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On the Relations between Lucas Sequence and Fibonacci-like Sequence by Matrix Methods

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Abstract

In the present paper first and foremost we introduce a generalization of a classical Fibonacci sequence which is called a Fibonacci-Like sequence and at hindmost we obtain some relationships between Lucas sequence and Fibonacci-Like sequence by using two cross two matrix representation to the Fibonacci-Like sequence. The most worth noticing cause of this article is our proof method, since all the identities are proved by using matrix methods.

Index Terms: Fibonacci Sequence, Lucas Sequence, Generalization of Fibonacci Sequence and Matrix Method.

Mathematics Subject Classification: 11B37, 11B39, 40C05.

1. Introduction

Fibonacci numbers have many applications as well as interesting properties almost in every field of science such as in Physics, Biology, Computer Science, Engineering, Mathematics (Algebra, Geometry and Number Theory itself). Furthermore Fibonacci and Lucas numbers have long interested mathematicians for their intrinsic theory and applications. Fibonacci numbers and Lucas numbers continue to provide invaluable opportunities for exploration, and contribute handsomely to the beauty of mathematics, especially number theory, one can see the citations [9, 10, 13].

The Fibonacci and Lucas sequences are defined by the recurrence relations:

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Definition 1. [13]

For the integer, the Fibonacci sequence is defined by the recurrence relation as

$$F_n = F_{n-1} + F_{n-2},\ n \ge 2,\ F_0 = 0,\ F_1 = 1 \tag{1}$$

Definition 2. [13]

For the integer, the Lucas sequence is defined by the recurrence relation as

$$L_n = L_{n-1} + L_{n-2},\ n \ge 2,\ L_0 = 2,\ L_1 = 1 \tag{2}$$

The generalized Fibonacci sequence $W_n = W_n(a, b;\ p, q)$ is defined as follows:

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = a,\ W_1 = b \tag{3}$$

where a, b, p and q are arbitrary complex numbers with $q \ne 0$. Since these numbers were first studied by Horadam [4], they are called Horadam numbers. Singh et al. in [11] delineated generalized identities on the relations between Fibonacci and Lucas sequences. Thongmoon in [12] gave identities about the common Factors of Fibonacci and Lucas numbers. Cerin in [2] obtained properties on the factors of summation of consecutive Fibonacci and Lucas numbers.

In 1960 Charles H. King introduced the matrix for classical Fibonacci numbers which is known as Q-matrix [9] and Q-matrix is given as

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Cerda [1] studied Horadam sequence (1.3) by matrix methods. Here the author considered two cases of $\{W_n\}$:

- $\{U_n\}$ is defined by $U_0 = 0$ and $U_1 = 1$
- $\{V_n\}$ is defined by $V_0 = 2$ and $V_1 = p$

Keskin and Demirturk [6] obtained some new identities for Fibonacci and Lucas numbers by matrix methods. Kilic [7] obtained some summation identities for Fibonacci numbers by matrix methods. In [8] koken and Bozkurt studied and defined a Lucas Q_L-matrix which is similar to the Fibonacci Q-matrix [9] and the Q_L-matrix is defined as

$$Q_L = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \text{ then } Q_L^n = \begin{cases} 5^{\frac{n}{2}} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, & \textit{for even } n \\ 5^{\frac{n-1}{2}} \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}, & \textit{for odd } n \end{cases}$$

where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively. Jun and Choi in [5] studied the properties of generalized Fibonacci numbers by matrix methods they defined the generalized Fibonacci sequence as

$$q_n = a^{1-\xi(n)} b^{\xi(n)} q_{n-1} + q_{n-2},\ n \geq 2,\ q_0 = 0,\ q_1 = 1$$

where a and b are positive real numbers and

$$\xi(n) = \begin{cases} 0 & if\ n\ \ is\ even \\ 1 & if\ n\ \ is\ odd \end{cases}$$

In [5] the authors defined the (2×2) matrix M for the above sequence.

$$M = \begin{bmatrix} ab & b \\ a & 0 \end{bmatrix} \text{ and } M^n = a^{\frac{n-2+\xi(n)}{2}} b^{\frac{n-\xi(n)}{2}} \begin{bmatrix} a^{1-\xi(n)} b^{\xi(n)} q_{n+1} & bq_n \\ aq_n & a^{1-\xi(n)} b^{\xi(n)} q_{n-1} \end{bmatrix}$$

In addition to this Dasdemir in [3] obtained some identities of Pell, Pell-Lucas and Modified Pell numbers by the matrix methods, in [3] the author defined some two cross two matrices as

$$N = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix},\ R = \begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix} \text{ and } R = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}$$

2. Fibonacci-Like sequence and its Matrix Representation

In this section we define a generalization of Fibonacci sequence which is called Fibonacci-Like sequence also we introduce a 2×2 matrix representation for Fibonacci-Like sequence.

Definition 3

For the integers $n \geq 2$ *and* $p \geq 1$, *the Fibonacci-Like sequence is defined by the recurrence relation as*

$$T_n = T_{n-1} + T_{n-2},\ n \geq 2,\ T_0 = p,\ T_1 = p \tag{4}$$

and a 2×2 matrix representation for Fibonacci-Like sequence and is given by

$$T = \begin{bmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

3. Main Results

In this section we present some main results of this article by using a matrix representation T to Fibonacci-Like sequence defined in equation (4).

Lemma 1 *If* X *is a square matrix with* $X^2 = X + I$ *then*

$$pX^n = T_{n-1}X + T_{n-2}\,I,\ n \geq 2 \tag{5}$$

Proof.

To prove the result we shall use induction on *n*

Let $n = 2$, we get

$$pX^2 = T_1 X + T_0\,I$$

$$pX^2 = pX + p\,I$$

$$X^2 = X + I$$

Hence the result is true for $n = 2$.

Assume that the result is true for *n*. Now we show that that the

$$pX^{n+1} = T_n\,X + T_{n-1}\,I$$

Therefore,

$$T_n\,X + T_{n-1}\,I = (T_{n-1} + T_{n-2})X + T_{n-1}\,I$$

$$T_n\,X + T_{n-1}\,I = (X + I)T_{n-1} + T_{n-2}\,X$$

$$T_n\,X + T_{n-1}\,I = X^2\,T_{n-1} + T_{n-2}\,X$$

$$T_n\,X + T_{n-1}\,I = X^2\,T_{n-1} + T_{n-2}\,X$$

$$T_n\,X + T_{n-1}\,I = X\,(X\,T_{n-1} + T_{n-2}\,I)$$

$$T_n\,X + T_{n-1}\,I = pX\,(X^n)$$

$$T_n\,X + T_{n-1}\,I = pX^{\,n+1}$$

as required.

Now we show that

$$pX^{(-n)} = T_{-n-1}\,X + T_{-n-2}\,I,\ n \geq 2$$

Let

$$Y = I - X = -X^{-1}$$

Therefore,

$$Y^2 = (I - X)^2$$

$$Y^2 = I^2 - 2X + X^2$$

$$Y^2 = I - 2X + X + I$$

$$Y^2 = I - X + I$$

$$Y^2 = Y + I$$

This show that

$$pY^n = T_{n-1}Y + T_{n-2}\, I$$

$$p\left(-X^{-1}\right)^n = T_{n-1}(I - X) + T_{n-2}\, I$$

$$p\left(-X^{-1}\right)^n = T_{n-1}(I - X) + T_{n-2}\, I$$

$$p(-1)^n X^{-n} = -T_{n-1}\, X + (T_{n-1} + T_{n-2})I$$

$$pX^{-n} = (-1)^{n+1}T_{n-1}\, X + (-1)^n T_n\, I$$

Since $(-1)^{n+1}T_{n-2} = T_{-n-1}$ and $(-1)^{n+2}T_n = T_{-n-2}$ then, we have

$$pX^{-n} = T_{-n-1}\, X + T_{-n-2}\, I$$

Hence the result.

Theorem 1 *Let* $T = \begin{bmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ *then*

$$T^n = \begin{bmatrix} \frac{L_n}{2} & \frac{5T_{n-1}}{2p} \\ \frac{T_{n-1}}{2p} & \frac{L_n}{2} \end{bmatrix} \qquad (6)$$

Proof. Since $T^2 = T + I$ then by lemma (3.1), we have

$$pT^n = \begin{bmatrix} \frac{L_n}{2} & \frac{5T_{n-1}}{2p} \\ \frac{T_{n-1}}{2p} & \frac{L_n}{2} \end{bmatrix} + \begin{bmatrix} T_{n-2} & 0 \\ 0 & T_{n-2} \end{bmatrix}$$

$$pT^n = \begin{bmatrix} \frac{T_{n-1}+2T_{n-2}}{2} & \frac{5T_{n-1}}{2} \\ \frac{T_{n-1}}{2} & \frac{T_{n-1}+2T_{n-2}}{2} \end{bmatrix}$$

$$pT^n = \begin{bmatrix} \frac{T_{n-1}+T_{n-2}+T_{n-2}}{2} & \frac{5T_{n-1}}{2} \\ \frac{T_{n-1}}{2} & \frac{T_{n-1}+T_{n-2}+T_{n-2}}{2} \end{bmatrix}$$

$$pT^n = \begin{bmatrix} \frac{T_n+T_{n-2}}{2} & \frac{5T_{n-1}}{2} \\ \frac{T_{n-1}}{2} & \frac{T_n+T_{n-2}}{2} \end{bmatrix}$$

Since $T_n + T_{n-2} = pL_n$ then

$$pT^n = \begin{bmatrix} \frac{pL_n}{2} & \frac{5T_{n-1}}{2} \\ \frac{T_{n-1}}{2} & \frac{pL_n}{2} \end{bmatrix}$$

$$T^n = \begin{bmatrix} \frac{L_n}{2} & \frac{5T_{n-1}}{2p} \\ \frac{T_{n-1}}{2p} & \frac{L_n}{2} \end{bmatrix}$$

Hence the result.

Theorem 2 *For the positive integer n, we have*

$$p^2L_n^2 - 5T_{n-1}^2 = 4p^2(-1)^n \tag{7}$$

Proof. Since

$$T = \begin{bmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \left|T^n\right| = (-1)^n$$

Again,

$$T^n = \begin{bmatrix} \frac{L_n}{2} & \frac{5T_{n-1}}{2p} \\ \frac{T_{n-1}}{2p} & \frac{L_n}{2} \end{bmatrix} \Rightarrow \left|T^n\right| = \frac{L_n^2}{4} - 5\frac{T_{n-1}^2}{4p^2}$$

Hence we conclude that

$$\frac{L_n^2}{4} - 5\frac{T_{n-1}^2}{4p^2} = (-1)^n \Rightarrow p^2 L_n^2 - 5T_{n-1}^2 = 4p^2(-1)^n$$

Hence the result.

Theorem 3

$$\begin{aligned} &2p^2 L_{n+s} = p^2 L_n L_s + 5T_{n-1} T_{s-1},\ n \geq 1,\ s \geq 1 \\ &2T_{n+s-1} = T_{n-1} L_s + L_n T_{s-1},\ n \geq 1,\ s \geq 1 \end{aligned} \tag{8}$$

Proof.

$$T^{n+s} = T^n T^s$$

$$T^{n+s} = \begin{bmatrix} \frac{L_n}{2} & \frac{5T_{n-1}}{2p} \\ \frac{T_{n-1}}{2p} & \frac{L_n}{2} \end{bmatrix} \begin{bmatrix} \frac{L_s}{2} & \frac{5T_{s-1}}{2p} \\ \frac{T_{s-1}}{2p} & \frac{L_s}{2} \end{bmatrix}$$

$$T^{n+s} = \begin{bmatrix} \frac{L_n}{2}\frac{L_s}{2} + 5\frac{T_{n-1}}{2p}\frac{T_{s-1}}{2p} & \frac{L_n}{2}\frac{5T_{s-1}}{2p} + 5\frac{T_{n-1}}{2p}\frac{L_s}{2} \\ \frac{T_{n-1}}{2p}\frac{L_s}{2} + \frac{L_n}{2}\frac{T_{s-1}}{2p} & 5\frac{T_{n-1}}{2p}\frac{T_{s-1}}{2p} + \frac{L_n}{2}\frac{L_s}{2} \end{bmatrix}$$

$$T^{n+s} = \begin{bmatrix} \frac{p^2 L_n L_s + 5T_{n-1} T_{s-1}}{4p^2} & \frac{5\left(L_n T_{s-1} + T_{n-1} L_s\right)}{4p} \\ \frac{T_{n-1} L_s + L_n T_{s-1}}{4p} & \frac{5T_{n-1} T_{s-1} + p^2 L_n L_s}{4p^2} \end{bmatrix}$$

But,

$$T^{n+s}=\begin{bmatrix}\frac{L_{n+s}}{2} & \frac{5T_{n+s-1}}{2p}\\ \frac{T_{n+s-1}}{2p} & \frac{L_{n+s}}{2}\end{bmatrix}$$

Gives,

$$\frac{L_{n+s}}{2}=\frac{p^2L_n L_s+5T_{n-1}T_{s-1}}{4p^2}$$

$$2p^2L_{n+s}=p^2L_n L_s+5T_{n-1}T_{s-1}$$

and,

$$\frac{T_{n+s-1}}{2p}=\frac{T_{n-1}L_s+L_nT_{s-1}}{4p}$$

$$2T_{n+s-1}=T_{n-1}L_s+L_nT_{s-1}$$

Hence the theorem.

Theorem 4

$$\begin{aligned}&2p^2(-1)^s L_{n-s}=p^2L_n L_s-5T_{n-1}T_{s-1},\ n\geq 1,\ s\geq 1\\ &2(-1)^s T_{n-s-1}=T_{n-1}L_s-L_nT_{s-1},\ 1\leq s\leq n\end{aligned} \tag{9}$$

Proof. Since

$$T^s=\begin{bmatrix}\frac{L_s}{2} & \frac{5T_{s-1}}{2p}\\ \frac{T_{s-1}}{2p} & \frac{L_s}{2}\end{bmatrix}\Rightarrow T^{-s}=\begin{bmatrix}\frac{L_s}{2} & \frac{5T_{s-1}}{2p}\\ \frac{T_{s-1}}{2p} & \frac{L_s}{2}\end{bmatrix}^{-1}$$

Then by theorem (2), we have

$$T^{-s}=\frac{1}{(-1)^s}\begin{bmatrix}\frac{L_s}{2} & -\frac{5T_{s-1}}{2p}\\ -\frac{T_{s-1}}{2p} & \frac{L_s}{2}\end{bmatrix}$$

$$T^{n-s}=T^{n}T^{-s}\Rightarrow T^{n-s}=\begin{bmatrix}\frac{L_n}{2} & \frac{5T_{n-1}}{2p}\\ \frac{T_{n-1}}{2p} & \frac{L_n}{2}\end{bmatrix}\frac{1}{(-1)^s}\begin{bmatrix}\frac{L_s}{2} & -\frac{5T_{s-1}}{2p}\\ -\frac{T_{s-1}}{2p} & \frac{L_s}{2}\end{bmatrix}$$

$$T^{n-s}=\frac{1}{(-1)^s}\begin{bmatrix}\frac{L_n}{2} & \frac{5T_{n-1}}{2p}\\ \frac{T_{n-1}}{2p} & \frac{L_n}{2}\end{bmatrix}\begin{bmatrix}\frac{L_s}{2} & -\frac{5T_{s-1}}{2p}\\ -\frac{T_{s-1}}{2p} & \frac{L_s}{2}\end{bmatrix}$$

$$T^{n-s}=(-1)^{-s}\begin{bmatrix}\frac{p^2L_nL_s-5T_{n-1}T_{s-1}}{4p^2} & \frac{5(T_{n-1}L_s-L_nT_{s-1})}{4p}\\ \frac{T_{n-1}L_s-L_nT_{s-1}}{4p} & \frac{p^2L_nL_s-5T_{n-1}T_{s-1}}{4p^2}\end{bmatrix}$$

But,

$$T^{n-s}=\begin{bmatrix}\frac{L_{n-s}}{2} & \frac{5T_{n-s-1}}{2p}\\ \frac{T_{n-s-1}}{2p} & \frac{L_{n-s}}{2}\end{bmatrix}$$

Then,

$$\frac{L_{n-s}}{2}=(-1)^{-s}\,\frac{p^2L_nL_s-5T_{n-1}T_{s-1}}{4p^2}$$

$$2p^2(-1)^sL_{n-s}=p^2L_nL_s-5T_{n-1}T_{s-1}$$

and,

$$\frac{T_{n-s-1}}{2p}=(-1)^{-s}\,\frac{T_{n-1}L_s-L_nT_{s-1}}{4p}$$

$$2(-1)^sT_{n-s-1}=T_{n-1}L_s-L_nT_{s-1}$$

Theorem 5

$$L_nL_s=L_{n+s}+(-1)^sL_{n-s},\ 1\le s\le n$$
$$2(-1)^sT_{n-s-1}=\frac{T_{n+s-1}+(-1)^sT_{n-s-1}}{T_{n-1}},\ 1\le s\le n \qquad (10)$$

Proof. Since

$$T^{n+s}=\begin{bmatrix}\frac{L_{n+s}}{2} & \frac{5T_{n+s-1}}{2p}\\ \frac{T_{n+s-1}}{2p} & \frac{L_{n+s}}{2}\end{bmatrix} \text{ and } (-1)^s T^{n-s}=\begin{bmatrix}(-1)^s\frac{L_{n+s}}{2} & (-1)^s\frac{5T_{n+s-1}}{2p}\\ (-1)^s\frac{T_{n+s-1}}{2p} & (-1)^s\frac{L_{n+s}}{2}\end{bmatrix}$$

Therefore,

$$T^{n+s}+(-1)^s T^{n-s}=\begin{bmatrix}\frac{L_{n+s}+(-1)^s L_{n-s}}{2} & \frac{5\left[T_{n+s-1}+(-1)^s T_{n-s-1}\right]}{2p}\\ \frac{T_{n+s-1}+(-1)^s T_{n-s-1}}{2p} & \frac{L_{n+s}+(-1)^s L_{n-s}}{2}\end{bmatrix}$$

On the other hand,

$$T^{n+s}+(-1)^s T^{n-s}=T^n\,T^s+(-1)^s T^n\,T^{-s}$$

$$T^{n+s}+(-1)^s T^{n-s}=T^n\left[T^s+(-1)^s T^{-s}\right]$$

$$T^{n+s}+(-1)^s T^{n-s}=T^n\left[\begin{pmatrix}\frac{L_s}{2} & \frac{5T_{s-1}}{2p}\\ \frac{T_{s-1}}{2p} & \frac{L_s}{2}\end{pmatrix}+\frac{(-1)^s}{(-1)^s}\begin{pmatrix}\frac{L_s}{2} & -\frac{5T_{s-1}}{2p}\\ -\frac{T_{s-1}}{2p} & \frac{L_s}{2}\end{pmatrix}\right]$$

$$T^{n+s}+(-1)^s T^{n-s}=\begin{bmatrix}\frac{L_n}{2} & \frac{5T_{n-1}}{2p}\\ \frac{T_{n-1}}{2p} & \frac{L_n}{2}\end{bmatrix}\begin{bmatrix}L_s & 0\\ 0 & L_s\end{bmatrix}$$

$$T^{n+s}+(-1)^s T^{n-s}=\begin{bmatrix}\frac{L_n\,L_s}{2} & \frac{5T_{n-1}\,L_s}{2p}\\ \frac{T_{n-1}\,L_s}{2p} & \frac{L_n\,L_s}{2}\end{bmatrix}$$

Hence,

$$\frac{L_{n+s}+(-1)^s L_{n-s}}{2}=\frac{L_n\,L_s}{2}$$

$$L_n\,L_s=L_{n+s}+(-1)^s L_{n-s}$$

and

$$\frac{T_{n+s-1}+(-1)^s T_{n-s-1}}{2p}=\frac{T_{n-1}L_s}{2p}$$

$$T_{n-1}L_s=T_{n+s-1}+(-1)^s T_{n-s-1}$$

$$L_s=\frac{T_{n+s-1}+(-1)^s T_{n-s-1}}{T_{n-1}}$$

Theorem 6

$$\begin{aligned}&4p^2L_{x+y+z}=p^2L_xL_yL_z+5\left(L_xT_{y-1}T_{z-1}+T_{x-1}L_yT_{z-1}+T_{x-1}T_{y-1}L_z\right),\ x\geq 1,\ y\geq 1,\ z\geq 1\\&4p^2T_{x+y+z-1}=p^2\left(L_xL_yT_{z-1}+T_{x-1}L_yL_z+L_xL_{y-1}L_z\right)+5T_{x-1}T_{y-1}T_{z-1}\\&\textit{where }x\geq 1,\ y\geq 1,\ z\geq 1\ \textit{ and }\ x+y+z\geq 1\end{aligned} \tag{11}$$

Proof. By the definition of the matrix T^n, we have

$$T^{x+y+z}=\begin{bmatrix}\frac{L_{x+y+z}}{2} & \frac{5T_{x+y+z-1}}{2p}\\ \frac{T_{x+y+z-1}}{2p} & \frac{L_{x+y+z}}{2}\end{bmatrix}$$

Again,

$$T^{x+y+z}=T^{x+y}T^z\Rightarrow T^{x+y+z}=\begin{bmatrix}\frac{L_{x+y}}{2} & \frac{5T_{x+y-1}}{2p}\\ \frac{T_{x+y-1}}{2p} & \frac{L_{x+y}}{2}\end{bmatrix}\begin{bmatrix}\frac{L_z}{2} & \frac{5T_{z-1}}{2p}\\ \frac{T_{z-1}}{2p} & \frac{L_z}{2}\end{bmatrix}$$

$$T^{x+y+z}=\begin{bmatrix}\frac{L_{x+y}}{2}\frac{L_z}{2}+5\frac{T_{x+y-1}}{2p}\frac{T_{z-1}}{2p} & \frac{L_{x+y}}{2}\frac{5T_{z-1}}{2p}+5\frac{T_{x+y-1}}{2p}\frac{L_z}{2}\\ \frac{T_{x+y-1}}{2p}\frac{L_z}{2}+\frac{L_{x+y}}{2}\frac{T_{z-1}}{2p} & 5\frac{T_{z-1}}{2p}\frac{T_{x+y-1}}{2p}+\frac{L_{x+y}}{2}\frac{L_z}{2}\end{bmatrix}$$

$$T^{x+y+z}=\begin{bmatrix}\frac{p^2L_{x+y}L_z+5T_{x+y-1}T_{z-1}}{4p^2} & \frac{5\left(L_{x+y}T_{z-1}+T_{x+y-1}L_z\right)}{4p}\\ \frac{T_{x+y-1}L_z+L_{x+y}T_{z-1}}{4p} & \frac{5T_{x+y-1}T_{z-1}+p^2L_{x+y}L_z}{4p^2}\end{bmatrix}$$

Equating corresponding terms of the the two matrices, we get

$$\frac{L_{x+y+z}}{2} = \frac{p^2 L_{x+y} L_s + 5T_{x+y-1} T_{z-1}}{4p^2}$$

$$4p^2 L_{x+y+z} = \left(2p^2 L_{x+y}\right) L_z + 5\left(2T_{x+y-1}\right) T_{z-1}$$

Using theorem (3), we have

$$4p^2 L_{x+y+z} = \left(p^2 L_x L_y + 5T_{x-1} T_{y-1}\right) L_z + 5\left(T_{x-1} L_y + L_x T_{y-1}\right) T_{z-1}$$

$$4p^2 L_{x+y+z} = p^2 L_x L_y L_z + 5T_{x-1} T_{y-1} L_z + 5T_{x-1} L_y T_{z-1} + 5L_x T_{y-1} T_{z-1}$$

$$4p^2 L_{x+y+z} = p^2 L_x L_y L_z + 5\left(L_x T_{y-1} T_{z-1} + T_{x-1} L_y T_{z-1} + T_{x-1} T_{y-1} L_z\right)$$

and,

$$\frac{T_{x+y+z-1}}{2p} = \frac{T_{x+y-1-1} L_z + L_{x+y} T_{z-1}}{4p}$$

$$4pT_{x+y+z-1} = p\left(2T_{x+y-1}\right) L_z + \left(2pL_{x+y}\right) T_{z-1}$$

$$4p^2 T_{x+y+z-1} = p^2\left(2T_{x+y-1}\right) L_z + \left(2p^2 L_{x+y}\right) T_{z-1}$$

Using theorem (3), we have

$$4p^2 T_{x+y+z-1} = p^2\left(T_{x-1} L_y + L_x T_{y-1}\right) L_z + \left(p^2 L_x L_y + 5T_{x-1} T_{y-1}\right) T_{z-1}$$

$$4p^2 T_{x+y+z-1} = p^2 T_{x-1} L_y L_z + p^2 L_x T_{y-1} L_z + p^2 L_x L_y T_{z-1} + 5T_{x-1} T_{y-1} T_{z-1}$$

$$4p^2 T_{x+y+z-1} = p^2\left(L_x L_y T_{z-1} + T_{x-1} L_y L_z + L_x L_{y-1} L_z\right) + 5T_{x-1} T_{y-1} T_{z-1}$$

Hence the theorem.

Theorem 7

$$L_y = \frac{(-1)^{-x}}{L_{z-x}} p^2 L_{x+y} L_z - 5T_{x-1} T_{y+z-1}, \; 1 \le x \le z, \; y \ge 1 \; y+z \ge 1$$
$$T_{y-1} = \frac{p^2 (-1)^{-x}}{L_{z-x}} L_x T_{y+z-1} - T_{z-1} L_{x+y}, \; 1 \le x \le z, \; y \ge 1 \; y+z \ge 1 \qquad (12)$$

Proof. Let us consider a product

$$B_1=\begin{bmatrix}\frac{L_x}{2} & \frac{5T_{x-1}}{2p}\\ \frac{T_{z-1}}{2p} & \frac{L_z}{2}\end{bmatrix}\begin{bmatrix}L_y\\ \frac{T_{y-1}}{p}\end{bmatrix}=\begin{bmatrix}\frac{L_x}{2}L_y+\frac{5T_{x-1}}{2p}\frac{T_{y-1}}{p}\\ \frac{T_{z-1}}{2p}L_y+\frac{L_z}{2}\frac{T_{y-1}}{p}\end{bmatrix}$$

$$B_1=\begin{bmatrix}\frac{p^2L_xL_y+5T_{x-1}T_{y-1}}{2p^2}\\ \frac{T_{z-1}L_y+L_zT_{y-1}}{2p}\end{bmatrix}$$

From theorem (3), we have

$$B_1=\begin{bmatrix}\frac{2p^2L_{x+y}}{2p^2}\\ \frac{2T_{y+z-1}}{2p}\end{bmatrix}=\begin{bmatrix}L_{x+y}\\ \frac{T_{y+z-1}}{p}\end{bmatrix}\text{ then}$$

$$B_1=\begin{bmatrix}\frac{L_x}{2} & \frac{5T_{x-1}}{2p}\\ \frac{T_{z-1}}{2p} & \frac{L_z}{2}\end{bmatrix}\begin{bmatrix}L_y\\ \frac{T_{y-1}}{p}\end{bmatrix}=\begin{bmatrix}L_{x+y}\\ \frac{T_{y+z-1}}{p}\end{bmatrix}$$

Let

$$B_2=\begin{vmatrix}\frac{L_x}{2} & \frac{5T_{x-1}}{2p}\\ \frac{T_{z-1}}{2p} & \frac{L_z}{2}\end{vmatrix}=\frac{L_xL_z}{4}-5\frac{T_{x-1}T_{z-1}}{4p^2}\Rightarrow B_2=\frac{p^2L_xL_z-5T_{x-1}T_{z-1}}{4p^2}$$

Again by theorem (4), we have

$$B_2=\frac{2(-1)^xL_{z-x}}{4p^2}=\frac{(-1)^xL_{z-x}}{2p^2}\neq 0,\quad for\ z\neq x$$

Therefore,

$$\begin{bmatrix}L_y\\ \frac{T_{y-1}}{p}\end{bmatrix}=\begin{bmatrix}\frac{L_x}{2} & \frac{5T_{x-1}}{2p}\\ \frac{T_{z-1}}{2p} & \frac{L_z}{2}\end{bmatrix}^{-1}\begin{bmatrix}L_{x+y}\\ \frac{T_{y+z-1}}{p}\end{bmatrix}$$

$$\begin{bmatrix} L_y \\ \frac{T_{y-1}}{p} \end{bmatrix} = \frac{2p^2(-1)^{-x}}{L_{z-x}} \begin{bmatrix} \frac{L_z}{2} & \frac{-5T_{x-1}}{2p} \\ \frac{-T_{z-1}}{2p} & \frac{L_x}{2} \end{bmatrix} \begin{bmatrix} L_{x+y} \\ \frac{T_{y+z-1}}{p} \end{bmatrix}$$

$$\begin{bmatrix} L_y \\ \frac{T_{y-1}}{p} \end{bmatrix} = \frac{2p^2(-1)^{-x}}{L_{z-x}} \begin{bmatrix} \frac{L_z}{2} L_{x+y} \frac{-5T_{x-1}}{2p} \frac{T_{y+z-1}}{p} \\ \frac{-T_{z-1}}{2p} L_{x+y} + \frac{L_x}{2} \frac{T_{y+z-1}}{p} \end{bmatrix}$$

$$\begin{bmatrix} L_y \\ \frac{T_{y-1}}{p} \end{bmatrix} = \frac{2p^2(-1)^{-x}}{L_{z-x}} \begin{bmatrix} \frac{p^2 L_z L_{x+y} - 5T_{x-1} T_{y+z-1}}{2p^2} \\ \frac{L_x T_{y+z-1} - T_{z-1} L_{x+y}}{2p} \end{bmatrix}$$

Therefore,

$$L_y = \frac{(-1)^{-x}}{L_{z-x}} p^2 L_z L_{x+y} - 5T_{x-1} T_{y+z-1}$$

$$\frac{T_{y-1}}{p} = \frac{2p(-1)^{-x}}{L_{z-x}} L_x T_{y+z-1} - T_{z-1} L_{x+y}$$

$$T_{y-1} = \frac{2p^2(-1)^{-x}}{L_{z-x}} L_x T_{y+z-1} - T_{z-1} L_{x+y}$$

Theorem 8

$$\begin{aligned} L_y &= \frac{(-1)^{-x}}{T_{z-x-1}} L_{x+y} T_{z-1} - T_{x-1} L_{y+z},\ 1 \le x \le z,\ y \ge 1 \\ T_{y-1} &= \frac{p^2(-1)^{-x}}{5T_{z-x-1}} L_x L_{y+z} - L_z L_{x+y},\ 1 \le x \le z,\ y \ge 1 \end{aligned} \tag{13}$$

Proof. Let us consider a product

$$B_3 = \begin{bmatrix} \frac{L_x}{2} & \frac{5T_{x-1}}{2p} \\ \frac{L_z}{2} & 5\frac{T_{z-1}}{2p} \end{bmatrix} \begin{bmatrix} L_y \\ \frac{T_{y-1}}{p} \end{bmatrix} = \begin{bmatrix} \frac{L_x}{2} L_y + \frac{5T_{x-1}}{2p} \frac{T_{y-1}}{p} \\ \frac{L_z}{2} L_y + 5\frac{T_{z-1}}{2p} \frac{T_{y-1}}{p} \end{bmatrix}$$

$$B_3 = \begin{bmatrix} \dfrac{p^2 L_x L_y + 5T_{x-1} T_{y-1}}{2p^2} \\ \dfrac{p^2 L_z L_y + 5T_{z-1} T_{y-1}}{2p^2} \end{bmatrix}$$

From theorem (3), we have

$$B_3 = \begin{bmatrix} \dfrac{2p^2 L_{x+y}}{2p^2} \\ \dfrac{2p^2 L_{y+z}}{2p^2} \end{bmatrix} = \begin{bmatrix} L_{x+y} \\ L_{y+z} \end{bmatrix} \text{ then}$$

$$B_3 = \begin{bmatrix} \dfrac{L_x}{2} & \dfrac{5T_{x-1}}{2p} \\ \dfrac{L_z}{2} & 5\dfrac{T_{z-1}}{2p} \end{bmatrix} \begin{bmatrix} L_y \\ \dfrac{T_{y-1}}{p} \end{bmatrix} = \begin{bmatrix} L_{x+y} \\ L_{y+z} \end{bmatrix}$$

Let

$$B_4 = \begin{vmatrix} \dfrac{L_x}{2} & \dfrac{5T_{x-1}}{2p} \\ \dfrac{L_z}{2} & 5\dfrac{T_{z-1}}{2p} \end{vmatrix} = \frac{L_x}{2} 5 \frac{T_{z-1}}{2p} - \frac{L_z}{2} 5 \frac{T_{x-1}}{2p} \Rightarrow B_4 = \frac{5\left(L_x T_{z-1} - L_z T_{x-1}\right)}{4p}$$

Again by theorem (4), we have

$$B_4 = \frac{5}{4p} \frac{\left[2(-1)^x T_{z-x-1}\right]}{4p} = \frac{5(-1)^x T_{z-x-1}}{2p} \neq 0$$

Therefore,

$$\begin{bmatrix} L_y \\ \dfrac{T_{y-1}}{p} \end{bmatrix} = \begin{bmatrix} \dfrac{L_x}{2} & \dfrac{5T_{x-1}}{2p} \\ \dfrac{L_z}{2} & 5\dfrac{T_{z-1}}{2p} \end{bmatrix}^{-1} \begin{bmatrix} L_{x+y} \\ L_{y+z} \end{bmatrix}$$

$$\begin{bmatrix} L_y \\ \dfrac{T_{y-1}}{p} \end{bmatrix} = \frac{2p(-1)^{-x}}{5T_{z-x-1}} \begin{bmatrix} 5\dfrac{T_{z-1}}{2p} & -\dfrac{5T_{x-1}}{2p} \\ -\dfrac{L_z}{2} & \dfrac{L_x}{2} \end{bmatrix} \begin{bmatrix} L_{x+y} \\ L_{y+z} \end{bmatrix}$$

$$\begin{bmatrix} L_y \\ \frac{T_{y-1}}{p} \end{bmatrix} = \frac{2p(-1)^{-x}}{5T_{z-x-1}} \begin{bmatrix} \frac{5T_{z-1}}{2p} L_{x+y} - 5\frac{T_{x-1}}{2p} L_{y+z} \\ -\frac{L_z}{2} L_{x+y} + \frac{L_x}{2} L_{y+z} \end{bmatrix}$$

$$\begin{bmatrix} L_y \\ \frac{T_{y-1}}{p} \end{bmatrix} = \frac{2p(-1)^{-x}}{5T_{z-x-1}} \begin{bmatrix} \frac{5\left(T_{z-1} L_{x+y} - T_{x-1} L_{y+z}\right)}{2p} \\ \frac{L_x L_{y+z} - L_z L_{x+y}}{2} \end{bmatrix}$$

Therefore,

$$L_y = \frac{2p(-1)^{-x}}{5T_{z-x-1}} \frac{5\left(T_{z-1} L_{x+y} - T_{x-1} L_{y+z}\right)}{2p}$$

$$L_y = \frac{(-1)^{-x}}{T_{z-x-1}} T_{z-1} L_{x+y} - T_{x-1} L_{y+z}$$

and

$$\frac{T_{y-1}}{p} = \frac{2p(-1)^{-x}}{5T_{z-x-1}} \frac{L_x L_{y+z} - L_z L_{x+y}}{2}$$

$$\frac{T_{y-1}}{p} = \frac{p(-1)^{-x}}{5T_{z-x-1}} L_x L_{y+z} - L_z L_{x+y}$$

$$T_{y-1} = \frac{p^2(-1)^{-x}}{5T_{z-x-1}} L_x L_{y+z} - L_z L_{x+y}$$

Hence the result

4. Conclusion

In this article we presented a generalized Fibonacci sequence called Fibonacci-Like sequence and after that some relations have been obtained between Lucas sequence and Fibonacci-Like sequence by matrix methods.

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