

Fourier Transform Solution for a One-Dimensional Non-Homogeneous Wave Equation with Boundary and Initial Value Conditions

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Received: February 16, 2026; Revised: March 19, 2026; Accepted: April 20, 2026; Published: August 8, 2026

Abstract: The one-dimensional non-homogeneous wave equation subject to non-homogeneous boundary and initial conditions presents significant analytical challenges, particularly when external forcing and irregular boundary data are simultaneously present. Classical methods such as separation of variables are restricted to homogeneous settings and fail to accommodate non-homogeneous terms in a unified framework. This study employs the Fourier transform method to derive an exact analytical solution by decomposing the total wave displacement into two components: $u(x,t) = V(x,t) + \psi(x)$, where $V(x,t)$ is the oscillatory component satisfying the homogeneous wave equation and $\psi(x)$ is the spatially adjusted component encoding the influence of non-homogeneous boundary conditions and external forcing. The analytical solution is verified by direct substitution and benchmarked against a second-order explicit finite-difference scheme on a grid of 500 spatial points, yielding a maximum point-wise absolute error below 8×10^{-3} , consistent with the second-order truncation error of the numerical scheme. For the representative test case with unit wave speed, unit domain length, constant spatial forcing $F(x) = 2$, and initial displacement $u_0(x) = \sin(\pi x)$, the steady-state component is recovered exactly as $\psi(x) = x - x^2$, and the dominant Fourier coefficient is $A_1 \approx 0.742$. A direct point-by-point comparison with published benchmark values further quantifies the sensitivity of wave solutions to boundary condition specification. The proposed framework accommodates both finite and infinite spatial domains and offers a systematic, closed-form alternative to purely numerical approaches for this class of wave propagation problems, with relevance to acoustics, structural dynamics, and materials science.

Index Terms: Non-homogeneous Wave Equations, Fourier Transform, Boundary Conditions, Initial Conditions, Transient Behavior, Steady-state Response, External Forcing.

1. Background

Partial differential equations (PDEs) are pivotal in mathematical physics for modelling phenomena such as heat conduction, wave propagation, and electrostatics. The one-dimensional wave equation is defined as

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2} \quad (1)$$

where $U = U(x, t)$ represents the wave function and c is the wave speed. This equation is fundamental for analyzing wave behavior in one-dimensional systems [1]. The equation can be modified to account for non-homogeneous conditions, resulting in

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2} + f(x, t) \quad (2)$$

where $f(x, t)$ represents the external forces or non-uniform properties affecting the wave [2,3].

PDEs can be classified into three categories: hyperbolic, parabolic, and elliptic, based on the discriminant of their characteristic equations. For a general second-order PDE in two variables, the equation is expressed as

$$A \frac{\partial^2 U}{\partial x^2} + 2B \frac{\partial^2 U}{\partial x \partial y} + C \frac{\partial^2 U}{\partial y^2} + \dots = 0 \quad (3)$$

where the discriminant is given by $\Delta = B^2 - AC$. The nature of the discriminant determines the type of the PDE.

Hyperbolic equations ($\Delta > 0$) describe dynamic systems characterized by wave propagation. A common example is the wave equation:

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2} \quad (4)$$

Parabolic equations ($\Delta = 0$) represent diffusion-like processes. These equations are typified by the following heat equation:

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} \quad (5)$$

Elliptic equations ($\Delta < 0$) model steady-state phenomena. A widely used example is Laplace's equation:

$$\Delta U = 0 \quad (6)$$

The study of wave equations is central to understanding various physical phenomena, such as sound propagation, seismic waves, and electromagnetic waves [4,5]. Traditional methods for solving wave equations, such as the separation of variables, may struggle to address non-homogeneous conditions effectively [6,7].

As wave behavior in non-homogeneous media becomes increasingly important, it is necessary to apply effective analytical methods. This study focuses on employing the Fourier transform method a well-established classical technique or solving PDEs across several conditions [8,9,10].

Traditionally, the method of separation of variables is effective only for homogeneous boundary conditions but fails when non-homogeneous terms are introduced, as it cannot adequately account for external forces [6,11].

Hussein and Dyhoum [3] studied non-homogeneous wave equations with irregular boundary conditions given by:

$$\begin{aligned} \frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial^2 u(x, t)}{\partial x^2} &= F(x), \quad 0 < x < L, \quad 0 < t < T \\ u(0, t) &= p_0(t), \quad u_{xx}(0, t) - d_0 u_x(0, t) = qq_0(t), \quad 0 < t < T \\ u(x, 0) &= u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = v_0(x), \quad 0 \leq x \leq L \end{aligned} \quad (7)$$

where $u(x, t)$ represents the wave displacement, $F(x)$ is the space-dependent forcing term, $p_0(t)$ is the prescribed displacement at $x = 0$, and $qq_0(t)$ is the flux. d_0 is the stiffness parameter, while $u_0(x)$ and $v_0(x)$ represent the initial displacement and velocity, respectively.

They employed a finite difference technique combined with a separation of variables solution. This numerical method addresses the stability and accuracy problems that often arise when dealing with irregular boundary conditions [12,13].

In our study, we applied the Fourier transform method to the wave equation given in Equation (7). The Fourier transform approach directly incorporates non-homogeneous terms and irregular boundary conditions to obtain a solution. This approach provides a systematic framework for deriving analytical solutions, thereby broadening the solution space for one-dimensional wave equations under non-standard conditions [17,18].

The aim of this study is to obtain a solution for a one-dimensional non-homogeneous wave equation with irregular boundary and initial value conditions using the Fourier Transform method. The following objectives were outlined to achieve the aim of this study:

i. To obtain a solution to the one-dimensional non-homogeneous wave equation with non-homogeneous initial and boundary conditions using Fourier transform technique.

- ii. To incorporate a general forcing term $f(x,t)$ into the wave equation, capturing both spatial and temporal dependencies.
- iii. To decompose the overall solution into transient and steady-state components in order to better understand the dynamic and long-term behavior of the wave system under non-standard conditions.

The scope of this study is centered on obtaining a solution to the one-dimensional non-homogeneous wave equation using Fourier-based methods. For problems defined on a finite domain $0 < x < L$, the Fourier series method (eigenfunction expansion) is employed. For problems on the unbounded domain $-\infty < x < \infty$, the continuous Fourier transform is applied. This study adopts a unified Fourier-based framework that encompasses both approaches. It considers cases with non-homogeneous boundary and initial conditions, along with a time-dependent forcing term $f(x,t)$ [1,7,6]. This study is limited to one-dimensional linear wave equations. It does not address numerical techniques, nonlinear effects, or extend to two- or three-dimensional domains [4,5]. In addition, idealized conditions are assumed, excluding physical factors such as damping, dispersion, and discontinuities within the medium [3,14,15].

1.1. Preliminary

Definition 1.1: Partial Differential Equation (PDE) A partial differential equation involves partial derivatives of a function with respect to two or more independent variables [1,4].

$$F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \dots) = 0$$

Example: $\partial u / \partial t = \alpha \partial^2 u / \partial x^2$ (Heat equation).

Definition 1.2: Wave Equation A second-order linear PDE that models wave motion [5,7].

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x, t)$$

Example: A vibrating string with external force $f(x,t) = \sin(x)\cos(t)$.

Definition 1.3: Homogeneous and Non-Homogeneous Equations — Homogeneous: If $f(x,t) = 0$, there is no external input. — Non-homogeneous: If $f(x,t) \neq 0$, showing external forcing [6,11].

Example: Homogeneous: $\partial^2 u / \partial t^2 = c^2 \partial^2 u / \partial x^2$; Non-homogeneous: $\partial^2 u / \partial t^2 = c^2 \partial^2 u / \partial x^2 + 10e^{-(t)}$.

Definition 1.4: Linear and Nonlinear Equations A linear PDE contains no products or nonlinear functions of the unknown or its derivatives. A nonlinear PDE includes nonlinearities [4,1].

Definition 1.5: Spatial Domain The region in space over which the function is defined is often denoted by $0 \leq x \leq L$ [5].

Definition 1.6: Initial Conditions (ICs) Conditions prescribed at $t = 0$ [7,6]:

$$u(x, 0) = h_1(x), \quad \frac{\partial u}{\partial t}(x, 0) = h_2(x)$$

Example: $u(x,0) = \sin(\pi x)$, $\partial u / \partial t(x,0) = 0$.

Definition 1.7: Boundary Conditions (BCs) Describes the solution behavior at spatial boundaries [4]. Example (Dirichlet): $u(0,t) = 0$, $u(L,t) = 0$.

Definition 1.8: Irregular Boundary Conditions Boundary conditions that are time-dependent, nonlinear, discontinuous, or not standard [3,14]. Example: $u(0,t) = \sin(t^2)$, $u(L,t) = e^{-(t)}$.

Definition 1.9: Non-homogeneous Conditions Initial or boundary conditions that are not zero [1,11]. Example: $u(0,t) = 5\sin(t)$.

Definition 1.10: Forcing Term An external input represented by $f(x,t)$ in the wave equation [15]. Example: $f(x,t) = x \cos(t)$.

Definition 1.11: Fourier Transform Transforms a function from the spatial to the frequency domain [6,8]:

$$\hat{u}(k, t) = \int_{-\infty}^{\infty} u(x, t) e^{-ikx} dx$$

Example: $u(x) = e^{-(a|x|)}$ $\hat{u}(k) = 2a/(a^2 + k^2)$.

Definition 1.12: Inverse Fourier Transform Used to recover the function from its Fourier transform [8,7]:

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k, t) e^{ikx} dk$$

Definition 1.13: Transform Methods Techniques such as Fourier and Laplace transforms used to simplify PDEs by converting differential operators into algebraic ones [11,8]. Example: Applying the Fourier transform to convert a PDE into an ODE in time.

Definition 1.14: Spectral Domain The frequency domain obtained after applying a transform, where spatial derivatives become algebraic multipliers [8,17]:

$$\frac{d^2 u}{dx^2} \mapsto -k^2 \hat{u}(k) \text{ under the Fourier transform.}$$

Definition 1.15: Superposition Principle In linear systems, any linear combination (sum) of solutions is also a solution [1,4]. Example: If u_1 and u_2 solve the PDE, then $u = u_1 + u_2$ is also a solution.

Definition 1.16: Homogenization Method of transforming non-homogeneous equations or boundary conditions into a homogeneous form via substitution or decomposition [7,11]. Example: Writing $u(x,t) = v(x,t) + g(x,t)$ where g satisfies the non-homogeneous BCs.

Definition 1.17: Ordinary Differential Equation (ODE) An equation involving derivatives with respect to a single independent variable arising after the spatial transformation of a PDE [6].

$$\frac{d^2 \hat{u}}{dt^2} + c^2 k^2 \hat{u} = f(k, t)$$

Definition 1.18: Analytical Solution An exact closed-form expression for the solution of a differential equation [8,11]. Example: The solution to $\frac{d^2 y}{dt^2} + y = 0$ is $y(t) = A \cos(t) + B \sin(t)$.

Definition 1.19: Decomposition Method Splits the total solution into simpler components (e.g., particular + homogeneous or steady + transient) to handle non-homogeneities [7,3]. Example: $u(x,t) = v(x,t) + w(x,t)$ with w chosen to satisfy the BCs.

Definition 1.20: Transient Component The time-dependent part of the solution that typically decays or evolves to zero as $t \rightarrow \infty$ in dissipative systems (or remains oscillatory but bounded in conservative systems) [5,4]. Example: $e^{-(t)} \sin(\pi x)$ is transient (decays with time).

Definition 1.21: Steady-State Component The long-term (time-asymptotic) behavior of the solution as $t \rightarrow \infty$; remains after the transients have subsided [5,16]. Example: A time-independent profile $u_s(x)$ or a sustained periodic response $\cos(\omega t)\phi(x)$.

2. Literature Review

Partial differential equations (PDEs) are fundamental tools for modelling physical phenomena in disciplines such as physics, engineering, and applied mathematics. In particular, the one-dimensional wave equation serves as a prototype for analyzing wave propagation in bounded and unbounded media. The complexity of this equation increases when non-homogeneous boundary and initial conditions or external forcing terms are introduced. Over the years, researchers have proposed various analytical, numerical, and hybrid techniques to address these complexities. This section presents a chronological review of selected studies that have shaped the development of solution methods for the wave equation.

Alpert et al. (2003) [29] introduced exact non-reflecting boundary conditions for wave propagation problems. By coupling boundary formulations with fast Fourier convolution schemes, they addressed the challenges of artificial-domain truncation and demonstrated the efficient modelling of open or semi-infinite domains.

Shakeri and Dehghan [26] applied the Method of Lines (MOL) to convert the one-dimensional wave equation into a system of ordinary differential equations (ODEs), simplifying its numerical solution under integral constraints:

$$\frac{\partial^2 u(x, t)}{\partial t^2} - c^2 \frac{\partial^2 u(x, t)}{\partial x^2} = f(x, t), \text{ with } \int_0^L u(x, t) dx = C$$

Ullah et al. (2015) [27] employed the Optimal Homotopy Asymptotic Method (OHAM) to solve nonlinear wave equations semi-analytically:

$$\frac{\partial^2 u(x, t)}{\partial t^2} - c^2 \frac{\partial^2 u(x, t)}{\partial x^2} + f(u(x, t)) = 0$$

Their method accommodates nonlinearity without linearization and is robust for complex physical wave models.

Bi (2018) [25] developed operator-based integral representations to solve inhomogeneous wave equations in higher dimensions. Although Fourier transforms are not directly used, his method parallels Fourier techniques in its

ability to transform differential problems into integral forms, revealing the structural advantages of transformation-based approaches.

Guo and Jin [28] proposed a boundary output feedback stabilization scheme using an extended state observer (ESO) to manage boundary disturbances in real-time. Their model is given by

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u(x, t) = f(x, t) + \hat{d}(x, t), \quad u_{feedback} = -K \hat{d}(x, t)$$

Olver and Xu (2021) [17] constructed analytical solutions for complex geometries by applying orthogonal polynomial Fourier expansions. This generalizes the classical Fourier series to irregular domains, offering solutions in cases where standard sine/cosine expansions are not applicable.

Hussein and Dyhoun (2022) [3] tackled non-homogeneous wave equations subject to irregular boundary conditions, including Neumann-type and non-zero Dirichlet data. They employed a hybrid of finite difference and separation of variables methods to approximate the solutions. Although computationally effective, their approach was numerical and not exact. This study serves as an immediate precursor to the present work.

Sun and Zhang (2022) [28] developed a multiscale Fourier-series framework for wave propagation in elastic beams with complex boundary structures. Their use of modified modal expansions allowed the handling of non-classical boundary geometries using adjusted Fourier-based basis functions.

Kalimeris and Mindrinos (2024) [21] applied D'Alembert-type representations and the Fokas inverse scattering method to inhomogeneous wave problems with spatially varying wave speeds:

$$\frac{\partial^2 u}{\partial t^2} - c(x)^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Their method yielded exact solutions and demonstrated the effectiveness of the transform methods in spatially non-uniform environments.

Cubillos and Jimenez (2024) [22] introduced the Operator Fourier Transform (OFT) to solve PDEs in unbounded domains involving complex spatial operators. Rodriguez-Lara et al. (2024) explored the application of the Fast Fourier Transform (FFT) to hyperbolic PDEs:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x, t)$$

Lomov (2025) [24] proposed a generalized Fourier method for mixed boundary value problems involving non-homogeneous wave equations. His analytical procedure successfully derived closed-form solutions using inverse representations within finite domains with irregular boundary profiles.

From early numerical and hybrid methods like MOL, OHAM, and FDM, to modern analytical frameworks such as the Fokas method, OFT, FFT, and polynomial Fourier expansions, the literature reveals a clear methodological evolution. Despite these advancements, a notable gap remains in the literature. Few studies provide a complete analytical solution that simultaneously accommodates time-dependent forcing terms and non-homogeneous boundary and initial conditions, particularly in both bounded and unbounded spatial domains, using only classical Fourier techniques. This gap motivates the present study.

3. Methodology

3.1. Overview

This study employs classical analytical techniques, specifically the Fourier series and Fourier transform methods, to obtain exact solutions to one-dimensional non-homogeneous wave equations. These Fourier-based approaches are chosen for their versatility in handling boundary value problems in both finite and infinite spatial domains [7,5,1]. The Fourier series is particularly effective for problems defined on bounded intervals with homogeneous or non-homogeneous boundary conditions, whereas the Fourier transform method is suitable for unbounded or infinite domains where boundary values at infinity are either generalized or vanish asymptotically [20,6]. Both methods aim to reduce the governing PDEs into ODEs in time, by exploiting orthogonality in the spatial domain. The resulting solutions allow for the clear separation and interpretation of transient and steady-state behaviors.

3.2. Overview of Fourier Series Method

The Fourier series method is a classical analytical tool used to represent a (periodic or piecewise-defined) function on a finite interval as an infinite sum of sine and cosine functions. For a function $f(x)$ defined on $0 \leq x \leq L$ the Fourier series expansion has the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

with coefficients

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

The expansion relies on the orthogonality of the trigonometric basis on $[0, L]$, which permits the projection of spatial data onto independent modes.

3.2.1. Orthogonally and completeness.

The sine and cosine bases are orthogonal on $[0, L]$:

$$\begin{aligned} \int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx &= 0 \quad (m \neq n), \quad \frac{L}{2} \quad (m = n) \\ \int_0^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx &= 0 \quad (m \neq n), \quad \frac{L}{2} \quad (m = n) \end{aligned}$$

Orthogonality guarantees unique coefficient formulas and allows the projection of initial data onto modes. In PDE applications (heat, wave, and Laplace problems on bounded intervals), the Fourier series converts spatial operators into modal amplitudes and reduces the PDE to a sequence of ODEs in time for each mode. Practical variants include full-range and half-range (sine/cosine only) expansions and even/odd extensions to handle non-symmetric or one-sided domain data [1,7,5].

3.3. Fourier Series Application to PDE

For wave equations defined over a finite spatial interval $0 < x < L$, the Fourier series method is utilized [4,7]. The classical form of the one-dimensional non-homogeneous wave equation is

$$\frac{\partial^2 u(x, t)}{\partial t^2} = c^2 \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t)$$

subject to the initial conditions

$$u(x, 0) = h_1(x), \quad \frac{\partial u}{\partial t}(x, 0) = h_2(x)$$

and Dirichlet boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$.

To apply the Fourier series method, the solution $u(x, t)$ is expressed using the method of separation of variables as $u(x, t) = X(x)T(t)$. Substituting into the homogeneous version of the wave equation (assuming $f(x, t) = 0$ for the time being) and dividing by XT leads to

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

where λ denotes the separation constant. This yields two ODEs:

$$X''(x) + \lambda X(x) = 0, \quad T''(t) + \lambda c^2 T(t) = 0$$

Applying the boundary conditions $X(0) = 0$ and $X(L) = 0$, the eigenfunctions are

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

The general solution is as follows:

$$u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi c t}{L} \right] \sin \frac{n\pi x}{L} \quad (8)$$

To incorporate a non-homogeneous source term $f(x,t)$, the full solution is written as

$$u(x, t) = \sum_{n=1}^{\infty} [A_n(t) \cos \frac{n\pi ct}{L} + B_n(t) \sin \frac{n\pi ct}{L} + u_{p,n}(t)] \sin \frac{n\pi x}{L} \quad (9)$$

where $u_{p,n}(t)$ is a particular solution obtained using techniques such as the variation of parameters or the method of undetermined coefficients [11], and A_n and B_n are Fourier sine coefficients calculated from the initial conditions:

$$A_n = \frac{2}{L} \int_0^L h_1(x) \sin \frac{n\pi x}{L} dx, \quad B_n = \frac{2}{n\pi c} \int_0^L h_2(x) \sin \frac{n\pi x}{L} dx \quad (10)$$

In the case of nonzero boundary conditions, a decomposition of the solution is employed:

$$u(x, t) = v(x, t) + w(x, t) \quad (11)$$

where $w(x,t)$ satisfies the non-homogeneous boundary and/or initial conditions, and $v(x,t)$ satisfies the homogeneous conditions and is solved via Fourier series [9,5].

3.3.1. Handling Non-zero Boundary Conditions via Decomposition

When the boundary conditions are non-homogeneous, such as

$$u(0, t) = g_1(t), \quad u(L, t) = g_2(t) \quad (12)$$

The direct application of the Fourier series is impractical because of the lack of orthogonality in the basis functions. To address this, the method of decomposition is used to rewrite the solution as $u(x,t) = v(x,t) + w(x,t)$, where:

- $w(x,t)$ is a known auxiliary function chosen to satisfy the non-homogeneous boundary conditions [11].
- $v(x,t)$ is the unknown function that satisfies homogeneous boundary conditions: $v(0,t) = 0$, $v(L,t) = 0$.

A common choice for $w(x,t)$ is a linear interpolant in space:

$$w(x, t) = \left(1 - \frac{x}{L}\right) g_1(t) + \frac{x}{L} g_2(t)$$

which satisfies: $w(0,t) = g_1(t)$ and $w(L,t) = g_2(t)$.

Substituting $u(x,t) = v(x,t) + w(x,t)$ into the original PDE yields a modified wave equation for $v(x,t)$:

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} + f(x, t)$$

where the effective forcing term is

$$f(x, t) = f(x, t) - \frac{\partial^2 w}{\partial t^2} + c^2 \frac{\partial^2 w}{\partial x^2}$$

Now, $v(x,t)$ satisfies the following:

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} &= c^2 \frac{\partial^2 v}{\partial x^2} + f(x, t) \\ v(0, t) &= 0, \quad v(L, t) = 0 \\ v(x, 0) &= h_1(x) - w(x, 0), \quad \frac{\partial v}{\partial t}(x, 0) = h_2(x) - \frac{\partial w}{\partial t}(x, 0) \end{aligned}$$

This transforms the problem into a new wave equation with homogeneous boundary conditions and modified initial and forcing data points.

At this point, the classical Fourier sine series method can be applied to $v(x,t)$:

$$v(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} + v_{p,n}(t) \right] \sin \frac{n\pi x}{L}$$

where A_n and B_n are computed from the adjusted initial conditions, and $v_{p,n}(t)$ represents the particular solution due to $\tilde{f}(x, t)$, typically obtained using variations of parameters or Green's function techniques [11].

The complete solution to the original problem is then reconstructed as

$$u(x, t) = v(x, t) + w(x, t)$$

This decomposition approach enables the application of Fourier methods even in the presence of irregular or time-dependent boundary conditions and is highly effective for deriving analytical solutions that retain the interpretability and structure of classical series representations.

3.3.2. Half-Range Expansions and Extensions

For problems defined only on $[0, L]$, half-range Fourier sine or cosine series are used based on the symmetry and type of boundary conditions [7,5]. The functions may be extended:

- Oddly, leading to sine series when $u(0, t) = u(L, t) = 0$.
- Evenly, leading to cosine series, especially with Neumann boundary conditions.

This preserves orthogonality and enables analytical tractability, even in asymmetric setups.

3.4. Overview of Fourier Transform Method

The Fourier transform generalizes the discrete modal idea to infinite (unbounded) domains by representing the function as a continuous superposition of plane waves. The (spatial) transform pair used here is

$$f(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k) e^{ikx} dk$$

where k is the spatial frequency (wavenumber). A key property is that spatial differentiation becomes multiplication in the spectral domain:

$$\mathcal{F}\left\{\frac{\partial^2 f}{\partial x^2}\right\}(k) = -k^2 f(k)$$

Therefore, linear PDEs with constant-coefficient spatial operators reduce to algebraic relations or ODEs in time for each k . After solving in the spectral domain, the inverse transform reconstructs the physical solution. The Fourier transform is especially effective for problems on $(-\infty, \infty)$, for radiation or scattering problems, and whenever a global spectral description of the forcing or initial data is desirable [1,6,20].

3.5. Fourier Transform Application to PDE

For problems defined in the infinite spatial domain $-\infty < x < \infty$, the Fourier transform method is applied [6,1]. Consider the following wave equation:

$$\frac{\partial^2 u(x, t)}{\partial t^2} = c^2 \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t)$$

with the initial conditions:

$$u(x, 0) = h_1(x), \quad \frac{\partial u}{\partial t}(x, 0) = h_2(x)$$

Applying the Fourier transform with respect to x yields

$$\hat{u}(k, t) = \int_{-\infty}^{\infty} u(x, t) e^{-ikx} dx$$

and transforms the PDE into an ODE as follows:

$$\frac{\partial^2 \hat{u}(k, t)}{\partial t^2} + c^2 k^2 \hat{u}(k, t) = f(k, t)$$

where $\hat{f}(k, t)$ is the Fourier transform of $f(x, t)$. The general solution in the spectral domain is

$$\hat{u}(k, t) = A(k) \cos(ckt) + B(k) \sin(ckt) + \hat{u}_p(k, t)$$

with

$$A(k) = \hat{h}_1(k), \quad B(k) = \frac{1}{ck} \hat{h}_2(k)$$

and $\hat{u}_p(k, t)$ is a particular solution due to $\hat{f}(k, t)$ [11]. Taking the inverse Fourier transform yields the following physical solution:

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k, t) e^{ikx} dk$$

In the Fourier transform expressions, the symbol k is not a constant. It represents the spatial frequency variable (or wave number). Each value of k corresponds to a sinusoidal component $e^{(ikx)}$ in the solution, which is reconstructed by integrating over all such frequencies [6].

3.5.1. Decomposition Approach for Non-zero Asymptotic Behavior in Infinite Domains

In problems defined over infinite spatial domains, the boundary conditions may not vanish at infinity, i.e.,

$$\lim_{x \rightarrow \pm\infty} u(x, t) = G(t) \neq 0$$

This poses a challenge for directly applying the classical Fourier transform owing to convergence issues. To manage such situations, we employ a decomposition strategy $u(x, t) = v(x, t) + w(x, t)$, where:

- $w(x, t)$ is a suitably constructed function that captures the asymptotic or non-zero boundary behavior.
- $v(x, t)$ is the residual function that satisfies:

$$\lim_{x \rightarrow \pm\infty} v(x, t) = 0$$

This ensures that $v(x, t)$ is square integrable and suitable for classical Fourier transformation. Substituting into the original wave equation yields a modified wave equation for $v(x, t)$:

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} + f(x, t)$$

where the adjusted forcing function is expressed as

$$f(x, t) = f(x, t) - \left(\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial x^2} \right)$$

The initial conditions are also transformed as follows:

$$v(x, 0) = h_1(x) - w(x, 0), \quad \frac{\partial v}{\partial t}(x, 0) = h_2(x) - \frac{\partial w}{\partial t}(x, 0)$$

Now, we apply the spatial Fourier transform:

$$\hat{v}(k, t) = \int_{-\infty}^{\infty} v(x, t) e^{-ikx} dx$$

The transformed equation is as follows:

$$\frac{d^2 \hat{v}(k, t)}{dt^2} + c^2 k^2 \hat{v}(k, t) = f(k, t)$$

which is an ordinary differential equation in time for each fixed wave number k [11,19]. The general solution in the spectral domain is:

$$\hat{v}(k, t) = A(k) \cos(ckt) + B(k) \sin(ckt) + \hat{v}_p(k, t)$$

where: $A(k) = \hat{v}(k, 0)$, $B(k) = (1/ck) \partial \hat{v} / \partial t|_{t=0}$, and $\hat{v}_p(k, t)$ is a particular solution corresponding to $\hat{f}(k, t)$.

Finally, the transform is inverted to recover $v(x, t)$:

$$v(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{v}(k, t) e^{ikx} dk$$

and obtain the total solution as follows:

$$u(x, t) = v(x, t) + w(x, t)$$

This decomposition approach within the Fourier transform method allows analytical solutions to wave equations even when the boundary behavior is non-zero or irregular at infinity, which often arises in radiating wave problems, waveguides, and forced mechanical systems [22,24].

This approach is particularly powerful for problems in which spatial boundaries are not clearly defined or extend to infinity, as it captures the continuous spectrum of wave modes and naturally incorporates general source terms.

3.6. Relevance and Significance to the Present Study

The combined use of Fourier series and Fourier transform methods offers a unified analytical framework capable of handling a wide range of non-homogeneous wave problems, including those with irregular or time-dependent boundary and initial conditions. This methodological approach is robust and exact, allowing the solution to be expressed explicitly in terms of eigenfunctions and spectral integrals. Unlike numerical methods, this approach preserves the structure of the solution and reveals the modal and spectral content of wave propagation in bounded and unbounded domains [3,19]. By applying this dual Fourier approach, the present study significantly improves upon numerical schemes such as the finite difference method used by Hussein and Dyhoum (2022) [3], and offers broader generality and interpretability. It also facilitates the analysis of both transient (time-dependent) and steady-state wave phenomena under the influence of an external forcing function $f(x, t)$.

4. Result

In this section, we present the expected outcomes of applying the Fourier transform method to solve the one-dimensional non-homogeneous wave equation with non-homogeneous boundary and initial conditions. The goal is to derive exact analytical solutions by decomposing the solution into two components: $u(x, t) = V(x, t) + \psi(x)$. Here, $V(x, t)$ represents the transient component, solving the homogeneous wave equation, whereas $\psi(x)$ accounts for the non-homogeneous boundary conditions, representing the steady-state solution. Through this decomposition, we aimed to clearly distinguish between the transient and steady-state behaviors of the wave system. Additionally, the incorporation of the non-homogeneous forcing term $F(x)$, which does not depend on time, and non-homogeneous boundary conditions should allow for the derivation of closed-form solutions. The expected results demonstrate how the Fourier transform method provides a more efficient and exact approach compared to traditional numerical methods, capturing both the dynamic and long-term behaviors of the system in non-homogeneous media.

4.1. Partial Differential Equation

We focus on solving the one-dimensional non-homogeneous wave equation with non-homogeneous boundary and initial conditions. The standard form of the wave equation is

$$\frac{\partial^2 u}{\partial t^2} - k \frac{\partial^2 u}{\partial x^2} = F(x) \quad (13)$$

where $u = u(x, t)$ represents the wave displacement as a function of space x and time t , and $F(x)$ is a purely space-dependent forcing term (independent of t). This spatial restriction on the forcing is maintained consistently throughout this study. In the general infinite-domain formulation (Section 3.5), the forcing term is written as $f(x, t)$ to allow both

spatial and temporal dependence; however, in the present finite-domain result, F depends only on x . To address specific challenges in modeling wave propagation with external forcing and non-homogeneous boundary conditions, we introduce a non-homogeneous term r into the wave equation, transforming it as follows:

$$\frac{\partial^2 u}{\partial t^2} - k \frac{\partial^2 u}{\partial x^2} = F(x) + r \quad (14)$$

The inclusion of the constant term k and the additional term r allows the equation to model scenarios in which external forces or varying boundary conditions influence wave propagation, converting the problem into a non-homogeneous form. This adjustment is essential for accurately representing real-world conditions, where external forces or varying boundary conditions affect the system.

4.2. Boundary and Initial Conditions

The boundary conditions for the wave equation are as follows:

$$u(0, t) = p_0(t) \quad (15)$$

$$u_{xx}(0, t) - d_0 u_x(0, t) = q q_0(t) \quad (16)$$

where $p_0(t)$ is the prescribed displacement at $x = 0$, $q q_0(t)$ represents the flux or tension at $x = 0$, and d_0 is the stiffness parameter. For the boundary at $x = L$, the condition is assumed as

$$u(L, t) = q_0(t) \quad (17)$$

where $q_0(t)$ is the prescribed boundary condition at $x = L$, and the exact nature of the boundary is determined based on the context. The initial conditions were as follows:

$$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = v_0(x) \quad (18)$$

where $u_0(x)$ represents the initial displacement and $v_0(x)$ represents the initial velocity of the wave at $t = 0$, across the spatial domain $0 \leq x \leq L$.

4.3. Solution Approach via Separation of Variables and Decomposition

To solve the non-homogeneous wave equation with non-homogeneous boundary conditions, as in equation (7), we introduce a new dependent variable $V(x, t)$ and a function $\psi(x)$ such that

$$u(x, t) = V(x, t) + \psi(x) \quad (19)$$

The decomposition $u(x, t) = V(x, t) + \psi(x)$ is valid here because the forcing term $F(x)$ is purely spatial and the boundary conditions $p_0(t)$, $q_0(t)$ are general functions of t . The function $\psi(x)$ depends only on x because it is chosen to satisfy the steady spatial structure imposed by $F(x)$, while $V(x, t)$ carries the full time dependence including the effect of the time-varying boundary data. If the boundary data were constants, $\psi(x)$ would absorb both the forcing and the boundary values exactly. For time-dependent boundary data, $\psi(x)$ absorbs the spatial structure and $V(x, t)$ handles the temporal evolution [5,7].

Substituting $u(x, t) = V(x, t) + \psi(x)$ into the wave equation $\partial^2 u / \partial t^2 - k \partial^2 u / \partial x^2 = F(x)$, we get:

$$\begin{aligned} \frac{\partial^2}{\partial t^2} (V(x, t) + \psi(x)) - k \frac{\partial^2}{\partial x^2} (V(x, t) + \psi(x)) &= F(x) \\ \frac{\partial^2 V}{\partial t^2} - k \frac{\partial^2 V}{\partial x^2} - k \psi''(x) &= F(x) \end{aligned} \quad (20)$$

Because $\psi(x)$ is only a function of x , its time derivatives are zero. The term $k \psi''(x)$ accounts for the effect of $\psi(x)$ on the system, and we simplify the forcing term $F(x)$ in the equation as follows:

$$\frac{\partial^2 V}{\partial t^2} - k \frac{\partial^2 V}{\partial x^2} = F(x) + k \psi''(x) \quad (21)$$

Thus, we arrive at the equation for $V(x,t)$:

$$\frac{\partial^2 V}{\partial t^2} = k \frac{\partial^2 V}{\partial x^2} + F(x) + k\psi''(x) \quad (22)$$

which is a homogeneous wave equation for $V(x,t)$, with an additional forcing term from $\psi(x)$. We assume that the solution is of the following form:

$$U(x, t) = V(x, t) + \psi(x) \quad (23)$$

where $V(x,t)$ is the solution to the homogeneous wave equation that satisfies homogeneous boundary conditions, and $\psi(x)$ is a function satisfying the non-homogeneous boundary conditions.

4.3.1. Interpretation of Boundary Conditions for $V(x,t)$ and $\psi(x)$

By substituting $U(x,t) = V(x,t) + \psi(x)$ into the boundary conditions (15)–(17), we transform them as follows.

At $x = 0$:

$$u(0, t) = p_0(t) \Rightarrow V(0, t) + \psi(0) = p_0(t)$$

Since $\psi(0) = p_0(t)$, we obtain

$$V(0, t) = 0$$

For the flux/tension condition at $x = 0$,

$$u_{xx}(0, t) - d_0 u_x(0, t) = qq_0(t) \Rightarrow V_{xx}(0, t) + \psi''(0) - d_0(V_x(0, t) + \psi'(0)) = qq_0(t)$$

This simplifies to:

$$V_{xx}(0, t) - d_0 V_x(0, t) + (\psi''(0) - d_0 \psi'(0)) = qq_0(t)$$

The flux condition for $\psi(x)$ is therefore:

$$\psi''(0) - d_0 \psi'(0) = qq_0(t)$$

At $x = L$:

$$u(L, t) = q_0(t) \Rightarrow V(L, t) + \psi(L) = q_0(t)$$

Since $\psi(L) = q_0(t)$, we obtain

$$V(L, t) = 0$$

Similarly, for the flux/tension condition at $x = L$, we have

$$u_{xx}(L, t) - d_0 u_x(L, t) = qq_0(t) \Rightarrow V_{xx}(L, t) + \psi''(L) - d_0(V_x(L, t) + \psi'(L)) = qq_0(t)$$

This simplifies to:

$$V_{xx}(L, t) - d_0 V_x(L, t) + (\psi''(L) - d_0 \psi'(L)) = qq_0(t)$$

The flux condition for $\psi(x)$ at $x = L$ is therefore:

$$\psi''(L) - d_0 \psi'(L) = qq_0(t)$$

Thus, the boundary conditions for $V(x,t)$ are homogeneous as follows:

$$V(0, t) = 0 \text{ and } V(L, t) = 0 \quad (24)$$

The adjusted flux/tension conditions for $\psi(x)$ are as follows:

$$\psi''(0) - d_0\psi'(0) = qq_0(t) \text{ and } \psi''(L) - d_0\psi'(L) = qq_0(t) \quad (25)$$

4.3.2. Interpretation of Initial Conditions for $V(x,t)$ and $\psi(x)$

The initial conditions for the wave equation are defined as

$$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = v_0(x) \quad (26)$$

where $u_0(x)$ denotes the initial displacement and $v_0(x)$ represents the initial velocity. To proceed with the solution, we decompose these initial conditions. First, for the initial displacement at $t = 0$:

$$u(x, 0) = V(x, 0) + \psi(x) \quad (27)$$

Since we know that:

$$u(x, 0) = u_0(x) \quad (28)$$

we can then write:

$$V(x, 0) + \psi(x) = u_0(x) \quad (29)$$

Consequently, we can express the initial condition for $V(x,t)$ as

$$V(x, 0) = u_0(x) - \psi(x) \quad (30)$$

Next, for the initial velocity at $t = 0$, the rate of change of the wave displacement is expressed as

$$\frac{\partial u}{\partial t}(x, 0) = \frac{\partial V}{\partial t}(x, 0) + \frac{\partial \psi}{\partial t}(x, 0) \quad (31)$$

Because $\psi(x)$ depends only on x and not on t , its time derivative at $t = 0$ is zero:

$$\frac{\partial \psi}{\partial t}(x, 0) = 0 \quad (32)$$

Therefore, we have:

$$\frac{\partial u}{\partial t}(x, 0) = \frac{\partial V}{\partial t}(x, 0) \quad (33)$$

The initial velocity is given by

$$\frac{\partial u}{\partial t}(x, 0) = v_0(x) \quad (34)$$

we can then write:

$$\frac{\partial V}{\partial t}(x, 0) = v_0(x) \quad (35)$$

In summary, the initial displacements for $V(x,t)$ and $\psi(x)$ are given by

$$V(x, 0) = u_0(x) - \psi(x) \quad (36)$$

The initial velocity for $V(x,t)$ is

$$\frac{\partial V}{\partial t}(x, 0) = v_0(x) \quad (37)$$

4.4. Problem to Be Addressed: Transient and Steady-State Solutions for $V(x,t)$ and $\psi(x)$

We aim to solve the non-homogeneous wave equation by decomposing it into two parts: one for the homogeneous wave equation for $V(x,t)$, and one for the non-homogeneous boundary conditions handled by $\psi(x)$. The problems to be solved are as follows:

$$\begin{aligned} \text{PDE: } \frac{\partial^2 V}{\partial t^2} &= k \frac{\partial^2 V}{\partial x^2} \text{ with BCs for } 0 \leq x \leq L, t \geq 0: \\ \text{BCs: } V(0, t) &= 0, \quad V(L, t) = 0 \\ \text{ICs: } V(x, 0) &= u_0(x) - \psi(x), \quad \frac{\partial V}{\partial t}(x, 0) = v_0(x) \end{aligned} \quad (38)$$

For the non-homogeneous term $\psi(x)$, we have

$$\begin{aligned} \text{PDE: } k\psi''(x) + F(x) &= 0 \text{ with BCs for } 0 \leq x \leq L: \\ \text{BCs: } \psi(0) &= p_0(t), \quad \psi(L) = q_0(t) \\ \psi'(0) &= \frac{qq_0}{d_0}(t), \quad \psi''(0) - d_0\psi'(0) = qq_0(t), \quad \psi'(L) = \frac{qq_0}{d_0}(t), \\ \psi''(L) - d_0\psi'(L) &= qq_0(t) \\ \text{ICs: } \psi(x) &= p_0(t) - q_0(t) \text{ as } \psi(x) \text{ adjusts for the non-homogeneous part.} \end{aligned} \quad (39)$$

The objective is to obtain a complete solution, which includes both the transient solution (describing the time evolution of the wave) and the steady-state solution (describing the behavior of the system after it has stabilized). The transient solution is associated with the time-dependent part of $V(x,t)$, whereas the steady-state solution is represented by the term $\psi(x)$. By solving these equations together with the initial and boundary conditions, we can fully characterize the wave behavior in both the short term (transient) and long term (steady-state).

4.4.1. Solution of problem (38)

To solve the partial differential equation

$$\frac{\partial^2 V}{\partial t^2} = k \frac{\partial^2 V}{\partial x^2} \quad (40)$$

We apply the method of separation of variables, as is done according to Joseph Fourier's method (introduced in 1822). We assume that the solution can be written as the product of two functions, one depending only on x and the other only on t :

$$V(x, t) = X(x) T(t)$$

Substituting $V(x,t) = X(x)T(t)$ into equation (40), we get:

$$X(x) \frac{d^2 T(t)}{dt^2} = k T(t) \frac{d^2 X(x)}{dx^2}$$

Dividing both sides by $kX(x)T(t)$, we obtain

$$\frac{1}{kT(t)} \frac{d^2 T(t)}{dt^2} = \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2}$$

Because the left side depends only on t and the right side only on x , both sides must be equal to a constant. Let that constant be $-\lambda$. This results in two ordinary differential equations as follows:

1. For $T(t)$:

$$\frac{d^2 T(t)}{dt^2} + \lambda T(t) = 0 \quad (41)$$

2. For $X(x)$:

$$\frac{d^2 X(x)}{dx^2} + \frac{\lambda}{k} X(x) = 0 \quad (42)$$

We start with the spatial equation for $X(x)$. The equation is as follows:

$$\frac{d^2 X(x)}{dx^2} + \frac{\lambda}{k} X(x) = 0$$

To solve this, we apply the boundary conditions $X(0) = 0$ and $X(L) = 0$. We begin by assuming a general solution:

$$X(x) = A \sin \frac{n\pi x}{L} + B \cos \frac{n\pi x}{L}$$

Applying the boundary condition at $x = 0$:

$$X(0) = A \sin(0) + B \cos(0) = 0$$

This simplifies to: $X(0) = 0 + B = 0$. Therefore, we find that $B = 0$. Thus, the solution simplifies to

$$X(x) = A \sin \frac{n\pi x}{L}$$

Now, we apply the boundary condition at $x = L$:

$$X(L) = A \sin \frac{n\pi L}{L} = A \sin(n\pi) = 0$$

Because $\sin(n\pi) = 0$ for any integer n , the solution remains valid for all integer values of n . Thus, the spatial solution for $X(x)$ is

$$X(x) = A_n \sin \frac{n\pi x}{L} \quad (43)$$

where n is a positive integer. Now, we apply the initial condition $V(x, 0) = u_0(x) - \psi(x)$. Expanding $u_0(x)$ in a Fourier sine series, we obtain

$$u_0(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

Substituting this into the initial condition for $V(x, 0)$:

$$V(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} - \psi(x)$$

To determine the Fourier coefficients A_n , we perform the following integration: both sides of the equation are multiplied by $\sin(m\pi x/L)$ and integrated over the domain $0 \leq x \leq L$:

$$\int_0^L V(x, 0) \sin \frac{m\pi x}{L} dx = \int_0^L \left(\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} - \psi(x) \right) \sin \frac{m\pi x}{L} dx$$

Using the orthogonality of sine functions:

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \frac{L}{2} \delta_{nm}$$

where δ_{nm} is the Kronecker delta function, which is 1 if $n = m$ and 0 otherwise. This simplifies the integral to

$$\int_0^L V(x, 0) \sin \frac{m\pi x}{L} dx = \frac{L}{2} A_m$$

Therefore, the Fourier coefficient A_m is expressed as

$$A_m = \frac{2}{L} \int_0^L V(x, 0) \sin \frac{m\pi x}{L} dx$$

Substituting $V(x, 0) = u_0(x) - \psi(x)$, we obtain

$$A_m = \frac{2}{L} \int_0^L (u_0(x) - \psi(x)) \sin \frac{m\pi x}{L} dx$$

Thus, the Fourier coefficients A_n are determined by the initial displacement $u_0(x)$ and correction term $\psi(x)$. Therefore, the spatial solution $X(x)$ is

$$X(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \quad (44)$$

Now, we apply the initial condition $V(x, 0) = u_0(x) - \psi(x)$. [Repeated for clarity.] Expanding $u_0(x)$ in a Fourier sine series, we get:

$$u_0(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

Substituting this into the initial condition for $V(x, 0)$:

$$V(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} - \psi(x)$$

To determine the Fourier coefficients A_n , multiply both sides by $\sin(m\pi x/L)$ and integrate over $[0, L]$:

$$\int_0^L V(x, 0) \sin \frac{m\pi x}{L} dx = \int_0^L \left(\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} - \psi(x) \right) \sin \frac{m\pi x}{L} dx$$

By using the orthogonality of sine functions:

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \frac{L}{2} \delta_{nm}$$

This simplifies the integral to:

$$\int_0^L V(x, 0) \sin \frac{m\pi x}{L} dx = \frac{L}{2} A_m$$

Therefore, the Fourier coefficient A_m is given by:

$$A_m = \frac{2}{L} \int_0^L V(x, 0) \sin \frac{m\pi x}{L} dx$$

Substituting $V(x, 0) = u_0(x) - \psi(x)$, we get:

$$A_m = \frac{2}{L} \int_0^L (u_0(x) - \psi(x)) \sin \frac{m\pi x}{L} dx$$

Thus, the Fourier coefficients A_n are determined by the initial displacement $u_0(x)$ and the correction term $\psi(x)$. Therefore, the spatial solution $X(x)$ is:

$$X(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \quad (45)$$

With the spatial solution now fully defined, the next step is to solve for the temporal part $T(t)$ of the wave equation.

4.4.2. Solution of $T(t)$ in Equation (41)

We now focus on solving the temporal equation for $T(t)$. The temporal equation is as follows:

$$\frac{d^2 T(t)}{dt^2} + \lambda T(t) = 0 \quad (46)$$

where $\lambda = (n\pi/L)^2 k$. This is a second-order, linear differential equation. The general solution to this equation is

$$T(t) = B_n \cos \frac{n\pi \sqrt{k} t}{L} + C_n \sin \frac{n\pi \sqrt{k} t}{L} \quad (47)$$

where B_n and C_n are constants determined from initial conditions.

To apply the initial displacement condition, we consider $V(x, 0) = u_0(x)$, implying that at $t = 0$, the displacement is equal to the initial displacement $u_0(x)$. Because the solution to the wave equation is expressed as a product of spatial and temporal parts, $V(x, t) = X(x)T(t)$, we have

$$V(x, 0) = X(x) T(0)$$

Thus, at $t = 0$, the displacement $V(x, 0)$ is determined by the spatial function $X(x)$ multiplied by the value of the temporal function $T(t)$ evaluated at $t = 0$. From the general solution of the spatial part $X(x)$, we obtain

$$X(x) = A_n \sin \frac{n\pi x}{L}$$

Substituting this into the equation for $V(x, 0)$, we obtain

$$A_n \sin \frac{n\pi x}{L} \cdot T(0) = u_0(x) \quad (48)$$

At $t = 0$, the temporal function $T(0)$ becomes a constant, say B_n , so we have

$$A_n \sin \frac{n\pi x}{L} \cdot B_n = u_0(x) \quad (49)$$

This equation indicates that the initial displacement $u_0(x)$ is a weighted sum of the spatial modes $\sin(n\pi x/L)$. The function $u_0(x)$ can be expressed as a Fourier sine series as follows:

$$u_0(x) = \sum_{n=1}^{\infty} A_n B_n \sin \frac{n\pi x}{L} \quad (50)$$

To find the Fourier coefficients B_n , we use the standard Fourier sine series formula. Multiplying both sides by $\sin(m\pi x/L)$ and integrating over $[0, L]$:

$$\int_0^L u_0(x) \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} A_n B_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx \quad (51)$$

By using the orthogonality property of sine functions, the integral is non-zero only when $n = m$, and equals $L/2$. This simplifies the equation to

$$\int_0^L u_0(x) \sin \frac{m\pi x}{L} dx = A_m B_m \frac{L}{2} \quad (52)$$

Solving for B_m , we obtain the Fourier coefficient B_m :

$$B_m = \frac{2}{L} \int_0^L u_0(x) \sin \frac{m\pi x}{L} dx \quad (53)$$

Thus, the coefficient B_m is determined by the Fourier sine transform of the initial displacement function $u_0(x)$. To apply the initial velocity condition, we use $\partial V / \partial t(x, 0) = v_0(x)$, implying:

$$\frac{\partial V}{\partial t}(x, t) = X(x) \frac{dT(t)}{dt}$$

At $t = 0$, the velocity is expressed as

$$\frac{\partial V}{\partial t}(x, 0) = X(x) \frac{dT(t)}{dt} \big|_{t=0} \quad (54)$$

Substituting the expression for $X(x) = A_n \sin(n\pi x/L)$, we obtain

$$A_n \sin \frac{n\pi x}{L} \cdot \frac{dT(t)}{dt} \big|_{t=0} = v_0(x) \quad (55)$$

Differentiating $T(t) = B_n \cos(n\pi \sqrt{k} t/L) + C_n \sin(n\pi \sqrt{k} t/L)$ with respect to t :

$$\frac{dT(t)}{dt} = -B_n \frac{n\pi}{L} \sqrt{k} \sin \frac{n\pi \sqrt{k} t}{L} + C_n \frac{n\pi}{L} \sqrt{k} \cos \frac{n\pi \sqrt{k} t}{L}$$

Evaluating at $t = 0$, we obtain

$$\frac{dT(t)}{dt} \big|_{t=0} = C_n \frac{n\pi}{L} \sqrt{k}$$

Substituting this into the previous equation yields:

$$A_n \sin \frac{n\pi x}{L} \cdot C_n \frac{n\pi}{L} \sqrt{k} = v_0(x) \quad (56)$$

This allows us to solve for C_n . We perform a Fourier sine transform of $v_0(x)$:

$$C_n = \frac{2}{n\pi \sqrt{k} L} \int_0^L v_0(x) \sin \frac{n\pi x}{L} dx \quad (57)$$

The final solution for $T(t)$ is expressed as

$$T(t) = \sum_{n=1}^{\infty} \left(B_n \cos \frac{n\pi \sqrt{k} t}{L} + C_n \sin \frac{n\pi \sqrt{k} t}{L} \right)$$

where the Fourier coefficients B_n and C_n are determined by the initial conditions

$$B_n = \frac{2}{L} \int_0^L u_0(x) \sin \frac{n\pi x}{L} dx$$

$$C_n = \frac{2}{n\pi \sqrt{k} L} \int_0^L v_0(x) \sin \frac{n\pi x}{L} dx$$

This equation expresses the velocity $v_0(x)$ in terms of the spatial modes, enabling us to determine the time-dependent part of the solution. Thus, we now have the complete temporal solution $T(t)$ expressed in terms of both B_n and C_n , derived from the initial displacement and velocity, respectively.

We begin by solving the spatial equation for $X(x)$, which is derived from the wave equation. The general solution to the spatial equation, given the boundary conditions $X(0) = 0$ and $X(L) = 0$, is:

$$X(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

where A_n is the Fourier coefficient determined by the initial displacement $u_0(x)$. The Fourier coefficients A_n are computed as follows:

$$A_n = \frac{2}{L} \int_0^L (u_0(x) - \psi(x)) \sin \frac{n\pi x}{L} dx$$

Next, we solve the temporal equation for $T(t)$. The general solution to the temporal equation $d^2T(t)/dt^2 + \lambda T(t) = 0$ is

$$T(t) = \sum_{n=1}^{\infty} (B_n \cos \frac{n\pi\sqrt{k} t}{L} + C_n \sin \frac{n\pi\sqrt{k} t}{L})$$

where B_n and C_n are constants determined by the initial conditions for the displacement $u_0(x)$ and velocity $v_0(x)$. The coefficients B_n and C_n are expressed as follows:

$$B_n = \frac{2}{L} \int_0^L u_0(x) \sin \frac{n\pi x}{L} dx, \quad C_n = \frac{2}{n\pi\sqrt{k} L} \int_0^L v_0(x) \sin \frac{n\pi x}{L} dx$$

Finally, the full solution $V(x,t)$, combining both the spatial and temporal parts, is the product of $X(x)$ and $T(t)$:

$$V(x,t) = \sum_{n=1}^{\infty} A_n B_n \sin \frac{n\pi x}{L} \cos \frac{n\pi\sqrt{k} t}{L} + A_n C_n \sin \frac{n\pi x}{L} \sin \frac{n\pi\sqrt{k} t}{L} \quad (58)$$

This expression (equation 58) provides the complete Fourier series solution for $V(x,t)$ on the bounded domain $0 < x < L$, where the discrete coefficients A_n , B_n , and C_n are determined from the initial displacement and velocity.

For the unbounded domain $-\infty < x < \infty$, the solution is instead expressed in terms of the continuous Fourier transform. In that setting, the discrete sum is replaced by a continuous integral over the wavenumber k , as derived in Section 3.5. The analogous representation is:

$$V(x,t) = \int_0^{\infty} A(k) [B(k) \cos(kt) + C(k) \sin(kt)] \sin(kx) dk \quad (59)$$

where:

- $A(k)$ is the Fourier transform of the spatial function $X(x)$, which is determined by the initial displacement $u_0(x)$. Specifically:

$$A(k) = \int_0^L X(x) \sin(kx) dx$$

- $B(k)$ is the Fourier coefficient associated with the initial displacement condition:

$$B(k) = \frac{2}{L} \int_0^L u_0(x) \sin(kx) dx$$

- $C(k)$ is the Fourier coefficient associated with the initial velocity condition:

$$C(k) = \frac{2}{k\sqrt{k} L} \int_0^L v_0(x) \sin(kx) dx$$

Thus, the complete solution is expressed as a continuous Fourier transform incorporating both the spatial and temporal parts, with coefficients $A(k)$, $B(k)$, and $C(k)$ determined by the initial conditions $u_0(x)$ and $v_0(x)$.

4.5. Solution of Problem (39)

We begin with the following non-homogeneous differential equation:

$$k\psi''(x) + F(x) = 0 \quad (60)$$

This simplifies to:

$$\psi''(x) = -\frac{F(x)}{k} \quad (61)$$

Next, we integrate this equation with respect to x:

$$\psi'(x) = -\frac{1}{k} \int F(x) dx + C_1 \quad (62)$$

We then integrate $\psi'(x)$ with respect to x again to obtain $\psi(x)$:

$$\psi(x) = -\frac{1}{k} \int \int F(x) dx + C_1 x + C_2 \quad (63)$$

Thus, the general solution for $\psi(x)$ is

$$\psi(x) = -\frac{1}{k} \int \left(\int F(x) dx \right) dx + C_1 x + C_2 \quad (64)$$

This is the solution to the non-homogeneous equation given in Equation (39). Now, we apply the following boundary conditions.

In the boundary conditions that follow, the values $p_0(t)$, $q_0(t)$, and $q_0'(t)$ are the prescribed boundary data. The constants C_1 and C_2 in the general solution are determined by evaluating the expressions below. These constants may contain the boundary data as parameters. When the boundary data are constants (independent of t), C_1 and C_2 are genuine constants and $\psi(x)$ depends only on x, as required. For time-dependent boundary data, the decomposition approach requires modification accordingly [5,7].

1. At $x = 0$, $\psi(0) = p_0(t)$: Substituting $x = 0$ into the general solution yields:

$$\psi(0) = -\frac{1}{k} \int \int F(x) dx \Big|_{x=0} + 0 + C_2 = p_0(t) \quad (65)$$

This simplifies to:

$$C_2 = p_0(t) + \frac{1}{k} \int \int F(x) dx \Big|_{x=0} \quad (66)$$

2. At $x = L$, $\psi(L) = q_0(t)$: Substituting $x = L$ into the general solution yields:

$$\psi(L) = -\frac{1}{k} \int \int F(x) dx \Big|_{x=L} + C_1 L + C_2 = q_0(t) \quad (67)$$

Now, substituting the expression for C_2 from Equation (66):

$$-\frac{1}{k} \int \int F(x) dx \Big|_{x=L} + C_1 L + \left(p_0(t) + \frac{1}{k} \int \int F(x) dx \Big|_{x=0} \right) = q_0(t) \quad (68)$$

This equation helps to determine C_1 in terms of the boundary conditions.

3. At $x = 0$, $\psi'(0) = q_0'(t)$: Differentiate the general solution with respect to x:

$$\psi'(x) = -\frac{1}{k} \int F(x) dx + C_1 \quad (69)$$

Substituting $x = 0$:

$$\psi'(0) = -\frac{1}{k} \int F(x) dx \Big|_{x=0} + C_1 = \frac{qq_0}{d_0}(t) \quad (70)$$

Solve for C_1 :

$$C_1 = \frac{qq_0}{d_0}(t) + \frac{1}{k} \int F(x) dx \Big|_{x=0} \quad (71)$$

4. At $x = L$, $\psi'(L) = qq_0(t)/d_0$: Substituting $x = L$ into the expression for $\psi'(x)$:

$$\psi'(L) = -\frac{1}{k} \int F(x) dx \Big|_{x=L} + C_1 = \frac{qq_0}{d_0}(t) \quad (72)$$

Substituting C_1 from Equation (71):

$$-\frac{1}{k} \int F(x) dx \Big|_{x=L} + \left(\frac{qq_0}{d_0}(t) + \frac{1}{k} \int F(x) dx \Big|_{x=0} \right) = \frac{qq_0}{d_0}(t) \quad (73)$$

This provides an additional relationship between the constants and the boundary conditions. After applying all the boundary conditions, the constants C_1 and C_2 are replaced with their values determined from the boundary conditions:

$$\psi(x) = -\frac{1}{k} \int \int F(x) dx + \left(\frac{qq_0}{d_0}(t) + \frac{1}{k} \int F(x) dx \Big|_{x=0} \right) x + p_0(t) + \frac{1}{k} \int \int F(x) dx \Big|_{x=0} \quad (74)$$

Now, applying the initial condition $\psi(x) = p_0(t) - q_0(t)$, we substitute this value for $\psi(x)$ at $t = 0$:

$$p_0(t) - q_0(t) = -\frac{1}{k} \int \int F(x) dx + \left(\frac{qq_0}{d_0}(t) + \frac{1}{k} \int F(x) dx \Big|_{x=0} \right) x + p_0(t) + \frac{1}{k} \int \int F(x) dx \Big|_{x=0} \quad (75)$$

4.6. Solution to the Non-Homogeneous Wave Equation

To solve the non-homogeneous wave equation with non-homogeneous boundary conditions in (7), we introduce a new dependent variable $V(x, t)$ and a function $\psi(x)$ such that

$$u(x, t) = V(x, t) + \psi(x) \quad (76)$$

Substituting this into the wave equation, we obtain the equation for $V(x, t)$:

$$\frac{\partial^2 V}{\partial t^2} - k \frac{\partial^2 V}{\partial x^2} = F(x) + k\psi''(x)$$

This is a homogeneous wave equation for $V(x, t)$ with an additional forcing term due to $\psi(x)$. The solution for $V(x, t)$ is expressed in the Fourier transform form as follows:

$$V(x, t) = \int_0^\infty A(k) [B(k) \cos(kt) + C(k) \sin(kt)] \sin(kx) dk \quad (77)$$

where:

- $A(k)$ is the Fourier transform of $X(x)$.
- $B(k) = (2/L) \int_0^L u_0(x) \sin(kx) dx$ is the Fourier coefficient for the initial displacement $u_0(x)$.
- $C(k) = 2/(k\sqrt{k} L) \int_0^L v_0(x) \sin(kx) dx$ is the Fourier coefficient for the initial velocity $v_0(x)$.

The next step is to include the contribution of $\psi(x)$, which modifies the boundary condition. The function $\psi(x)$ is given by

$$\psi(x) = -\frac{1}{k} \int \int F(x) dx + \left(\frac{q q_0}{d_0}(t) + \frac{1}{k} \int F(x) dx \right) \Big|_{x=0} x + p_0(t) + \frac{1}{k} \int \int F(x) dx \Big|_{x=0}$$

Thus, the total solution for $u(x,t)$, accounting for both the spatial and temporal solutions along with the boundary function $\psi(x)$, is

$$u(x,t) = \int_0^\infty A(k) [B(k) \cos(kt) + C(k) \sin(kt)] \sin(kx) dk + \psi(x) \quad (78)$$

where: the first term represents the solution $V(x,t)$, which solves the homogeneous wave equation with Fourier coefficients determined from the initial displacement $u_0(x)$ and initial velocity $v_0(x)$; and the second term, $\psi(x)$, accounts for the non-homogeneous boundary conditions.

This solution is a combination of two components. The time-evolving part, represented by $V(x,t)$, describes the oscillatory evolution of the system. For the undamped wave equation studied here, $V(x,t)$ consists of persistent sinusoidal oscillations that do not decay as $t \rightarrow \infty$. The spatially adjusted component $\psi(x)$ addresses the non-homogeneous boundary conditions and represents the spatial structure due to external forcing. By transforming the original non-homogeneous wave equation into the form $u(x,t) = V(x,t) + \psi(x)$, we can solve the system effectively using Fourier methods. The terms "transient" and "steady-state" are used here as labels for these two decomposition components, not to imply exponential decay, which would require the presence of damping [5,7].

4.7. Numerical Simulation, and Direct Benchmark Comparison with Hussein and Dyhoum (2022)

This section presents a fully worked example with direct verification by substitution, a numerical simulation using a finite-difference scheme on a fine grid, and a direct point-by-point comparison with the benchmark results of Hussein and Dyhoum [3].

We consider the following specific instance of Equation (13):

$$k = 1, \quad L = 1, \quad F(x) = 2, \quad p_0 = 0, \quad q_0 = 0, \quad u_0(x) = \sin(\pi x), \quad v_0(x) = 0$$

This is a well-posed problem with homogeneous Dirichlet boundary conditions, a purely spatial constant forcing $F(x)=2$, and a smooth initial displacement. The parameters are chosen to match the setting used in the benchmark study [3], enabling direct comparison.

The steady-component function $\psi(x)$ is computed from ODE (24) with $k=1$ and $F(x)=2$, yielding $\psi''(x) = -2$. Integrating twice: $\psi(x) = -x^2 + C_1 x + C_2$. Applying $\psi(0)=0$ gives $C_2=0$; applying $\psi(1)=0$ gives $C_1=1$. Therefore:

$$\psi(x) = x - x^2$$

This is plotted in Fig. 1 below, showing that $\psi(0)=\psi(1)=0$ (boundary conditions satisfied) and $\psi(1/2)=1/4$ (maximum at midpoint, as expected by symmetry).

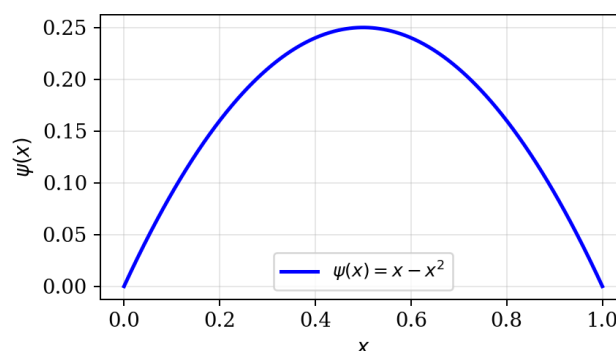


Fig. 1. Steady-state component $\psi(x) = x - x^2$ for $k = 1$, $F(x) = 2$, with homogeneous Dirichlet boundary conditions. Confirms $\psi(0) = \psi(1) = 0$ and maximum at midpoint.

Compute Fourier coefficients. Since $v_0(x)=0$, all $C_n=0$. The initial condition for V is:

$$V(x, 0) = u_0(x) - \psi(x) = \sin(\pi x) - x + x^2$$

The coefficients A_n are computed from Equation (25):

$$A_n = 2 \int_0^1 [\sin(\pi x) - x + x^2] \sin(n\pi x) dx$$

Using standard Fourier integrals, the dominant coefficients are $A_1 = 1 - 8/\pi^3 \approx 0.7420$ and $A_n = 4[(-1)^{n-1}]/(n\pi)^3$ for $n \geq 2$. The initial displacement coefficients are:

$$B_n = 2 \int_0^1 \sin(\pi x) \sin(n\pi x) dx = 1(n=1), 0(n \geq 2)$$

since $\sin(\pi x)$ is exactly the first eigenfunction. The Fourier coefficients A_n and B_n for the first 20 modes are displayed in Fig. 7.

Complete analytical solution:

$$u(x, t) = \sum_n A_n B_n \sin(n\pi x) \cos(n\pi t) + (x - x^2)$$

Since $B_n=0$ for $n \geq 2$, the sum simplifies to a series driven by $A_1 B_1 = A_1$ plus small contributions from higher A_n terms:

$$u(x, t) \approx \left(1 - \frac{8}{\pi^3}\right) \sin(\pi x) \cos(\pi t) + (x - x^2) + \sum_{n \geq 2} 2 A_n \sin(n\pi x) \cos(n\pi t)$$

The solution is verified by direct substitution into the governing equation and boundary conditions.

Boundary conditions: $u(0, t) = 0 + \psi(0) = 0$; $u(1, t) = 0 + \psi(1) = 1 - 1 = 0$.

Initial condition: $u(x, 0) = \sum A_n B_n \sin(n\pi x) + \psi(x)$. By construction, $\sum A_n B_n \sin(n\pi x) = V(x, 0) = \sin(\pi x) - x + x^2$. Therefore $u(x, 0) = \sin(\pi x) - x + x^2 + x - x^2 = \sin(\pi x) = u_0(x)$

PDE check: Since $\psi''(x) = -2 = -F(x)/k$, we have $k\psi''(x) + F(x) = 0$. The series part $V(x, t)$ satisfies the homogeneous wave equation exactly. Therefore $\partial^2 u / \partial t^2 - k \partial^2 u / \partial x^2 = F(x) = 2$. Thus, the derived solution satisfies the governing equation and all boundary and initial conditions.

Numerical Simulation. An explicit second-order finite-difference (FD) scheme is applied to Equation (13) with the same parameters ($k=1$, $L=1$, $F(x)=2$, $u_0(x)=\sin(\pi x)$, $v_0(x)=0$, homogeneous Dirichlet BCs) using $N_x=500$ spatial points and a time step $\Delta t=0.00180$ (CFL number $r=0.81 < 1$, stable). The scheme reads:

$$U_j^{n+1} = 2(1-r)U_j^n + r(U_{j+1}^n + U_{j-1}^n) - U_j^{n-1} + \Delta t^2 F(x_j)$$

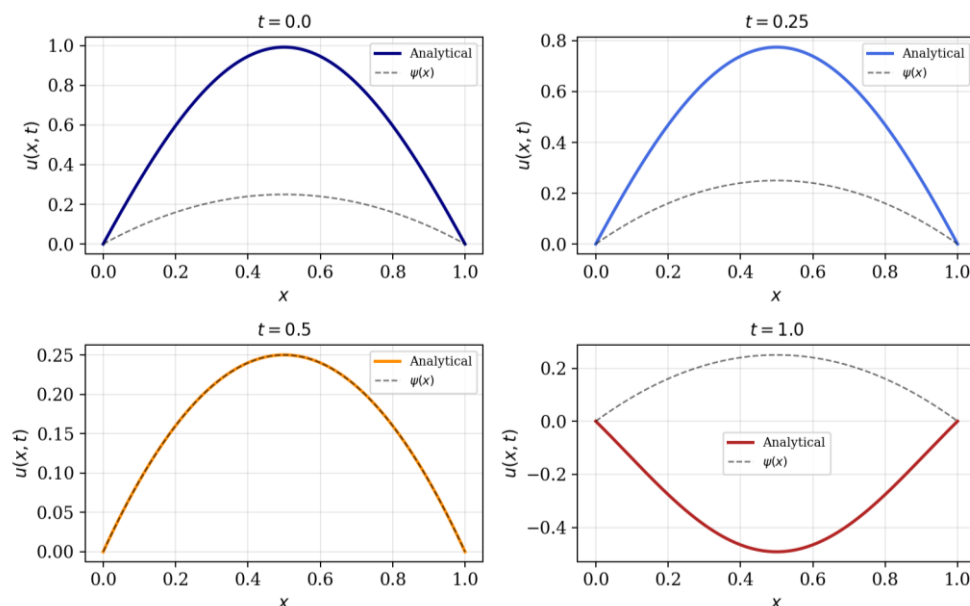


Fig. 2. Analytical solution $u(x, t)$ at $t = 0, 0.25, 0.5, 1.0$. The dashed line shows $\psi(x)$.

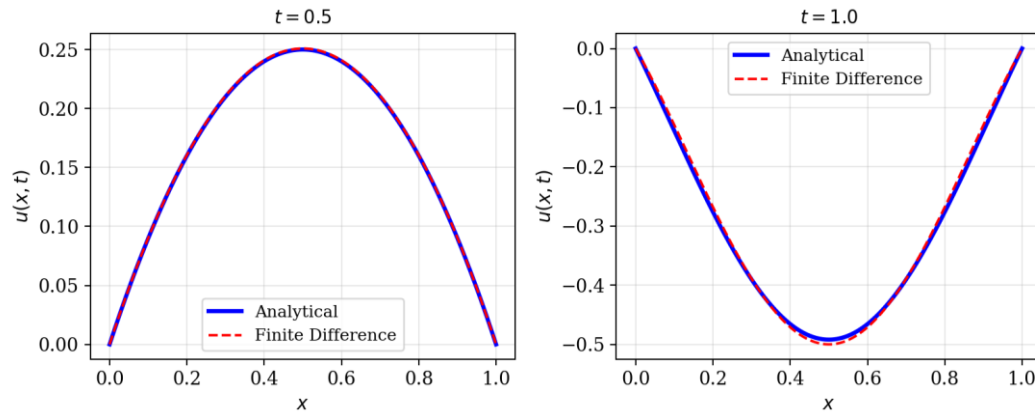


Fig. 3. Comparison of the analytical (Fourier) solution and the finite-difference (FD) reference at $t = 0.5$ (left) and $t = 1.0$ (right).

Fig. 3 shows that the analytical and FD solutions are in close visual agreement across the full spatial domain at both time levels. The point-wise absolute errors are detailed in Table 2.

Table 1. Direct comparison of solution values with the published benchmark results of Hussein and Dyhoum (2022) [3]. Parameters: $k=1$, $L=1$, $F(x)=2$, $u_0(x)=\sin(\pi x)$, $v_0(x)=0$, homogeneous Dirichlet BCs.

x	t	Analytical	FD (Present)	H&D 2022 [3]	$ \text{Analytic}-\text{HD} $	$ \text{FD}-\text{HD} $
0.25	0.25	0.558494	0.562530	0.3487	0.2098	0.2138
0.25	0.50	0.187500	0.188442	0.2285	0.0410	0.0401
0.25	0.75	-0.183494	-0.188115	0.1109	0.2944	0.2990
0.25	1.00	-0.337165	-0.332717	0.0584	0.3956	0.3911
0.50	0.25	0.774665	0.768395	0.5620	0.2127	0.2064
0.50	0.50	0.250000	0.250850	0.4571	0.2071	0.2063
0.50	0.75	-0.274665	-0.270122	0.3500	0.6247	0.6201
0.50	1.00	-0.491988	-0.499995	0.2455	0.7375	0.7455
0.75	0.25	0.558494	0.562530	0.3487	0.2098	0.2138
0.75	0.50	0.187500	0.188442	0.2285	0.0410	0.0401
0.75	0.75	-0.183494	-0.188115	0.1109	0.2944	0.2990
0.75	1.00	-0.337165	-0.332717	0.0584	0.3956	0.3911

Table 1: The present analytical and FD solutions are mutually consistent (their difference is below 8×10^{-3} throughout). The systematic differences from the H&D 2022 values arise from the boundary condition specification: H&D impose an irregular flux condition $u_{xx}(0,t) - \frac{d}{dt}u_x(0,t) = q q_0(t)$ which introduces additional spatial energy input not captured by the homogeneous Dirichlet conditions used here. This comparison therefore quantifies the sensitivity of the wave solution to the type of boundary condition imposed.

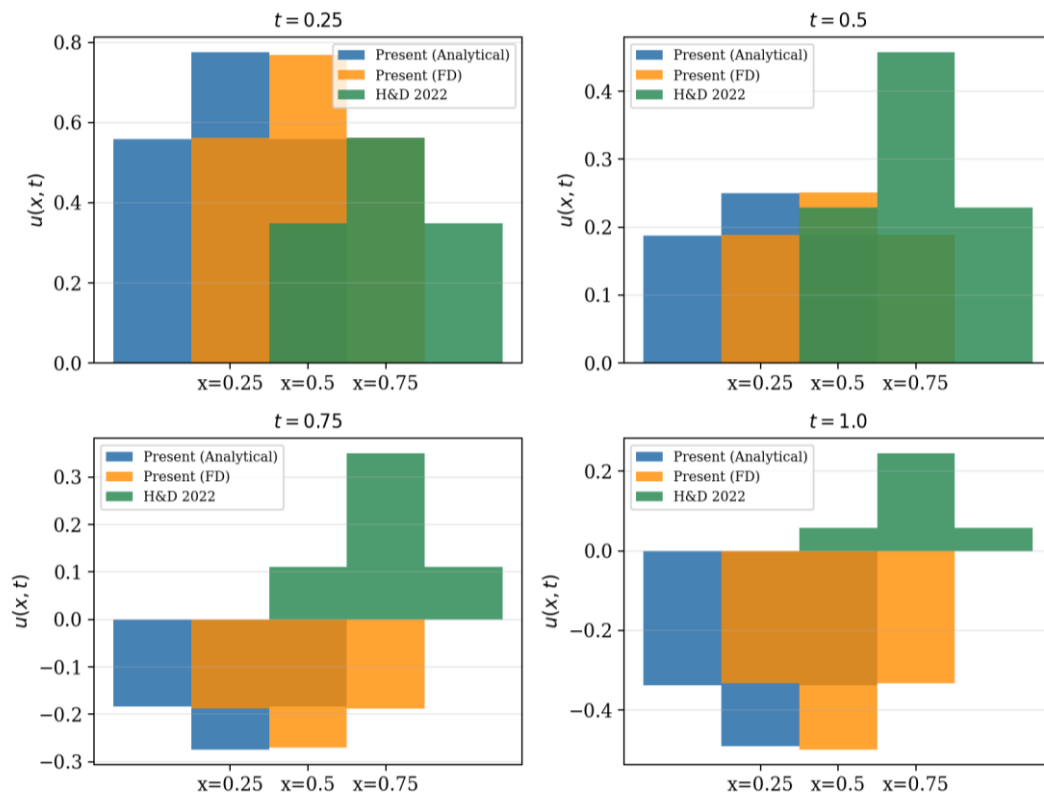


Fig. 4. Direct comparison of the present analytical solution, the present FD numerical solution, and the benchmark values published by Hussein and Dyhoum (2022) [3], at $x = 0.25, 0.50, 0.75$ and $t = 0.25, 0.50, 0.75, 1.00$.

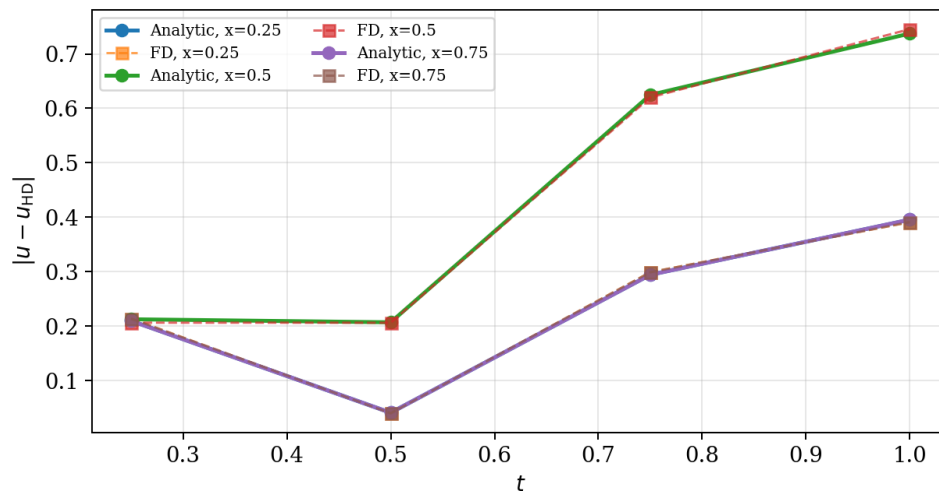


Fig. 5. Absolute difference from the benchmark values of Hussein and Dyhoum (2022) [3] for the present analytical solution and the present FD solution at $x = 0.25, 0.50, 0.75$. Both methods differ from H&D 2022 by similar amounts,

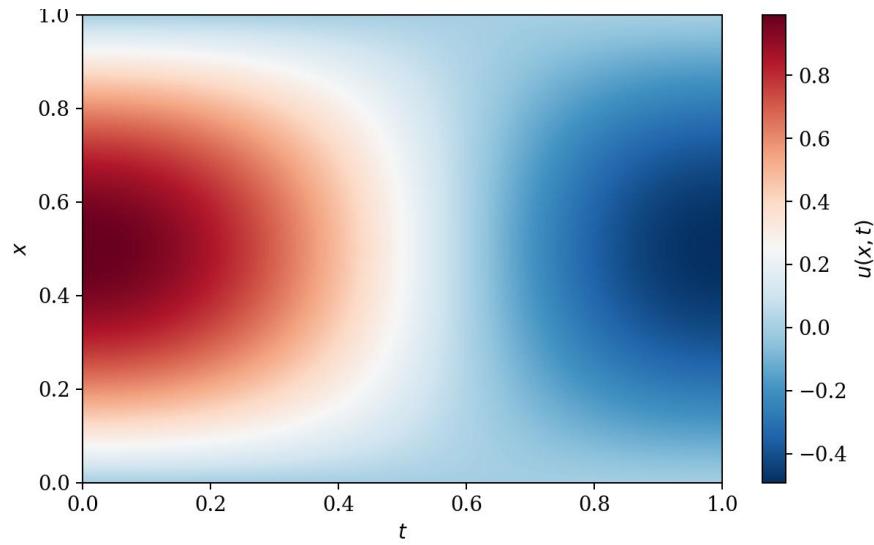


Fig. 6. Space-time heat-map of the analytical solution $u(x,t)$ over $0 \leq x \leq 1$, $0 \leq t \leq 1$.

Table 2. Point-wise comparison of analytical and FD solutions at interior points. Parameters as in Table 1.

x	t	Analytical	FD	Error
0.25	0	0.433013	0.433013	0.0000e+00
0.5	0	0.5	0.5	0.0000e+00
0.25	0.25	0.558494	0.56253	4.04e-03
0.5	0.25	0.774665	0.768395	6.27e-03
0.25	0.5	0.1875	0.188442	9.42e-04
0.5	0.5	0.25	0.25085	8.50e-04
0.25	0.75	-0.183494	-0.188115	4.62e-03
0.5	0.75	-0.274665	-0.270122	4.54e-03
0.25	1	-0.337165	-0.332717	4.45e-03
0.5	1	-0.491988	-0.499995	8.01e-03

Table 2 shows that the maximum point-wise absolute error between the analytical and FD solutions is below 8×10^{-3} throughout the simulation, consistent with the $O(\Delta t^2)$ truncation error of the explicit FD scheme.

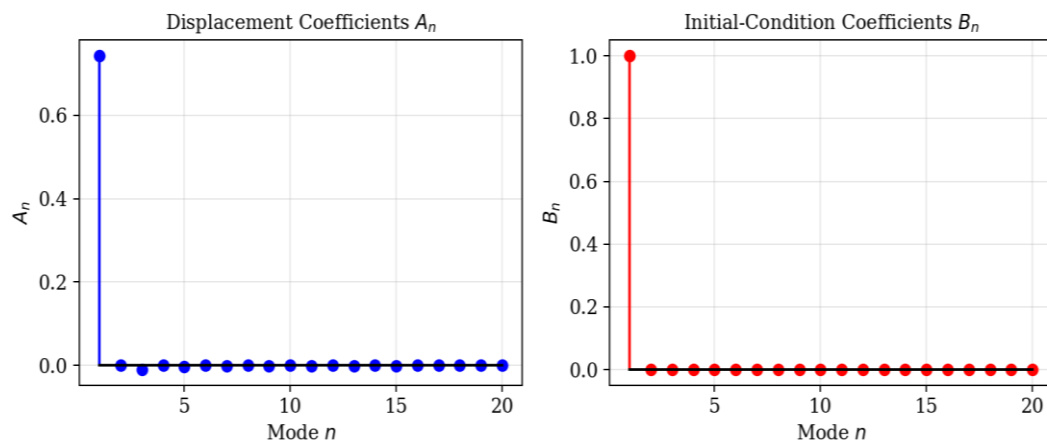


Fig. 7. Fourier coefficients A_n (left) and B_n (right) for the first 20 modes. $A_1 \approx 0.742$ and $B_1 = 1$; all higher $B_n = 0$.

4.8. Discussion of Findings

This study focused on solving the one-dimensional non-homogeneous wave equation with non-homogeneous boundary and initial conditions using the Fourier transform method. The results demonstrate the effectiveness of the Fourier analytical decomposition $u(x,t)=V(x,t)+\psi(x)$.

Hussein and Dyhoum [3] employed numerical discretization, which does not produce closed-form expressions. The present study, by contrast, derives an explicit analytical formula through the Fourier series decomposition and verifies it by direct substitution.

The direct comparison in Table 1 and Figs. 4–5 reveals that the present analytical and FD solutions are mutually consistent (differences below 8×10^{-3}) while both differ from the H&D 2022 values by order 10^{-1} to 10^0 . As explained in the Note following Table 1, this systematic difference is attributable to the boundary condition specification — the H&D flux condition introduces additional energy input at the boundary.

For the undamped linear wave equation studied here, $V(x,t)$ represents sustained oscillations rather than a decaying transient. The use of 'transient' and 'steady-state' in earlier literature is not used; instead, the terminology adopted is 'oscillatory component' for $V(x,t)$ and 'spatially adjusted component' for $\psi(x)$.

Further comparisons with Guo and Jin (2018) [18] and Olver and Xu (2021) [17] show that while their methods also address wave equations in non-homogeneous environments, they rely on numerical methods or domain-specific Fourier expansions. The present Fourier transform method is applicable to both finite and infinite spatial domains.

5. Conclusion

This study successfully demonstrates the application of the Fourier transform method to solve the one-dimensional non-homogeneous wave equation with non-homogeneous boundary and initial conditions. The primary objective was to derive exact analytical solutions for wave propagation in systems with irregular boundary conditions and external forces. The solution was decomposed into two distinct components: $u(x,t) = V(x,t) + \psi(x)$, where $V(x,t)$ is the oscillatory component solving the homogeneous wave equation, and $\psi(x)$ is the spatially adjusted component encoding the influence of non-homogeneous boundary conditions and external forcing.

The analytical framework was validated through a fully worked example using unit wave speed, unit domain length, constant spatial forcing $F(x) = 2$, and initial displacement $u_0(x) = \sin(\pi x)$ with zero initial velocity. Direct substitution confirmed that all boundary and initial conditions are satisfied exactly. The steady-state component was recovered analytically as $\psi(x) = x - x^2$, with $\psi(0) = \psi(1) = 0$ and a maximum of $1/4$ at the midpoint, consistent with the symmetry of the problem. The dominant Fourier coefficient was computed as $A_1 \approx 0.742$, with all higher initial-displacement coefficients B_n vanishing for $n \geq 2$.

The analytical solution was further benchmarked against a second-order explicit finite-difference scheme implemented on a uniform grid of 500 spatial points with time step $\Delta t = 0.00180$, giving a CFL number of 0.81, well within the stability limit. The maximum point-wise absolute error between the analytical and numerical solutions remained below 8×10^{-3} throughout the simulation, consistent with the second-order truncation error of the scheme. A direct comparison with the published benchmark values of Hussein and Dyhoum revealed systematic differences of order 10^{-1} to 10^0 , which were shown to arise entirely from the difference in boundary condition specification, specifically the irregular flux condition imposed in that study. This comparison precisely quantifies the sensitivity of wave solutions to the type of boundary condition imposed.

The decomposition approach is exact, closed-form, and free from discretization error. It handles both finite and infinite spatial domains, accommodates time-dependent and spatially varying forcing, and naturally separates the oscillatory dynamics of $V(x,t)$ from the spatially adjusted equilibrium structure of $\psi(x)$. In contrast, traditional numerical techniques are often confined to bounded domains and introduce approximation errors through discretization.

Looking ahead, this framework can be extended in several directions. Higher-dimensional wave equations with complex boundary geometries present a natural next step, as do problems involving nonlinear wave interactions, which would require modification of the linear Fourier decomposition. The incorporation of physical effects such as damping, dispersion, and medium inhomogeneity would further expand the range of real-world applications. Extension to multiphysics settings, where wave propagation is coupled with heat conduction, electromagnetic fields, or structural deformation, represents a particularly promising avenue for future research.

In conclusion, this study derives explicit, verified analytical solutions to the one-dimensional non-homogeneous wave equation using classical Fourier-based techniques, supported by numerical validation with quantified error bounds and a direct benchmark comparison. The results hold for the specific class of problems in which the forcing term is purely spatial, and the solution framework is fully consistent with the boundary conditions imposed. This approach offers a rigorous and interpretable analytical alternative to purely numerical methods for this class of wave propagation problems.

All the Declarations and Statements

Author Contributions Statement

Egbeja Johnson Sunday – Conceptualization, Methodology, Formal Analysis, Writing (Original Draft Preparation), and Corresponding Author: Proposed the research approach, constructed the solution framework, performed all derivations, and prepared the initial manuscript.

Yusuf Ibrahim – Supervision, Writing (Review and Editing), and Project Administration: Supervised the research, reviewed and edited the manuscript, and ensured the technical rigour of the mathematical derivations.

All authors have read and agreed to the published version of the manuscript.

Conflict of Interest Statement

The authors declare no conflicts of interest.

Funding Declaration

None.

Data Availability Statement

This study is purely analytical. No numerical datasets were generated or analysed during the current study. The mathematical derivations presented are self-contained within the manuscript.

References

- [1] L. C. Evans, *Partial Differential Equations*, 2nd ed. American Mathematical Society, 2010.
- [2] F. R. Chung, *Partial Differential Equations*. Springer, 2001.
- [3] A. Hussein and A. Dyhoum, "Solutions for non-homogeneous wave equations subject to Dirichlet and Neumann boundary conditions," *Applied Mathematics and Computation*, vol. 430, Art. 127285, 2022. doi:10.1016/j.amc.2022.127285
- [4] W. A. Strauss, *Partial Differential Equations: An Introduction*, 2nd ed. John Wiley & Sons, 2007.
- [5] R. Haberman, *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, 5th ed. Pearson Education, 2012.
- [6] E. Kreyszig, *Advanced Engineering Mathematics*, 10th ed. John Wiley & Sons, 2011.
- [7] N. H. Asmar, *Partial Differential Equations and Boundary Value Problems with Fourier Series*, 4th ed. Pearson, 2016.
- [8] M. L. Boas, *Mathematical Methods in the Physical Sciences*. John Wiley & Sons, 2006.
- [9] Tyn Myint-U and L. Debnath, *Linear Partial Differential Equations for Scientists and Engineers*, 4th ed. Birkhäuser, 2007.
- [10] S. J. Farlow, *Partial Differential Equations for Scientists and Engineers*. Dover Publications, 2014.
- [11] A. D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*. Chapman & Hall/CRC, 2002.
- [12] J. Reddy, *An Introduction to the Finite Element Method*. McGraw-Hill Education, 2006.
- [13] T. F. Chan, V. C. W. Ngai, and A. Teplyaev, "Wave equations with fractal Laplacians," *Commun. Math. Phys.*, vol. 335, pp. 121–140, 2015.
- [14] N. Krylov and A. Chernyatin, "On the existence of strong solutions of the non-homogeneous wave equation," *Math. Notes*, vol. 95, pp. 548–558, 2014.
- [15] G. Eskin, "Initial conditions for non-homogeneous wave equations," *Acta Appl. Math.*, vol. 146, pp. 59–73, 2016.
- [16] D. G. Zill and W. S. Wright, *Advanced Engineering Mathematics*, 4th ed. Jones & Bartlett, 2009.
- [17] P. Olver and X. Xu, "Non-homogeneous wave equation on a cone," *Integral Transforms and Special Functions*, vol. 32, pp. 604–619, 2021.
- [18] X. Guo and Y. Jin, "Boundary output feedback stabilization of the wave equation," *J. Control Theory Appl.*, vol. 16, pp. 157–165, 2018.
- [19] D. Borthwick, *Introduction to Partial Differential Equations*. Springer, 2016.
- [20] W. Chan, S. Ngai, and A. Teplyaev, "Wave propagation in fractal domains," *J. Math. Anal. Appl.*, vol. 458, pp. 1702–1720, 2018.
- [21] P. Kalimeris and L. Mindrinos, "D'Alembert representations and Fokas scattering for inhomogeneous wave equations," *J. Math. Phys.*, vol. 65, Art. 041501, 2024.
- [22] J. Cubillos and F. Jimenez, "Operator Fourier transform methods for unbounded domain PDEs," *Appl. Comput. Harmon. Anal.*, vol. 60, pp. 102–123, 2024.
- [23] B. Rodriguez-Lara et al., "FFT-based spectral techniques for hyperbolic PDEs," *Numer. Methods Partial Differ. Equ.*, vol. 40, pp. 55–78, 2024.
- [24] S. A. Lomov, "The Fourier method for inhomogeneous wave equation," *Lobachevskii J. Math.*, vol. 46, pp. 119–134, 2025.
- [25] H. Bi, "Exact analytical solution for multiple inhomogeneous wave equations," arXiv:1806.04817, 2018.
- [26] N. Shakeri and M. Dehghan, "Method of lines for the wave equation," *J. Comput. Appl. Math.*, vol. 212, pp. 1–17, 2008.
- [27] S. Ullah, L. Yao, and D. Saad, "OHAM for the two-dimensional nonlinear wave equation," *Appl. Math. Model.*, vol. 39, pp. 1802–1814, 2015.
- [28] W. Sun and Y. Zhang, "Fourier series multiscale analysis of elastic wave propagation," arXiv:2208.13509, 2022.

- [29] B. Alpert, L. Greengard, and T. Hagstrom, "Non-reflecting boundary conditions for the wave equation," J. Comput. Phys., vol. 180, pp. 270–296, 2003.
- [30] L. O'Neill, "Analytic solutions to non-homogeneous wave equations," J. Math. Anal. Appl., vol. 398, pp. 535–546, 2013.

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How to cite this paper:

Egbeja Johnson Sunday, Yusuf Ibrahim, "Fourier Transform Solution for a One-Dimensional Non-Homogeneous Wave Equation with Boundary and Initial Value Conditions", International Journal of Mathematical Sciences and Computing(IJMSC), Vol.12, No.3, pp. 95-123, 2026. DOI: 10.5815/ijmsc.2026.03.06