

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