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Labeling a kind of Cubic Graphs by Subgraph Embedding Method

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Abstract: Based on a problem raised by Gao et. al. (Bull. Malays. Math. Sci. Soc., 41 (2018) 443–453.), we construct a family of cubic graphs which are double-edge blow-up of ladder graphs. We determine the full friendly index sets of these cubic graphs by embedding labeling graph method. At the same time, the corresponding labeling graphs are provided.

Index Terms: Vertex labeling, Friendly labeling, Double-edge blow-up, P_2-embedding, C_4-embedding.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A *vertex 0-1 labeling* f induces an *edge 0-1 labeling* f^+, given by $f^+(uv) \equiv f(u)+f(v) \pmod 2$ for each $uv \in E(G)$. For $i \in \{0,1\}$, a vertex (resp. an edge) is called an i*-vertex* (resp. i*-edge*} if it is labeled by i. An edge is also called an (i, j) *-edge* if it is incident with both an i-vertex and a j-vertex.

Define $v_f(i) = |\{u \in V(G): f(u) = i\}|$ and $e_f(i) = |\{e \in E(G): f^+(e) = i\}|$. A vertex labeling f is said to be *friendly* if $|v_f(1) - v_f(0)| \le 1$. For a friendly labeling f of G, $I_f(G) = e_f(1) - e_f(0)$ is called the *friendly index* of G; and if $e_f(1) - e_f(0) = a$ then the labeling graph G is denoted by $G_f(a)$. The *full friendly index s*et, introduced by Shiu and Kwong[16], is the set $FFI(G) = \{e_f(1) - e_f(0): f \text{ is a friendly labeling of } G\}$. When the context is clear, we will drop the subscript f. In our figures of labeling graphs, the black points are labeled by 1 and the white points are labeled by 0.

Graph labeling theory has numerous applications in the field of parallel computing, very-large-scale integration design, etc. Generally speaking, it is difficult to obtain the full friendly index sets of an arbitrary graph. Researchers usually focus their study on some specific graphs. For example, the full friendly index sets of $P_2 \times P_n$ was determined by Shiu and Kwong[16]. The full friendly index sets of $C_m \times C_n$ was determined by Shiu and Ling [13]. In [12], Shiu and Lee determined the full friendly index sets of twisted cylinders. In [14] and [15], Sinha and Kaur investigated the full friendly index sets complete graphs, cycles, fans, double fans, wheels, double stars, $P_3 \times P_n$, and the tensor product of P_2 and P_n. Law [8] investigated the full friendly index sets of spiders and disproved a conjecture by Salehi and Lee [11] that the friendly index sets of a tree forms an arithmetic progression. Sun, Gao, and Lee [11] determined the full friendly index sets for the twisted product of Möbius ladders and they also determined the full friendly index sets of $P_m \times P_n$[4]. More results about full friendly index sets can be found in [10, 17-19].

Recently, Gao et. al. [2, 3] introduced the embedding labeling graph method and obtained the full friendly index sets of some kinds of cubic graphs. They asked in [2] for other types of cubic graphs, whose full friendly index sets can be obtained by embedding labeling graph method? In this paper, we obtain the full friendly index sets of a family of cubic graphs which are double-edge blow-up of $P_2 \times P_n, n \ge 2$.

Definition 1. ([3]) Let G and H be two graphs such that u and v are two particular vertices of H. An edge xy of G is blown-up by H at u and if xy is replaced by H by identifying x and u, and y and v respectively.

Let $P_2 \times P_n (n \geq 2)$ be the ladder graph with vertex set $V = \{u_i, v_i : 1 \leq i \leq n\}$ and edge set $E = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$, such that $|V| = 2n$ and $|E| = 3n-2$.

Let $K_{3,3}^-$ be the bipartite graph obtained by deleting an edge from $K_{3,3}$, suppose the two 2-degree vertices arrived are u and v.

Definition 2. The double-edge blow-up graph of $P_2 \times P_n$, denoted by H_{n-2}, is obtained by blowing up the edges $u_1 v_1$ and $u_n v_n$ of $P_2 \times P_n$ by two disjoint copies of $K_{3,3}^-$ at its u, v vertices respectively.

For easy of notations, we let $m = n-2$. By definition 2, we know that H_m is a cubic graph of order $2m+12$ and size $3m+18$. See H_1 in Figure 1 as an example.

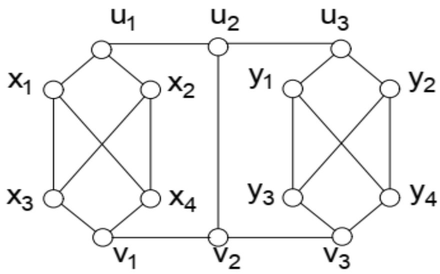

Fig.1. Graph H_1.

We end this section with some notations. As usual, $[k]$ stands for $\{1, 2, \cdots, k\}$. For a graph G and two vertex subsets X, Y of $V(G)$, we use $e(X)$ to denote the number of edges with both ends in X. We use $e(X, Y)$ to denote the number of edges with one end in X and the other in Y.

2. Some Basic Properties

The following lemma is concerned with the number of 1-edges in a labeling cycle. It says each cycle meets each edge cut with an even number of edges. We provide a proof by using the graph partitioning theory, which is closed related to graph labelings. For more problems and results on graph partitions, we refer the reader to [5, 6].

Lemma 3 ([17])**.** Let f be a labeling of a graph G that contains a cycle C as its subgraph. If C contains a 1-edge, then the number of 1-edges in C is a positive even number.

Proof: For $i = 1, 2$, let V_i be the set of vertices with label i. Then

$$e(1) = e(V_0, V_1) = \sum_{v \in V_0} d(v) - 2e(V_0) = 2|V_0| - 2e(V_0) \tag{1}$$

which is an even integer.

Staring with a friendly labeling of a graph, to get all possible friendly indices, an effective way is to exchanging vertices labels. The following lemma shows how the friendly index changes by changing vertices labels.

Lemma 4. Let f_1 be a friendly labeling of a cubic graph and f_2 is obtained from f_1 by exchanging two distinct vertices labels, then

$$\left(e_{f_1}(1) - e_{f_1}(0)\right) - \left(e_{f_2}(1) - e_{f_2}(0)\right) \equiv 0 \pmod 4. \tag{2}$$

Proof: Let V_1 (resp., $V_1^{'}$) be the set of vertices labelled 1 under f_1 (resp., f_2), then $|V_1|=|V_1^{'}|$. By considering the sum of degrees of vertices in V_1 and $V_1^{'}$, we obtain

$$\sum_{v\in V_1} d(v) = 3|V_1| = 2e(V_1) + e_{f_1}(1), \tag{3}$$

and

$$\sum_{v\in V_1^{'}} d(v) = 3|V_1^{'}| = 2e(V_1^{'}) + e_{f_2}(1). \tag{4}$$

Since $e_{f_1}(1) - e_{f_1}(0) = 2e_{f_1}(1) - |E|$, we have

$$\begin{aligned}
&\left(e_{f_1}(1) - e_{f_1}(0)\right) - \left(e_{f_2}(1) - e_{f2}(0)\right) \\
&= 2\left(e_{f_1}(1) - e_{f_2}(1)\right) \\
&= 2\left[\left(3|V_1| - 2e(V_1)\right) - \left(3|V_1^{'}| - 2e(V_1^{'})\right)\right] \\
&= 4\left(e(V_1^{'}) - e(V_1)\right) \\
&= 0(\text{mod } 4).
\end{aligned}$$

Let $H_m(a)$ be the friendly labeling graph H_m with $e(1) - e(0) = a$. We record the labels of a path-subgraph u_iv_i in a labeled H_m by a vector $\begin{bmatrix} f(u_i) \\ f(v_i) \end{bmatrix}$; and record the labels of a cycle-subgraph $u_iu_{i+1}v_{i+1}v_i$ by a matrix $\begin{bmatrix} f(u_i) & f(u_{i+1}) \\ f(v_i) & f(v_{i+1}) \end{bmatrix}$. Now we introduce the embedding method.

Definition 5. A P_2-embedding onto $H_m(a)$ at edges u_iu_{i+1} and v_iv_{i+1} is obtained by replacing u_iu_{i+1} and v_iv_{i+1} by two length 2 paths $u_ia_1u_{i+1}$ and $v_ia_2v_{i+1}$ respectively and jointing vertices a_1 and a_2. A P_2-embedding is denoted by

$$\begin{bmatrix} f(a_1) \\ f(a_2) \end{bmatrix} + \begin{bmatrix} f(u_i) & f(u_{i+1}) \\ f(v_i) & f(v_{i+1}) \end{bmatrix}. \tag{5}$$

For example, in $H_0(-14)$, embed $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ obtain $H_1(-15)$, see Figures 2 and 4.

Definition 6. A C_4-embedding onto $H_m(a)$ at edges u_iu_{i+1} and v_iv_{i+1} is obtained by replacing u_iu_{i+1} and v_iv_{i+1} by two length 3 paths $u_ib_1b_2u_{i+1}$ and $v_ib_3b_4v_{i+1}$ respectively. Then jointing vertices b_1, b_3 and vertices b_2, b_4 respectively. A C_4-embedding is denoted by

$$\begin{bmatrix} f(b_1) & f(b_2) \\ f(b_3) & f(b_4) \end{bmatrix} + \begin{bmatrix} f(u_i) & f(u_{i+1}) \\ f(v_i) & f(v_{i+1}) \end{bmatrix}. \tag{6}$$

For example, in $H_0(-14)$ embed $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ we obtain $H_2(-20)$, see Figures2 and 3.

Clearly, a P_2-embedding (resp., C_4-embedding) will adds three (resp., six) extra edges to a graph.

3. The Full Friendly index sets of H_m

In this section, we obtain the full friendly index sets of H_m by a sequence of embedding process. Firstly, we show the minimum value and maximum value of $e(1)$ among all friendly labelings.

Lemma 7. For any friendly labeling f of H_m, we have

(1) $2 \le e(1) \le 3m+18$ if $m \ge 0$ is even;

(2) $3 \le e(1) \le 3m+18$ if $m > 0$ is odd.

Proof: Let x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4 be the 3-degree vertices of the two $K_{3,3}^-$-subgraphs in H_m respectively (see Figure 1 for the case $m=1$).

Since H_m is connected, we have $e(1) \ge 1$. Note that each edge belongs to some cycle of H_m, by Lemma 3, we know that $e(1) \ge 2$.

When m is even, since $|V(H_m)| = 2(m+6)$ and the labeling f of H_m is friendly, we know $v(1) = v(0) = m+6$. To construct the labeling with $e(1) = 2$, we label x_1, x_2, x_3, x_4 and $u_i, v_i, (1 \le i \le \frac{m+2}{2})$ by 1 and the remainders by 0. The maximum value $3m+18$ can be attained by labeling $x_1, x_2, y_3, y_4, u_2, u_4, \ldots, u_{m+2}, v_1, v_3, \ldots, v_{m+1}$ by 1 and the remainders by 0.

When m is odd, if $e(1) = 2$ then by Lemma 3 each cycle contains one 1-edge must contain the other. Therefore, both of the two 1-edges will belong the same side (left or right) of edge $u_{\frac{m+3}{2}} v_{\frac{m+3}{2}}$, which implies the labeling f of H_m is not friendly. So $e(1) \ge 3$. The labeling with $e(1) = 3$ can be get by labeling x_1, x_2, x_3, x_4 and $v_i (1 \le i \le \frac{m+1}{2}), u_i (1 \le i \le \frac{m+3}{2})$ by 1 and the remainders by 0. The maximum value of $e(1)$ is $3m+18$ can be attained by labeling $x_3, x_4, y_3, y_4, u_1, u_3, \ldots, u_{m+2}, v_2, v_4, \ldots, v_{m+1}$ by 1 and the remainders by 0. This completes the proof of Lemma 7.

When m is even, first we show the $m=0$ and $m=2$ cases, then our result is proved by induction. When m is odd, our proof is carried out by doing P_2-embedding based on the even cases. Note that we only do embedding of friendly labeled P_2 or C_4-subgraphs. Hence, if the labeling of H_m is friendly, then so do H_{m+1} and H_{m+2}.

Lemma 8. When $m \ge 0$ is even,

$$FFI(H_m) = \{-3m - 18 + 4i : 1 \le i \le \frac{3m+18}{2}\}. \quad (7)$$

Proof: By Lemma 4 and Lemma 7, the left hand is a subset of the right. The labeling graphs in Figure 2 show that the equality holds when $m=0$.

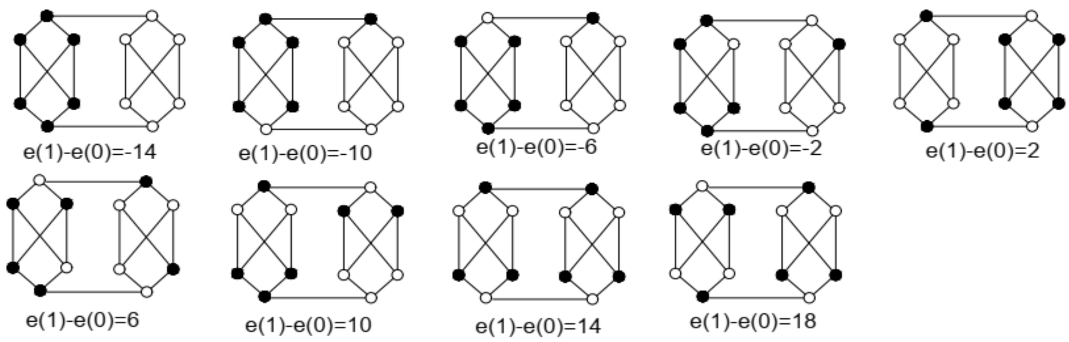

Fig.2. Labeling graphs H_0.

(i) $m = 2$

We show that there exist labeling graphs of $H_2(-24+4i), 1 \le i \le 12$, by doing the following C_4-embeddings:

(1). In $H_0(-14)$ embed $\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$, then $e(1)-e(0)$ decreased by 6, we obtain $H_2(-20)$. In this labeling graph, there exists the labeling subgraph $\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$.

(2). In $H_0(-14)$, $H_0(-2)$, $H_0(2)$, $H_0(10)$ embed

$$\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix},$$

then $e(1)-e(0)$ decreased by 2, we obtain $H_2(-16)$, $H_2(-4)$, $H_2(0)$, $H_2(8)$. In these labeling graphs, there exists the labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

(3). In $H_0(-10)$, $H_0(14)$, embed $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}+\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$, then $e(1)-e(0)$ decreased by 2, we obtain $H_2(-12)$, $H_2(12)$. In these labeling graphs, there exists the labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

(4). In $H_0(-6)$, $H_0(6)$, $H_0(18)$, embed $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}+\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, then $e(1)-e(0)$ decreased by 2, we obtain $H_2(-8)$, $H_2(4)$, $H_2(16)$. In these labeling graphs, there exists the labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

(5). In $H_0(14)$, embed $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$, then $e(1)-e(0)$ increased by 6, we obtain $H_2(20)$. In this labeling graph, there exists the labeling subgraph $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

(6). In $H_0(18)$, embed $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, then $e(1)-e(0)$ increased by 6, we obtain $H_2(24)$. In this labeling graph, there exists the labeling subgraph $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$.

The labeling graphs of H_2 are shown in Figure 3.

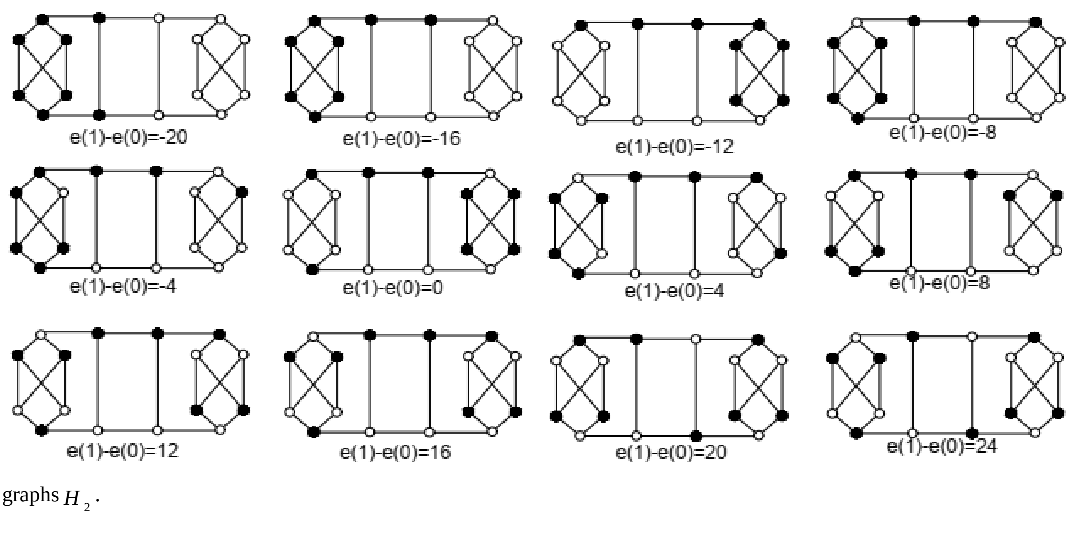

Fig.3. Labeling graphs H_2.

In conclusion, $FFI(H_2)=\{-24+4i:1\le i\le 12\}$.

(ii) $m=4$

(1). In $H_2(-20)$, embed $\begin{bmatrix}1&0\\1&0\end{bmatrix}+\begin{bmatrix}1&0\\1&0\end{bmatrix}$ and $\begin{bmatrix}1&1\\0&0\end{bmatrix}+\begin{bmatrix}1&0\\1&0\end{bmatrix}$, then $e(1)-e(0)$ decreased by 6 and 2 respectively, we obtain $H_4(-26)$ and $H_4(-22)$. In these labeling graphs, there exists the labeling subgraphs $\begin{bmatrix}1&0\\1&0\end{bmatrix}$ and $\begin{bmatrix}1&1\\0&0\end{bmatrix}$ respectively.

(2). In $H_2(-24+4i),2\le i\le 11$, there exists labeling subgraph $\begin{bmatrix}1&1\\0&0\end{bmatrix}$. We do embedding $\begin{bmatrix}1&1\\0&0\end{bmatrix}+\begin{bmatrix}1&1\\0&0\end{bmatrix}$ then $e(1)-e(0)$ decrease by 2, we obtain $H_4(-30+4i),3\le i\le 12$. In these labeling graphs, there exists the labeling subgraph $\begin{bmatrix}1&1\\0&0\end{bmatrix}$.

(3). In $H_2(20)$, embed $\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}1&1\\0&0\end{bmatrix}$, then $e(1)-e(0)$ increase by 6, we obtain $H_4(26)$. In this labeling graph, there exists the labeling subgraph $\begin{bmatrix}1&1\\0&0\end{bmatrix}$.

(4). In $H_2(24)$, embed $\begin{bmatrix}1&1\\0&0\end{bmatrix}+\begin{bmatrix}0&1\\1&0\end{bmatrix}$ and $\begin{bmatrix}0&1\\1&0\end{bmatrix}+\begin{bmatrix}0&1\\1&0\end{bmatrix}$, then $e(1)-e(0)$ decrease by 2 and $e(1)-e(0)$ increase by 6 respectively, we get $H_4(22)$ and $H_4(30)$. In thiese labeling graphs, there exists the labeling subgraphs $\begin{bmatrix}1&1\\0&0\end{bmatrix}$ and $\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

Hence, $FFI(H_4)=\{-30+4i:1\le i\le 15\}$.

(iii) $m\ge 6$

We assume that for even $k\ge 6$, $FFI(H_k)=\{-3k-18+4i:1\le i\le \frac{3k+18}{2}\}$. We do embeddings as follow.

(1). In $H_k(-3k-14)$, embed $\begin{bmatrix}1&0\\1&0\end{bmatrix}+\begin{bmatrix}1&0\\1&0\end{bmatrix}$ and $\begin{bmatrix}1&1\\0&0\end{bmatrix}+\begin{bmatrix}1&0\\1&0\end{bmatrix}$, then $e(1)-e(0)$ decreased by 6 and 2 respectively, we get $H_{k+2}(-3(k+2)-14)$ and $H_{k+2}(-3(k+2)-10)$. In these labeling graphs, there exists the labeling subgraphs $\begin{bmatrix}1&0\\1&0\end{bmatrix}$ and $\begin{bmatrix}1&1\\0&0\end{bmatrix}$ respectively.

(2). In $H_k(-3k-18+4i),2\le i\le \frac{3k+16}{2}$, there exists labeling subgraph $\begin{bmatrix}1&1\\0&0\end{bmatrix}$. We do embedding $\begin{bmatrix}1&1\\0&0\end{bmatrix}+\begin{bmatrix}1&1\\0&0\end{bmatrix}$, then $e(1)-e(0)$ decrease by 2, we obtain $H_{k+2}(-3(k+2)-18+4i),3\le i\le \frac{3(k+2)+12}{2}$. In these labeling graphs, there exists the labeling subgraph $\begin{bmatrix}1&1\\0&0\end{bmatrix}$.

(3). In $H_k(3k+14)$, embed $\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}1&1\\0&0\end{bmatrix}$ then $e(1)-e(0)$ increased by 6. Hence, we obtain $H_{k+2}(3(k+2)+14)$. In this labeling graph, there exists the labeling subgraph $\begin{bmatrix}1&1\\0&0\end{bmatrix}$.

(4). In $H_k(3k+18)$, embed $\begin{bmatrix}1&1\\0&0\end{bmatrix}+\begin{bmatrix}0&1\\1&0\end{bmatrix}$ and $\begin{bmatrix}0&1\\1&0\end{bmatrix}+\begin{bmatrix}0&1\\1&0\end{bmatrix}$, then $e(1)-e(0)$ decreased by 2 and increase $e(1)-e(0)$ by 6 respectively, we get $H_{k+2}(3(k+2)+10)$ and $H_{k+2}(3(k+2)+18)$. In these labeling graphs, there exists the labeling subgraphs $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

By induction, we have $FFI(H_{k+2}) = \{-3(k+2) - 18 + 4i : 1 \le i \le \frac{3(k+2)+18}{2}\}$.

Lemma 9. When $m \ge 1$ is odd,

$$FFI(H_m) = \{-3m - 16 + 4i : 1 \le i \le \frac{3m+17}{2}\}. \tag{8}$$

Proof: By Lemma 4 and Lemma 7, the left hand is a subset of the right. We prove the opposite direction by doing P_2-embeddings. We first show that the equality holds when $m = 1, 3$.

(i) $m = 1$

We show that there exist labeling graphs of $H_1(-19+4i), 1 \le i \le 10$, by doing the following P_2-embeddings:

(1). In $H_0(-14)$, $H_0(-2)$, $H_0(2)$, $H_0(10)$ embed $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, then $e(1) - e(0)$ decreased by 1, we obtain $H_1(-15)$, $H_1(-3)$, $H_1(1)$, $H_1(9)$.

(2). In $H_0(-6)$, $H_0(6)$, $H_0(18)$, embed $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $e(1) - e(0)$ decreased by 1, we obtain $H_1(-7)$, $H_1(5)$, $H_1(17)$.

(3). In $H_0(-10)$, $H_0(14)$, embed $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $e(1) - e(0)$ decreased by 1, we obtain $H_1(-11)$, $H_1(13)$.

(4). In $H_0(14)$, embed $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $e(1) - e(0)$ increased by 7, we obtain $H_1(21)$.

The labeling graphs of H_1 are shown in Figure 4.

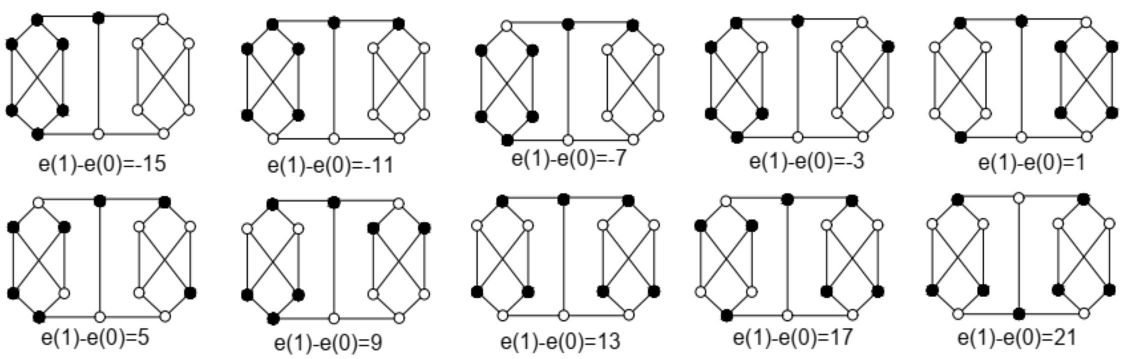

Fig.4. Labeling graphs H_1.

In summary, $FFI(H_1) = \{-19 + 4i : 1 \le i \le 10\}$.

(ii) $m = 3$

(1). In $H_2(-20)$, embed $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, then $e(1) - e(0)$ decreased by 1, we obtain $H_3(-21)$.

(2). In $H_2(-24+4i), 2 \le i \le 11$, there exists labeling subgraph $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. We do embedding $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $e(1) - e(0)$ decreased by 1, we obtain $H_3(-25+4i), 2 \le i \le 11$.

(3). In $H_2(24)$, embed $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $e(1) - e(0)$ decreased by 1, we obtain $H_3(23)$.

(4). In $H_2(20)$, embed $\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}1&1\\0&0\end{bmatrix}$, then $e(1)-e(0)$ increased by 7, we obtain $H_3(27)$.

The labeling graphs of H_3 are shown in Figure 5.

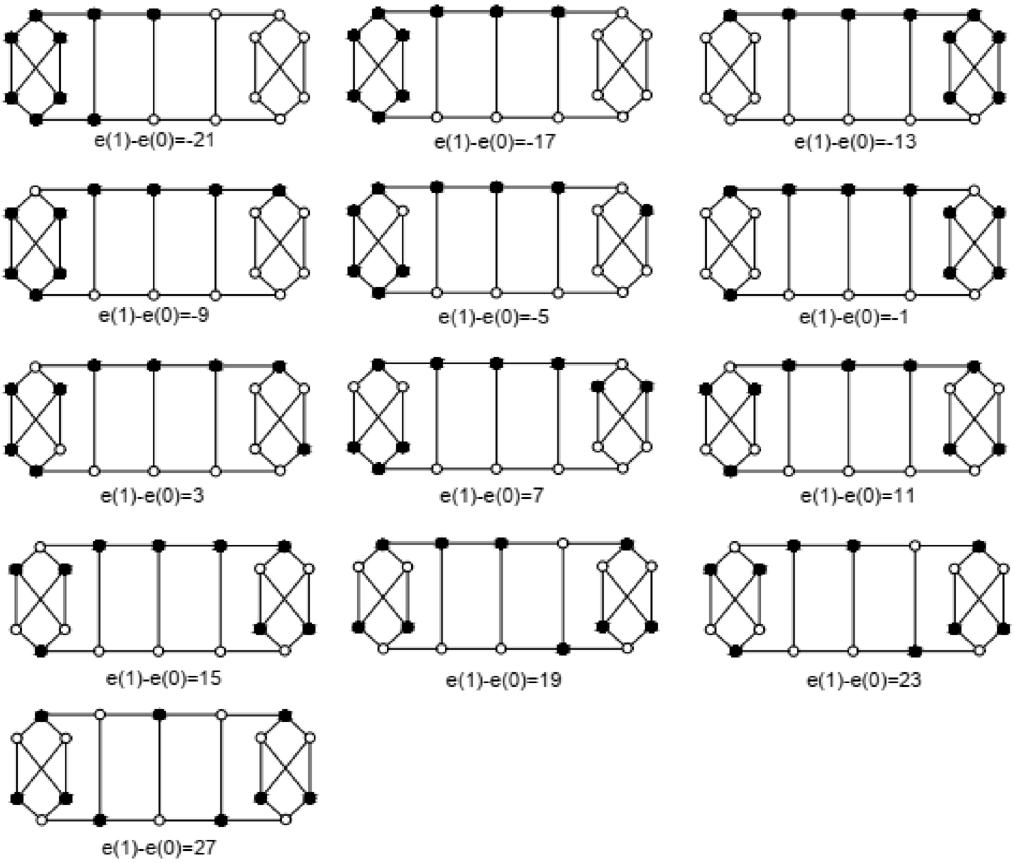

Fig.5. Labeling graphs H_3.

Hence, $FFI(H_3)=\{-24+4i:1\le i\le 13\}$.

(iii) $m\ge 5$

We assume that for even $k\ge 4$ so $k+1$ is odd and $k+1\ge 5$, $FFI(H_k)=\{-3k-18+4i:1\le i\le \frac{3k+18}{2}\}$. We do embeddings as follow.

(1). In $H_k(-3k-14)$, embed $\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1&0\\1&0\end{bmatrix}$, then $e(1)-e(0)$ decreased by 1, we obtain $H_{k+1}(-3(k+1)-12)$.

(2). In $H_k(-3k-18+4i), 2\le i\le \frac{3k+16}{2}$, there exists labeling subgraph $\begin{bmatrix}1&1\\0&0\end{bmatrix}$. We do embedding $\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1&1\\0&0\end{bmatrix}$, then $e(1)-e(0)$ decreased by 1, we obtain $H_{k+1}(-3(k+1)-16+4i), 2\le i\le \frac{3(k+1)+13}{2}$.

(3). In $H_k(3k+18)$, embed $\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0&1\\1&0\end{bmatrix}$, then $e(1)-e(0)$ decreased by 1, we obtain $H_{k+1}(3(k+1)+14)$.

(4). In $H_k(3k+14)$, embed $\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}1&1\\0&0\end{bmatrix}$, then $e(1)-e(0)$ increased by 7, we obtain $H_{k+1}(3(k+1)+18)$.

As mentioned above, we have $FFI(H_{k+1})=\{-3(k+1)-16+4i:1\le i\le \frac{3(k+1)+17}{2}\}$, which complete the proof of Lemma 9.

Combining Lemma 8 and Lemma 9,

Theorem 10. The full friendly index sets of H_m is

$$FFI(H_m)=\begin{cases}-3m-18+4i: \quad 1\le i\le \dfrac{3m+18}{2} & m \quad is \quad even,\\ -3m-16+4i: \quad 1\le i\le \dfrac{3m+17}{2} & m \quad is \quad odd.\end{cases} \qquad (9)$$

In [7], Hovey introduced the notion of A -cordial labelings, which generalized the concept of cordial graphs of Cahit [7]. A graph G is *cordial* if the number of edges labeled 0 and the number of edges labeled 1 differs by at most by 1 under some friendly labeling f .

By Theorem 10, we can derive the following corollary.

Corollary 11. H_m is cordial iff $m \neq 0 \pmod 4$.

4. Conclusion

We obtained the full friendly index sets of a family of cubic graph graphs by subgraph embedding method. In the future, we hope get more results about cubic graphs and use this method solve other problems of graph labeling.

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