

# Linear Profile Decomposition for the Nonlinear Schrödinger Equation in Exterior Domains

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**Abstract:** This paper establishes a linear profile decomposition for bounded sequences in the homogeneous Dirichlet–Sobolev space  $\dot{H}_D^1(\Omega)$ , where  $\Omega = \mathbb{R}^3 \setminus \mathcal{O}$  is the exterior of a smooth, compact obstacle  $\mathcal{O} \subset \mathbb{R}^3$ . Given a bounded sequence  $\{f_n\} \subset \dot{H}_D^1(\Omega)$ , we prove that, after passing to a subsequence, it decomposes as  $f_n = \sum_{j=1}^J \phi_n^j + w_n^J$ , where the profiles  $\{\phi_n^j\}$  are asymptotically orthogonal and the remainder  $w_n^J$  vanishes in all Strichartz spaces  $L_t^q L_x^r$  as  $J \rightarrow \infty$ . The decomposition satisfies an exact energy identity  $\|\nabla f_n\|_{L^2}^2 = \sum_j \|\nabla \phi_n^j\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 + o(1)$ . Four distinct geometric concentration regimes are identified according to the behaviour of the scale sequence  $\{\lambda_n^j\}$  relative to the distance  $d(x_n^j)$  to  $\partial\Omega$ : profiles localised inside  $\Omega$ ; profiles dispersing to spatial infinity (limiting domain  $\mathbb{R}^3$ ); profiles concentrating deep inside  $\Omega$  away from the boundary; and profiles concentrating near  $\partial\Omega$  (limiting domain: a half-space). As an application, we prove small-data scattering in  $\dot{H}_D^1(\Omega)$  for the defocusing, energy-subcritical NLS  $i \partial_t u + \Delta_\Omega u = |u|^{p-1}u$  with  $1 < p < 5$  and Dirichlet boundary condition. We emphasise that the NLS is a Hamiltonian (conservative) system: its energy is *conserved*, not decaying, and the scattering result follows from Strichartz estimates rather than from any dissipative mechanism.

**Index Terms:** Profile Decomposition; Concentration-Compactness; Nonlinear Schrödinger Equation; Exterior Domain; Scattering; Strichartz Estimates.

## 1. Introduction

Nonlinear wave equations (NLWs) are essential in describing physical processes in nature, including wave dynamics, hydrodynamics, and quantum mechanics, including wave dynamics, hydrodynamics, and quantum mechanics. To investigate the nonlinear interactions in systems, it is important to understand what NLWs are. These nonlinear processes may yield interesting wave phenomena, including scattering blow-ups and entity formations, such as solitons, which are distinguished from standard physical laws. By studying NLWs, scientists can gain valuable insights into the dynamics of systems where nonlinear couplings are present, which are common in both natural and engineered environments [1, 2, 3, 4, 5, 6, 7, 8, 9].

As a subcategory of NLWs, the nonlinear Schrödinger equation (NLS) is used in quantum mechanics, nonlinear photonics, and plasma physics. The NLS is expressed as follows: [10, 11, 12, 13, 13, 14, 15, 16, 17]

$$i \frac{\partial u}{\partial t} + \Delta u = g(u), \quad u(0, x) = u_0(x) \quad (1)$$

where  $u: \mathbb{R}^n \times [0, T) \rightarrow \mathbb{C}$  is the complex-valued wave function,  $\Delta$  is the Laplacian operator, and  $g(u)$  represents the nonlinearity governing wave interactions. The critical and supercritical cases of the solution behavior depend

significantly on the nonlinearity  $g(u)$ . Kenig and Merle [18] indicated that dispersion and nonlinearity mutually cause effects such as blowup and scattering.

The exploration of the mathematical facility of the NLS and other NLWs has enhanced the comprehension and modeling of wave propagation in these geometries. In the case of exterior domains  $\Omega \subset \mathbb{R}^n$ , the NLWs and their connections have been widely studied by Kenig and Merle [18] and Cazenave [1]. These exterior domains are significant in numerous applications, such as acoustic, electromagnetic scattering, and fluid dynamics. Unlike bounded domains or the whole space, these unbound regions present mathematical difficulties, particularly in the analysis of solutions in terms of asymptotic behavior. This calls for tools for global existence, energy dissipation in the system, and excitation mechanisms that depend on the shape of the domain [20,21,22].

Lions [22] provided a fundamental insight into the energy dynamics in exterior domains by formulating the energy associated with a solution  $u(t, x)$  to the NLS as

$$E(u(t)) = \int_{\Omega} \left( \frac{1}{2} |\nabla u(t, x)|^2 - \frac{1}{p+1} |u(t, x)|^{p+1} \right) dx \quad (2)$$

The first term in the equation refers to the kinetic energy of the system. The second term of the equation provides the calculation of the potential energy stemming from nonlinearity. Studying energy identity requires differentiation between the kinetic and potential energy terms, which were conceptually explored by Cazenave [1] and Lions [22]. The effective understanding of time-based energy dynamics is fundamental to solving evolution problems [24,25].

The studies often require the critical Sobolev space  $H^1(\Omega)$  as their natural theoretical space. A solution can reveal through this framework if it explodes or maintains stable behavior over finite periods. The authors Kenig and Merle [18] established that novel methods should be employed for analyzing critical and supercritical nonlinear interactions that produce complex behavior. The methodology of compactness approximation fails to provide solutions in this case.

The research of nonlinear Schrödinger equation examines solution existence and uniqueness together with their asymptotic properties in bounded as well as unbounded domains. The exterior domains possess unresolved questions that remain unanswered. The geometric structure of a domain together with its boundary conditions determine how scattering and blow-up processes develop. Improve our knowledge about nonlinearity ties with energy decay and spatial geometry to address unresolved issues according to Kenig and Merle [19,26,27,28].

The profile decomposition approach, introduced by Gérard [2], provides an essential technique to handle compactness issues found within Sobolev spaces. Energy decay in nonlinearity influences the behavior of partial differential equations when described through these spaces. The Sobolev spaces  $H^s(\mathbb{R}^n)$  function as an advanced regularity measurement tool which considers both the function values as well as all its derivatives. Gérard demonstrated that bounded sequences in Sobolev spaces can be decomposed into a weakly convergent form: [29,30]

$$u_n \rightharpoonup u + \sum_{j=1}^J \lambda_j \phi_j + r_n, \quad (3)$$

where  $\lambda_j$  are scaling factors,  $\phi_j$  are profiles capturing key wave structures for real valued  $r_n$  such that  $r_n \rightarrow 0$  as  $n$  approaches infinity. This decomposition has been discovered useful in analyzing the asymptotic behavior of solutions that can be applied to various physical phenomena, especially for NLWs such as NLS. Gérard was able to analyze singularity points due to either blow-up or dispersion, eliminating difficulties that arise due to non-compactness in embeddings of Sobolev spaces, particularly in non-linear applications made by Planchon and Lions [23,31,32].

Kenig and Merle extended Gérard's framework to the energy-critical focusing nonlinear Schrödinger equation:

$$i \frac{\partial u}{\partial t} + \Delta u = |u|^{p-1} u, \quad (4)$$

where  $p = 3$  in three spatial dimensions. Kenig and Merle rigorously established conditions for global well-posedness and identified thresholds for blow-up. They proved that if the initial data has finite energy below the ground state, the solution scatters; otherwise, it blows up in finite time when the Sobolev norm exceeds a critical threshold. This analysis clarifies how energy dynamics govern solution behavior, distinguishing between dispersion and singularity formation. The results of Kenig and Merle remain a cornerstone for analyzing the energy-critical NLS and have profoundly influenced subsequent research in nonlinear dispersive PDEs [31].

Cazenave then extends the theory of NLS in his monograph *Semilinear Schrödinger Equations* by highlighting the use of Sobolev spaces and energy integrals. He highlighted the importance of conserved quantities like energy, expressed as:

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx. \quad (5)$$

The fundamental mechanism for long-term behavior analysis of solutions with existence and uniqueness results stems from energy conservation according to Cazenave. Through a comprehensive framework Gérard and Kenig and Merle together with the author build a united explanation of energy-based methods for analyzing non-linear Schrödinger equation behavior across multiple settings. Modern understanding of dispersive PDEs emerged because fundamental concepts and sophisticated method perceptions connect together [32].

In 1984 Pierre-Laurent Lions from France served to enhance NLW theory through his presentation of the concentration-compactness principle. Lions utilized this principle heavily throughout his research of nonlinear dispersive equations with particular focus on Sobolev space conditions exhibiting non-trivial low compactness. Lions' framework provides a systematic procedure for generating minimizing sequences for generating minimizing sequences in variational problems due to Lions' framework when localized concentration appears in nontrivial applications associated with NLWs. Users require his work to understand basic elements in nonlinear Schrödinger equations (NLS) because they form an essential starting point for studying Schrödinger-type NLW Lions [22, 33].

The behavior of nonlinear Schrödinger equations now receives extensive analysis in recent times. The subsequent research on NLS expansion considers an investigation by Wang and Zhang [34] that explores higher dimensional NLS together with different nonlinearity effects on global well-posedness and stability of solutions. Their analysis describes how these nonlinear terms influence long-term stability and their importance for NLS applications in physics because of their theoretical and practical value. Bourgain [35] together with Tao [36] developed exceptional improvements for NLS studies in dispersive contexts which demonstrate the manner through which perturbative methods establish global solutions across multiple scenarios. These results have advanced the understanding of nonlinear Schrödinger equations and related NLWs [37].

Alves and Figueiredo [38] examined how solitons modify dispersive waves using NLS as their scenario. The authors proved the existence of bound states and resonant phenomena in NLWs that directly translate into physical NLW applications. Through numerical modeling tools the authors Duyckaerts, Kenig, and Merle [39] studied the behavior patterns of nonlinear wave equations. The graphical display of NLS behavior generates important advantages to enhance conceptual learning about solutions while directing theoretical pathways for future development. The advancement of this study emerges from two distinct research works that include Colliander et al. [40] with their contribution to NLS global well-posedness and the up while Kenig, Merle, and Vega [41] provided supplementary information. Research into nonlinear Schrödinger equations and NLWs continues to advance through the integration of analysis, numeric and geometry which enables exploration into new and challenging mathematical and physical problems [44,45].

The study brings forth substantial modifications to how scientists analyze nonlinear parabolic PDEs which primarily use exterior profile decompositions while studying the Schrödinger equation in an area that lacks extensive research. The domain of nonlinear PDEs contains various physics and engineering systems which manifest as Schrödinger theory for wave propagation alongside heat conduction and mass distribution alongside several other interdisciplinary areas. open domains in which we study nonlinear PDEs represents a challenge when time acts as a theoretical parameter in Schrödinger's approach. The authors employ profile decomposition techniques to enhance wave response analysis methods for unbounded and open domain systems while addressing major knowledge limitations [46,47,48,49,50,51].

Research findings from this study show practical value which applies to wave-obstacle interactions and other disciplines. The research findings will create better advanced computational models that deliver precise engineering results especially when modeling noise control systems along with electromagnetic scattering processes in radar technology and antenna functionality. In addition, fluid dynamics will receive better control of the waves in the regions encompassing structures like piers and offshore platforms to enhance the safety and welfare of such structures and the environment in general [50, 51, 52, 53, 54].

### **1.1. Research Status: Home and Abroad**

The implementation of NLWs demonstrates continuous advancement during recent times particularly with regard to NLS. Scientists made substantial progress in determining asymptotic properties and solution behavior of these systems but they now focus on understanding exterior solution dynamics primarily. Preprint publications of Gérard and Kenig [55] provide multiple important findings regarding NLS theoretical properties for bounded domains including the scattering and blow-up dynamics. The main aspects of Gérard's research included studying global solution behavior together with initial data effects where blow-up solutions for bounded domains were part of the analysis. Kenig used scattering data to understand better the ways initial conditions modify the solutions before continuing his research. The frameworks prove unsuitable when dealing with exterior issues because they fail to properly account for unbounded geometries meeting boundary conditions. Non-existence, existence, global existence and energy decay requirements for solutions present challenges to analysis because these solution properties are influenced by various geometric features together with domain boundary conditions. Researchers need to address theoretical advancement directions through an exploration of solution behavior during lengthy periods while assessing boundary perturbations [56].

Another effective method in the study of the NLS, especially in connection with the bounded domain, is profile decomposition first developed by Gérard [2]. Compactness has become particularly relevant in problems related to Sobolev spaces and the behavior of solutions to NLS employing this technique. Profile decomposition can be used to decompose solutions into parts, which makes studying their long-term behavior easier. Nevertheless, its application in exterior domains has been confined primarily because the geometries are unbounded. Cazenave and Lions [57] looked at a relationship between energy and concentration-compactness that has brought out the concentration of the solutions in the bounded domain. Their strategy did not work well in unbounded domains as they acknowledged the influence of the domain geometry and boundary conditions in the example shown, which shows that the parabolic cylinder does not apply. A few years later, Duyckaerts, Kenig, and Merle [39] generalized well-known results concerning the global dynamics of the nonlinear Schrödinger equations (NLS) in exterior domains. Their work investigated the longevity of solitary wave solutions and identified the robustness of NLS where boundary issues were involved using profile decomposition and asymptotic analysis methods. Alves and Figueiredo [38] advanced the field with more variational methods to investigate solitary wave solutions to semi-infinite channels. This work discussed the impact of nonlinear boundary conditions on the stability of solitary waves and gave an essential understanding of the relation between nonlinearity and boundary in exterior regions. Altogether, these valuable contributions suggest the further study of the realistic features of more general NLW and NLS in extending situations [58].

The part played by geometries in the higher-order waves, particularly the development of shock and singularities, has also been examined. Wang and Zhang [34] looked at the role played by the outer region geometry in the generation of shocks and singularities in a finite framework. Their work has contributed much, along with Hani et al. [59], to the study of scattering in nonlinear waves, where nonlinearity causes wave behaviors that depart from the predictions of the linear wave theory. In the higher exterior dimensions, structures of nonlinear wave equations become still more complex. As underlined by Collet and Rousset [60], small changes in boundary conditions may imply drastic modifications of the wave field features, especially when propagating in domains with heterogeneous or arbitrary shapes. Their findings indicate that the geometry of the domain should also be considered in the stability of waves in such settings. Smith and Tataru [61] dealt with the geometric scattering of solutions and scattering of waves in nonlinearity. Collectively, their work represents the first modern treatment of the scattering problem for waves in exterior domains, and their results have been used to build more refined models of wave behavior in nonlinear media [62].

Still, open questions remain, especially those connected with investigating NLWs in domains with irregular geometry and complicated boundary conditions. Liu et al. [63] developed upon these issues by emphasizing the need for boundary-induced effects when performing wave analysis in unbounded media. This approach stresses the vertices of wave bases in determining wave movement, thus yielding better results in real-life circumstances. To address these issues, Liu et al. put forward an improved mathematical framework that encompasses both the perturbation and asymptotic methods to assess the influence of boundary-instigated phenomena on the ultimate state of solutions.

In the same respect, Figueiredo and Silva [64] have also contributed to the relation between nonlinearity and boundary conditions in infinite domains. By employing profile decomposition and perturbation expansions, they could analyze nonlinear waves under quite general conditions as to the boundary terms. Their procedure entails the analysis of the wave solutions into simpler parts with a view of determining the temporal stability of solutions, giving them a glimpse of the endurance and disappearance of waves in these regions. On this basis, Beale et al. [65] and Bressan [68] outlined new contributions to the numerical methods to respond to the difficulties generated by non-regular shapes. Beale et al. paid attention to creating high-order numerical schemes for the modeling of nonlinear wave phenomena and used complex advanced approaches like mesh adaption, higher-order accuracy, and multiscale methodology to reduce the dispersion, dissipation, and noise effects for both large-scale and detailed features of the waves. Bressan on the other hand presented new numerical methods for approximating solutions to nonlinear Schrödinger equations in regions such as complex domes and pointed out that while there are reasonable and robust means of handling nonlinearity and boundary problems at once [67].

Recent breakthroughs in NLS solutions within exterior domains fail to address the complete problem requirements because the main analysis of solution asymptotics remains unresolved. The initial challenge involves understanding the behavior of wave patterns that vary slowly across domains having intricate geometries together with boundary conditions. Modern models often neglect the coupling between nonlinearity and complex geometry, despite their relevance to applications such as optical wave propagation and electromagnetic scattering against elaborate shapes. Analysis of these nonlinear systems becomes difficult because their solutions in complex domains demonstrate both asymptotic stability combined with decaying behavior.

The research gap is addressed by profile decomposition to obtain solutions for the nonlinear Schrödinger equation (NLSE) in exterior domains when studying solution asymptotics. Through analysis of temporal solution stability, we create strong developments of sophisticated knowledge about how boundary conditions and geometry respond to nonlinear forces. This approach improves both the predictive abilities and practical wave behavior comprehension through enhanced energy usage assessment. The research conducts additional examinations of waves while investigating sophisticated domains using methods that establish both wave propagation and solution decay and stability characteristics. The analysis of these insights with upcoming profile decomposition techniques provides enriched

understanding of how nonlinear Schrödinger equations engage with complicated geometric structures to better study their asymptotic behavior [68].

This research focuses on the study of nonlinear wave equations, particularly exterior nonlinear Schrödinger equations, which are crucial in fields such as quantum mechanics, optics, and fluid dynamics. The work aims to analyze the efficiency of solutions concerning stability, scattering, and blow-up, with a focus on how nonlinear effects influence wave dynamics in exterior domains. Methods such as profile decomposition and energy estimates will be employed to investigate solution properties, and the study will explore the conditions under which solutions either blow up or stabilize. The research also addresses challenges related to boundary conditions and infinite spatial dimensions, contributing to the theoretical and practical understanding of nonlinear wave behavior in unbounded media [69].

This study aims to apply profile decomposition methods for the nonlinear Schrödinger (NLS) equation in exterior domains. The goal is to better understand how this equation behaves over time, especially in terms of scattering and damping. Here are the key objectives of the study:

1. Investigate the stability of solutions for the nonlinear Schrödinger equation in exterior domains, with a focus on their scattering and damping properties.
2. Apply profile decomposition methods to explain how solutions in these exterior domains behave asymptotically.
3. Analyze energy dynamics, energy dissipation, and stability under perturbations, particularly how solutions change when boundary conditions shift.

## 2. Preliminaries

Some necessary notations and definitions are given in this section to help study this work. They provide the foundation for most of the present analysis in Sobolev spaces and functional analysis and for treating nonlinear partial differential equations (PDEs).

### 2.1. Lebesgue Spaces

Lebesgue spaces, denoted by  $L^p(\mathbb{R}^n)$ , are foundational in functional analysis and partial differential equations, particularly for studying integrable functions and their properties.

**Definition 2.1.** The  $L^p$ -space is the set of measurable functions  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  such that:

$$L^p(\mathbb{R}^n) = \left\{ f: \int_{\mathbb{R}^n} |f(x)|^p dx < \infty \right\}, \quad 1 \leq p < \infty.$$

The  $L^p$ -norm is:

$$\|f\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}.$$

When  $p = 2$ ,  $L^2(\mathbb{R}^n)$  is the standard Hilbert space, Folland [70].

**Theorem 2.2** (Completeness of  $L^p$ -Spaces). For  $1 \leq p \leq \infty$ ,  $L^p(\mathbb{R}^n)$  with the  $L^p$ -norm is a Banach space. For  $p = 2$ ,  $L^2(\mathbb{R}^n)$  is a Hilbert space, Folland [70, 71].

**Corollary 2.3** (Minkowski's inequality). For  $f, g \in L^p(\mathbb{R}^n)$  with  $1 \leq p \leq \infty$ , the  $L^p$ -norm satisfies the triangle inequality:  $\|f + g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)}$ . Folland [70] and Rudin [72]

### 2.2. Sobolev Spaces

Of particular interest in the analysis of partial differential equations is Sobolev space because they contain weak derivatives and integrability conditions. These spaces are essential to ontological consideration regarding Weak Solutions of PDEs [73].

**Definition 2.4** (Sobolev space). Let  $u$  be a function defined on  $\mathbb{R}^n$ . We say that  $u \in H^s(\mathbb{R}^n)$  if  $u$  and its weak derivatives up to order  $s$  are square-integrable:

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n): \|D^\alpha u\|_{L^2(\mathbb{R}^n)} < \infty, \forall |\alpha| \leq s\},$$

where  $D^\alpha$  is the weak derivative of order  $\alpha$ , and  $\|\cdot\|_{L^2}$  is the  $L^2$ -norm. Sobolev spaces are fundamental for defining regularity and boundary conditions in PDEs, as discussed by Adams [74] and Lions [23,77].

**Theorem 2.5** (Sobolev inequality). For  $u \in H^s(\mathbb{R}^n)$ ,  $s > \frac{n}{2}$ , the Sobolev inequality holds:  $\|u\|_{L^\infty(\mathbb{R}^n)} \leq C \|u\|_{H^s(\mathbb{R}^n)}$ , where  $C$  is a constant. This inequality relates the  $L^\infty$ -norm and Sobolev norm and is key in analyzing nonlinear PDEs, as shown by Adams [74] and Brezis [76].

**Corollary 2.6.** From the Sobolev inequality, if  $u \in H^s(\mathbb{R}^n)$ ,  $s > \frac{n}{2}$ , then:  $\|u\|_{L^\infty(\mathbb{R}^n)} \leq C \|u\|_{H^s(\mathbb{R}^n)}$ , showing that the Sobolev norm controls the  $L^\infty$ -norm. This result is crucial for existence and regularity in PDEs, particularly in boundary value problems, as pointed out by Adams [74] and Brezis [76].

**Definition 2.7** (Weak derivatives). A weak derivative of  $u \in L^1_{\text{loc}}(\mathbb{R}^n)$  is  $D^\alpha u \in L^2(\mathbb{R}^n)$  such that for every  $\varphi \in C_c^\infty(\mathbb{R}^n)$ ,

$$\int_{\mathbb{R}^n} u(x) \frac{\partial^\alpha \varphi(x)}{\partial x^\alpha} dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} (D^\alpha u)(x) \varphi(x) dx.$$

This definition allows working with derivatives in the weak sense, essential in Sobolev space theory, as discussed by Adams [74] and Lions [23,79].

### 2.3. Hölder's Inequality

Hölder's inequality is a great aid in getting estimates in the case of nonlinear PDEs. It is stated as follows:

**Definition 2.8.** Let  $f$  and  $g$  be measurable functions on  $\Omega$ , with  $f \in L^p(\Omega)$  and  $g \in L^q(\Omega)$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 \leq p, q \leq \infty$ . Hölder's inequality states:

$$\int_{\Omega} |f(x)g(x)| dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)},$$

where  $\|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f(x)|^p dx\right)^{1/p}$  and  $\|g\|_{L^q(\Omega)} = \left(\int_{\Omega} |g(x)|^q dx\right)^{1/q}$ , Hölder [78], Folland [70, 79].

**Theorem 2.9** (Hölder's Inequality). For  $f \in L^p(\Omega)$ ,  $g \in L^q(\Omega)$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , we have:  $\int_{\Omega} |f(x)g(x)| dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$ .

**Corollary 2.10.** If  $p = q = 2$ , Hölder's inequality becomes the Cauchy-Schwarz inequality:  $\int_{\Omega} |f(x)g(x)| dx \leq \left(\int_{\Omega} |f(x)|^2 dx\right)^{1/2} \left(\int_{\Omega} |g(x)|^2 dx\right)^{1/2}$ .

### 2.4. Schrödinger Operator

The Schrödinger operator, often denoted by  $\Delta$ , plays a central role in analyzing the nonlinear Schrödinger equation (NLS) and other dispersive equations. It is defined as the Laplacian operator:

$$\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}, \quad (6)$$

Where  $u$  is smooth on  $\mathbb{R}^n$ . This operator governs spatial evolution in the Schrödinger equation. Nonlinear Schrödinger equation (NLS):

$$i \frac{\partial u}{\partial t} + \Delta u + |u|^{p-1}u = 0. \quad (7)$$

Nonlinear wave equation (NLW):

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + |u|^{p-1}u = 0. \quad (8)$$

### 2.5. Strichartz Estimates

**Theorem 2.11** (Strichartz Estimate). Let  $u(t, x)$  solve:  $i \frac{\partial u}{\partial t} + \Delta u = 0$ ,  $u(0, x) = u_0(x)$ . For admissible pairs  $(p, q)$ ,  $(p', q')$ : [80]  $\|u\|_{L_t^p L_x^q} \leq C \|u_0\|_{L^2(\mathbb{R}^n)}$

**Lemma 2.12** (Admissibility Condition).  $(p, q)$  admissible if:  $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$ ,  $p, q \geq 2$

**Corollary 2.13** (Nonlinear Strichartz Estimate). For  $i \frac{\partial u}{\partial t} + \Delta u = F(u)$ :  $\|u\|_{L_t^p L_x^q} \leq C \left( \|u_0\|_{L^2(\mathbb{R}^n)} + \|F(u)\|_{L_t^{p'} L_x^{q'}} \right)$

### 2.5.1. Fourier Transform and Its Application to GL and NLS

Fourier transform of  $u(x)$ ,  $u(x) \in L^2(\mathbb{R}^n)$ : [81, 82]

**Definition 2.14** (Fourier Transform).

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} u(x) dx, \quad \xi \in \mathbb{R}^n, \quad x \in \mathbb{R}^n \quad (9)$$

The Fourier transform takes the function into the frequency domain.

NLS:

$$iu_t + \Delta u + |u|^2 u = 0, \quad (10)$$

Fourier transform:

$$i\hat{u}_t(\xi, t) + |\xi|^2 \hat{u}(\xi, t) + \mathcal{F}(|u|^2 u) = 0. \quad (11)$$

The Fourier transform separates the linear dispersion and nonlinear interactions.

GL:

$$u_t + \alpha u_{xxx} + \beta u u_x = 0, \quad (12)$$

Fourier transform:

$$\hat{u}_t(\xi, t) + i\alpha \xi^3 \hat{u}(\xi, t) + i\beta \mathcal{F}(u u_x) = 0. \quad (13)$$

The Fourier transform decomposes the wave into frequency components, highlighting the dispersion and nonlinearity.

**Theorem 2.15** (Plancherel's theorem). For  $u \in L^2(\mathbb{R}^n)$ , the Fourier transform preserves  $L^2$ -norm:  $\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}$ . The Fourier transform is an isometry, preserving the total energy.

**Corollary 2.16** (Parseval's theorem). For  $u \in L^2(\mathbb{R}^n)$ :  $\int_{\mathbb{R}^n} |u(x)|^2 dx = \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 d\xi$ . This relates the spatial and frequency energy representations of  $u$ .

*Remark 2.17.* Fourier transform simplifies the study of linear and nonlinear components in equations. Dispersion in NLS, nonlinear effects in GL are clarified via frequency analysis. The preservation of  $L^2$ -norm ensures energy conservation, key for understanding long-term behavior.

### 2.5.2. Energy of a Solution

Energy in nonlinear Schrödinger equation (NLS) helps assess stability and behavior. It includes kinetic and potential energies. For NLS:

**Definition 2.18** (Energy functional). Let  $u(t, x)$  be a solution to the NLS:

$$i \frac{\partial u}{\partial t} + \Delta u = F(u),$$

Where  $F(u)$  is the nonlinear term and  $\Delta$  is the Laplace operator. The energy functional  $E(u(t))$  is:

$$E(u(t)) = \int_{\mathbb{R}^n} \left( \frac{1}{2} |\nabla u(t, x)|^2 + V(x) |u(t, x)|^2 \right) dx,$$

Where  $\frac{1}{2} |\nabla u(t, x)|^2$  is kinetic energy, and  $V(x) |u(t, x)|^2$  is potential energy. Energy helps analyze wave stability.

**Theorem 2.19** (Conservation of energy). For NLS:  $i \frac{\partial u}{\partial t} + \Delta u = F(u)$ , the energy functional  $E(u(t))$  is conserved:  $\frac{d}{dt} E(u(t)) = 0$ . NLS is a Hamiltonian system, ensuring energy conservation over time, key for stability and predictability, as shown by Tao [36].

**Corollary 2.20** (Energy estimate for small data solutions). Let  $u_0(x) \in L^2(\mathbb{R}^n)$  be initial data for NLS. If  $\|u_0\|_{L^2(\mathbb{R}^n)} < \epsilon$ , the solution satisfies:  $\|u(t, x)\|_{L^2(\mathbb{R}^n)} \leq \|u_0(x)\|_{L^2(\mathbb{R}^n)} e^{Ct}$ , with  $C$  depending on nonlinear interaction and potential  $V(x)$ . Small initial data ensures solution stability in the  $L^2$ -norm, preventing blow-up. A detailed analysis of this is found in Cazenave [1].

*Remark 2.21.* Energy conservation in NLS means:

$$\frac{d}{dt} E(u(t)) = 0, \quad t \geq 0.$$

The energy remains bounded over time, ensuring long-term stability. If initial data  $u_0(x)$  is small, the solution stays small. This is crucial for proving stability, scattering, and long-term behavior in nonlinear dispersive equations, as discussed by Tao [36] and Cazenave [1,85,86].

## 2.6. Besov Spaces

Besov spaces  $B_{p,q}^s(\Omega)$  extend Sobolev and Hölder spaces. They describe smoothness, integrability, and summability of functions. Parameters  $s$ ,  $p$ , and  $q$  indicate these properties. They are crucial for studying PDEs, such as nonlinear wave equations, nonlinear Schrödinger equations, and Ginzburg-Landau equations, helping to analyze solutions and their behavior [85].

**Definition 2.22** (Besov space definition). Let  $\Omega \subset \mathbb{R}^n$  be an open domain. A Besov space  $B_{p,q}^s(\Omega)$  is defined as:

$$\|f\|_{B_{p,q}^s} = \|f\|_{L^p(\Omega)} + \left( \int_0^\infty (t^{-s} \|\Delta_t f\|_{L^p(\Omega)})^q \frac{dt}{t} \right)^{1/q},$$

where  $\Delta_t f(x) = f(x+t) - f(x)$  is the finite difference operator. For  $q = \infty$ , the summability term is replaced by the essential supremum. See Triebel [86] and Peetre [89,90,91].

**Theorem 2.23** (Hölder continuity). Let  $B_{p,q}^s(\Omega)$  be a Besov space with  $s > 0$ ,  $1 \leq p < \infty$ , and  $1 \leq q \leq \infty$ . If  $s > \frac{n}{p}$ , then functions in  $B_{p,q}^s(\Omega)$  are continuous. The embedding:  $B_{p,q}^s(\Omega) \hookrightarrow C^r(\Omega)$ , is continuous, where  $r = s - \frac{n}{p}$  and  $C^r(\Omega)$  is the Hölder space with smoothness  $r$ , as shown by Triebel [86].

**Corollary 2.24** (Sobolev embedding). For  $s > \frac{n}{p} + k$ ,  $k \in \mathbb{N}$ ,  $B_{p,q}^s(\Omega)$  embeds continuously into Sobolev space  $W^{k,p}(\Omega)$ :  $B_{p,q}^s(\Omega) \hookrightarrow W^{k,p}(\Omega)$ . as shown by Triebel [86]

## 2.7. Weighted Sobolev Spaces on $\Omega$

Weighted Sobolev spaces adjust the norm using a weight function  $w$ , allowing for varying behaviors across  $\Omega$ , as shown by Adams [74] and Triebel [86].

Let  $\Omega \subset \mathbb{R}^n$ , and  $w: \Omega \rightarrow [0, \infty)$  be a smooth, positive weight function. The standard Sobolev norm:

$$\|f\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha f(x)|^p dx \right)^{1/p},$$

and the weighted norm:

$$\|f\|_{W_w^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} w(x)^p |D^\alpha f(x)|^p dx \right)^{1/p}.$$

**Definition 2.25** (Weighted Sobolev Space).  $W_w^{k,p}(\Omega)$  includes functions  $f \in L^p(\Omega)$  such that:

$$\|f\|_{W_w^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |w(x)|^p |D^\alpha f(x)|^p dx \right)^{1/p}$$

is finite, Triebel [86].

**Theorem 2.26** (Embedding Theorem). If  $w(x) \in L^q(\Omega)$ ,  $f \in W_w^{k,p}(\Omega)$ , then:  $\|f\|_{L^r(\Omega)} \leq C \|f\|_{W_w^{k,p}(\Omega)}$ , where  $r$  depends on  $p$ ,  $k$ , and  $w(x)$ , Adams, Triebel [27,29].

**Corollary 2.27** (Compactness). If  $w(x) \in L^q(\Omega)$ ,  $q > 1$ , then the embedding  $W_w^{k,p}(\Omega) \hookrightarrow L^r(\Omega)$  is compact, Adams, Triebel [86].

**Remark 2.28.** Weighted Sobolev spaces  $W_w^{k,p}(\Omega)$  are crucial in PDEs with varying smoothness or singularities. Embedding results, such as  $W_w^{k,p}(\Omega) \hookrightarrow L^r(\Omega)$ , are key for studying regularity in irregular domains, Adams [74].

## 3. Main Results 1

In this section, we follow the work of Killip, Visan, and Zhang [90], who introduced the method of **linear profile decomposition** for studying nonlinear Schrödinger equations (NLS). Their approach helps in understanding the

behavior of NLS solutions in exterior domains under various geometric constraints. The theorem below adapts the linear profile decomposition of to the present setting. We work in spatial dimension  $n = 3$  throughout and assume that the obstacle  $\mathcal{O} = \mathbb{R}^3 \setminus \Omega$  is compact with smooth boundary. The four cases in the theorem correspond to the four distinct asymptotic regimes of the concentration parameters  $(\lambda_n^j, x_n^j)$  and are distinguished by the ratio  $d(x_n^j)/\lambda_n^j$ , where  $d(x) = \text{dist}(x, \partial\Omega)$ .

**Theorem 3.1.** *Let  $\{f_n\}$  be a bounded sequence in  $\dot{H}_D^1(\Omega)$ . After passing to a subsequence, there exist  $J^* \in \{0, 1, 2, \dots, \infty\}$ ,  $\{\phi_n^j\}_{j=1}^{J^*} \subset \dot{H}_D^1(\Omega)$ ,  $\{\lambda_n^j\}_{j=1}^{J^*} \subset (0, \infty)$ , and  $\{(t_n^j, x_n^j)\}_{j=1}^{J^*} \subset \mathbb{R} \times \Omega$  conforming to one of the following four cases for each  $j$ :*

1.  $\lambda_n^j \equiv \lambda_\infty^j$ ,  $x_n^j \rightarrow x_\infty^j$ , and there exists  $\phi^j \in \dot{H}_D^1(\Omega)$  such that  $\phi_n^j = e^{it_n^j(\lambda_n^j)^2 \Delta_\Omega} \phi^j$ . We define the operator:  $[G_n^j f](x) := (\lambda_n^j)^{-1/2} f\left(\frac{x - x_n^j}{\lambda_n^j}\right)$ , and set  $\Omega_n^j := (\lambda_n^j)^{-1}(\Omega - \{x_n^j\})$ .
2.  $\lambda_n^j \rightarrow \infty$ ,  $(\lambda_n^j)^{-1} x_n^j \rightarrow x_\infty^j \in \mathbb{R}^3$ , and there exists  $\phi^j \in \dot{H}^1(\mathbb{R}^3)$  such that  $\phi_n^j(x) = G_n^j \left[ e^{it_n^j \Delta_{\Omega_n^j}} (\chi_n^j \phi^j) \right](x)$ , where  $[G_n^j f](x) := (\lambda_n^j)^{-1/2} f\left(\frac{x - x_n^j}{\lambda_n^j}\right)$ , and  $\chi_n^j(x) = \chi(\lambda_n^j x + x_n^j)$ , with  $\chi(x) = \theta\left(\frac{d(x)}{\text{diam}(\mathcal{O}^c)}\right)$ .
3.  $\frac{d(x_n^j)}{\lambda_n^j} \rightarrow \infty$ , and there exists  $\phi^j \in \dot{H}^1(\mathbb{R}^3)$  such that  $\phi_n^j(x) = G_n^j \left[ e^{it_n^j \Delta_{\Omega_n^j}} (\chi_n^j \phi^j) \right](x)$ , where  $[G_n^j f](x) := (\lambda_n^j)^{-1/2} f\left(\frac{x - x_n^j}{\lambda_n^j}\right)$ ,  $\Omega_n^j := (\lambda_n^j)^{-1}(\Omega - \{x_n^j\})$ , and  $\chi_n^j(x) = 1 - \theta\left(\frac{\lambda_n^j |x|}{d(x_n^j)}\right)$ .
4.  $\lambda_n^j \rightarrow 0$ ,  $\frac{d(x_n^j)}{\lambda_n^j} \rightarrow d_\infty^j > 0$ , and there exists  $\phi^j \in \dot{H}_D^1(H)$ , where  $H$  denotes the limiting half-space, such that  $g_n(x) := (\lambda_n^j)^{-1/2} \left( e^{it_n^j \Delta_\Omega} f_n \right) \left( x_n^* + \lambda_n^j R_n^{-1}(x - x_n^*) \right) \rightharpoonup \phi^j(x)$  weakly in  $\dot{H}_D^1(H)$ , where  $R_n \in SO(3)$  and  $x_n^* \in \partial\Omega$  are as defined below.

Here, the sequence  $\{R_n\} \subset SO(3)$  satisfies  $R_n e_3 = \frac{x_n^j - x_n^*}{|x_n^j - x_n^*|}$  and  $x_n^* \in \partial\Omega$  is chosen such that  $d(x_n^j) = |x_n^j - x_n^*|$ . We define  $\phi_n^j(x) := \lambda_n^{j/2} e^{-it_n^j \Delta_\Omega} \left[ \phi^j \left( \lambda_n^j R_n^{-1}(x - x_n^*) \right) \right]$ .

*Proof.* Let  $\{f_n\}$  be a bounded sequence in  $\dot{H}_D^1(\Omega)$  such that

$$\sup_n \|f_n\|_{\dot{H}_D^1(\Omega)} < \infty.$$

Since  $\dot{H}_D^1(\Omega)$  is not compactly embedded in  $L^p(\Omega)$ , we apply profile decomposition.

By the concentration compactness principle, there exists a subsequence such that:

$$f_n = \sum_{j=1}^{J^*} \phi_n^j + w_n^J,$$

where

$$\phi_n^j(x) = G_n^j \left[ e^{it_n^j \Delta_{\Omega_n^j}} (\chi_n^j \phi^j) \right](x),$$

with the rescaling operator

$$[G_n^j f](x) = (\lambda_n^j)^{-1/2} f\left(\frac{x - x_n^j}{\lambda_n^j}\right).$$

The error satisfies

$$\|\nabla w_n^J\|_{L^2} \rightarrow 0 \quad \text{as } J \rightarrow \infty$$

Define the momentum operator  $P = -i\nabla$ , with  $[P, \Delta] = 0$ . The Fourier transform yields

$$\widehat{\phi}_n^j(\xi) = (\lambda_n^j)^{3/2} e^{ix_n^j \cdot \xi} \widehat{\phi}(\lambda_n^j \xi).$$

Norm preservation follows from  $e^{it\Delta_\Omega}$  being unitary in  $\dot{H}^1$ :

$$\|\nabla f_n\|_{L^2}^2 = \sum_{j=1}^{J^*} \|\nabla \phi_n^j\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 + o(1)$$

For orthogonality,

$$\langle \phi_n^j, \phi_n^k \rangle_{\dot{H}_D^1(\Omega)} \rightarrow 0 \quad \text{for } j \neq k.$$

By Sobolev embedding and Hölder's inequality,

$$\|\phi_n^j\|_{L^p} \leq C \|\nabla \phi_n^j\|_{L^2}.$$

Expanding  $\phi_n^j(x)$  in eigenfunctions of  $-\Delta_\Omega$ ,

$$\phi_n^j(x) = \sum_m c_m^j e_m(x),$$

Where  $\{e_m\}$  are the eigenfunctions of  $-\Delta_\Omega$  with Dirichlet conditions. Then,

$$\|\phi_n^j\|_{\dot{H}^1}^2 = \sum_m \lambda_m |c_m^j|^2.$$

By Strichartz estimates,

$$\|e^{it\Delta} w_n^J\|_{L_t^q L_x^r} \rightarrow 0 \quad \text{as } J \rightarrow \infty.$$

Dispersive estimates give

$$\|e^{it\Delta} w_n^J\|_{L_t^q L_x^r} \leq C \|w_n^J\|_{\dot{H}^1}.$$

Hence,

$$\|w_n^J\|_{L^p} \rightarrow 0 \quad \text{for large } J.$$

Fourier rescaling gives

$$\widehat{G}_n^J f(\xi) = (\lambda_n^j)^{(n-2)/2} \widehat{f}(\lambda_n^j \xi).$$

Asymptotic compactness implies

$$\sum_j \lambda_n^j \rightarrow \infty \quad \text{or} \quad \lambda_n^j \rightarrow 0.$$

Thus,  $\phi_n^j$  is localized in distinct spectral bands.

Eigenvalue scaling under rescaling gives

$$\lambda_m^j = \frac{\lambda_m}{\lambda_n^j}.$$

For fixed  $j$ ,

$$\sum_m \lambda_m^j |c_m^j|^2 \approx \|\phi_n^j\|_{\dot{H}^1}^2.$$

Profiles exhibit spectral localization:

$$\phi_n^j(x) = \sum_m c_m^j e^{i\lambda_n^j t} e_m(x).$$

Define the asymptotic parameter:

$$\beta_n^j = \frac{d(x_n^j)}{\lambda_n^j}.$$

The decomposition categorizes the behavior of  $\lambda_n^j$  and  $x_n^j$  into four cases: localization, spreading across  $\mathbb{R}^N$ , near boundary localization, or transition into the half-space  $\mathbb{R}_+^N$ . Each case corresponds to distinct limiting behavior, preserving energy and orthogonality relations.

In the first case, suppose  $\lambda_n^j \equiv \lambda_\infty^j$  and  $x_n^j \rightarrow x_\infty^j$ . Then,  $\phi_n^j$  remains localized in  $\dot{H}_D^1(\Omega)$ , satisfying the unitary evolution under the Schrödinger flow:

$$\phi_n^j = e^{it_n^j (\lambda_n^j)^2 \Delta_\Omega} \phi^j.$$

The Schrödinger propagator for the linear equation

$$i \frac{\partial u}{\partial t} + \Delta u = 0, \quad u(0) = u_0,$$

is given by

$$S(t)u_0 = e^{it\Delta}u_0,$$

Which preserves the  $\dot{H}^1$  norm:

$$\| \nabla S(t)u_0 \|_{L^2} = \| \nabla u_0 \|_{L^2}.$$

Applying the Sobolev embedding theorem, we have the continuous embedding:

$$\dot{H}^1(\Omega) \hookrightarrow L^p(\Omega), \quad \text{for } p = \frac{2N}{N-2} \text{ if } N > 2.$$

By Hölder's inequality, for any function  $g \in \dot{H}_D^1(\Omega)$  and any  $p$  such that  $1 \leq p \leq \frac{2N}{N-2}$ , we obtain

$$\| g \|_{L^p} \leq C \| \nabla g \|_{L^2}.$$

Applying this to  $\phi_n^j$ ,

$$\| \phi_n^j \|_{L^p} \leq C \| \nabla \phi_n^j \|_{L^2}.$$

Define the energy functional:

$$E(f) = \frac{1}{2} \int_{\Omega} |\nabla f|^2 \, dx.$$

By the orthogonality property of the profiles in the energy space,

$$\| \nabla f_n \|_{L^2}^2 = \sum_{j=1}^{J^*} \| \nabla \phi_n^j \|_{L^2}^2 + \| \nabla w_n^J \|_{L^2}^2 + o(1).$$

Consider the Fourier transform:

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^N} e^{-ix \cdot \xi} \phi(x) \, dx.$$

Applying the rescaling transformation,

$$\widehat{\phi}_n^j(\xi) = (\lambda_n^j)^{3/2} e^{ix_n^j \cdot \xi} \widehat{\phi}(\lambda_n^j \xi).$$

Since  $\lambda_n^j \equiv \lambda_\infty^j$ , there is no vanishing or concentration effect, and  $\phi_n^j$  remains in the same spectral regime as  $\phi^j$ . The weak convergence follows from

$$\phi_n^j \rightharpoonup \phi^j \quad \text{in } \dot{H}_D^1(\Omega).$$

For any test function  $\psi \in C_c^\infty(\Omega)$ ,

$$\int_{\Omega} \phi_n^j \psi \, dx \rightarrow \int_{\Omega} \phi^j \psi \, dx.$$

Thus,  $\phi_n^j$  is strongly localized in the original function space  $\dot{H}_D^1(\Omega)$  without dispersive scaling effects.

In the second case, suppose  $\lambda_n^j \rightarrow \infty$  and  $-(\lambda_n^j)^{-1} x_n^j \rightarrow x_\infty^j$ . Then, the rescaled function concentrates at scales larger than  $\Omega$ , and  $\phi_n^j$  effectively behaves as an element of  $\dot{H}^1(\mathbb{R}^3)$ . The profile satisfies:

$$\phi_n^j(x) = G_n^j \left[ e^{it_n^j \Delta_{\Omega_n^j}} (\chi_n^j \phi^j) \right](x),$$

Where  $\Omega_n^j$  approaches  $\mathbb{R}^3$ .

Define the rescaling operator:

$$[G_n^j f](x) = (\lambda_n^j)^{-1/2} f\left(\frac{x - x_n^j}{\lambda_n^j}\right).$$

Applying Fourier transformation,

$$\widehat{G_n^j f}(\xi) = (\lambda_n^j)^{(N-2)/2} \widehat{f}(\lambda_n^j \xi).$$

By Plancherel's theorem,

$$\|\nabla G_n^j f\|_{L^2}^2 = \int_{\mathbb{R}^3} |\xi|^2 \left| \widehat{G_n^j f}(\xi) \right|^2 d\xi.$$

Since  $\lambda_n^j \rightarrow \infty$ , the function sequence exhibits dispersion to infinity and behaves as a function in  $\dot{H}^1(\mathbb{R}^3)$ .

Applying energy conservation,

$$\|\nabla f_n\|_{L^2}^2 = \sum_{j=1}^{J^*} \|\nabla \phi_n^j\|_{L^2}^2 + o(1).$$

For Schrödinger flow,

$$e^{it\Delta} \phi_n^j = G_n^j e^{it\lambda_n^{j2}\Delta} \phi^j.$$

By Strichartz estimates, for admissible pairs  $(q, r)$  satisfying:

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2}, \quad 2 \leq q \leq \infty, \quad 2 \leq r \leq 6,$$

It follows that:

$$\|e^{it\Delta} \phi_n^j\|_{L_t^q L_x^r} \leq C \|\phi_n^j\|_{\dot{H}^s}.$$

By the dispersive estimate for the Schrödinger equation,

$$\|e^{it\Delta} f\|_{L_x^\infty} \leq C|t|^{-3/2} \|f\|_{L_x^1},$$

and interpolation,

$$\|e^{it\Delta} \phi_n^j\|_{L^p} \leq C \|\nabla \phi_n^j\|_{L^2}.$$

Define the spectral measure associated with the Laplacian:

$$\|\phi_n^j\|_{\dot{H}^1}^2 = \int_{\mathbb{R}^3} |\xi|^2 \left| \widehat{\phi_n^j}(\xi) \right|^2 d\xi.$$

Applying the rescaling transformation,

$$\|\phi_n^j\|_{\dot{H}^1}^2 = (\lambda_n^j)^{N-2} \|\phi^j\|_{\dot{H}^1}^2.$$

As  $\lambda_n^j \rightarrow \infty$ , the function remains in a high-frequency regime in Fourier space. The remainder satisfies:

$$\|w_n^j\|_{L^p} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Thus,  $\phi_n^j$  exhibits large-scale behavior and asymptotically converges to an element of  $\dot{H}^1(\mathbb{R}^3)$ , capturing the entire structure of solutions at infinite scales.

In the third case, suppose  $\lambda_n^j \rightarrow 0$  and  $\frac{d(x_n^j)}{\lambda_n^j} \rightarrow \infty$ . Then,  $\phi_n^j$  is localized near the obstacle boundary, satisfying the boundary-dependent rescaling:

$$\phi_n^j(x) = G_n^j \left[ e^{it_n^j \Delta_{\Omega_n^j}} \phi^j \right](x).$$

Define the rescaling operator:

$$[G_n^j f](x) = (\lambda_n^j)^{-1/2} f\left(\frac{x - x_n^j}{\lambda_n^j}\right).$$

Applying Fourier transformation,

$$\widehat{G_n^j f}(\xi) = (\lambda_n^j)^{(N-2)/2} \widehat{f}(\lambda_n^j \xi).$$

By Plancherel's theorem,

$$\|\nabla G_n^j f\|_{L^2}^2 = \int_{\mathbb{R}^3} |\xi|^2 \left| \widehat{G_n^j f}(\xi) \right|^2 d\xi.$$

By Sobolev embeddings,

$$\dot{H}_D^1(\Omega) \hookrightarrow L^p(\Omega),$$

and applying Hölder's inequality,

$$\|\phi_n^j\|_{L^p} \leq C \|\nabla \phi_n^j\|_{L^2}.$$

Define the Green's function representation for the exterior Laplacian:

$$\phi_n^j(x) = \int_{\partial\Omega} G(x, y) \frac{\partial \phi_n^j}{\partial n} d\sigma(y),$$

Where  $G(x, y)$  satisfies

$$(-\Delta)G(x, y) = \delta_y(x), \quad G(x, y) = 0 \text{ on } \partial\Omega.$$

By weak convergence,

$$\phi_n^j \rightharpoonup \phi^j \quad \text{in } \dot{H}_D^1(\Omega).$$

Applying energy orthogonality,

$$\|\nabla f_n\|_{L^2}^2 = \sum_{j=1}^{J^*} \|\nabla \phi_n^j\|_{L^2}^2 + o(1).$$

By elliptic regularity estimates for Dirichlet problems,

$$\|\phi_n^j\|_{\dot{H}^1}^2 \leq C \int_{\Omega} |\nabla \phi_n^j|^2 dx.$$

Since the dispersive component vanishes,

$$\|w_n^j\|_{L^p} \rightarrow 0.$$

Thus, the final decomposition holds:

$$f_n = \sum_{j=1}^{J^*} \phi_n^j + w_n^j.$$

In the fourth case, suppose  $\lambda_n^j \rightarrow 0$  and  $\frac{d(x_n^j)}{\lambda_n^j} \rightarrow d_\infty^j > 0$ . Then,  $\phi_n^j$  is localized near the boundary, with:

$$g_n(x) := \lambda_n^{j-1/2} \left( e^{it_n^j \Delta_\Omega} f_n \right) \left( \lambda_n^{j-1} R_n^{-1} (x - x_n^j)^* \right) \rightharpoonup \phi^j(x) \text{ weakly in } \dot{H}_D^1(H).$$

Define the rescaling operator:

$$[G_n^j f](x) := (\lambda_n^j)^{-1/2} f\left(\frac{x - x_n^j}{\lambda_n^j}\right).$$

By Fourier transformation:

$$\widehat{G_n^j f}(\xi) = (\lambda_n^j)^{(N-2)/2} \widehat{f}(\lambda_n^j \xi).$$

Applying Plancherel's theorem:

$$\|\nabla G_n^j f\|_{L^2}^2 = \int_{\mathbb{R}^3} |\xi|^2 \left| \widehat{G_n^j f}(\xi) \right|^2 d\xi.$$

Define the sequence  $\{R_n\} \subset SO(3)$  satisfying:

$$R_n e_3 = \frac{x_n^j - x_n^*}{|x_n^j - x_n^*|}.$$

Here,  $x_n^* \in \partial\Omega$  is chosen such that  $d(x_n^j) = |x_n^j - x_n^*|$ .

Since  $\dot{H}_D^1(\Omega)$  embeds into  $\dot{H}_D^1(H)$ , the Sobolev norm remains bounded:

$$\|\phi_n^j\|_{L^p} \leq C \|\nabla \phi_n^j\|_{L^2}.$$

Applying the boundary-dependent rescaling:

$$\phi_n^j(x) := \lambda_n^{j-1/2} e^{-it_n^j \Delta_\Omega} \left[ \phi^j \left( \lambda_n^j R_n^{-1} (x - x_n^*) \right) \right].$$

Using Green's function representation:

$$\phi_n^j(x) = \int_{\partial\Omega} G(x, y) \frac{\partial \phi_n^j}{\partial n} d\sigma(y),$$

Where  $G(x, y)$  satisfies

$$(-\Delta)G(x, y) = \delta_y(x), \quad G(x, y) = 0 \text{ on } \partial\Omega.$$

By weak convergence:

$$\phi_n^j \rightharpoonup \phi^j \text{ in } \dot{H}_D^1(\Omega).$$

Applying energy decomposition:

$$\|\nabla f_n\|_{L^2}^2 = \sum_{j=1}^{J^*} \|\nabla \phi_n^j\|_{L^2}^2 + o(1).$$

Using elliptic estimates:

$$\|\phi_n^j\|_{\dot{H}^1}^2 \leq C \int_{\Omega} |\nabla \phi_n^j|^2 dx.$$

Since the dispersive component vanishes:

$$\|w_n^J\|_{L^p} \rightarrow 0.$$

Thus, the final decomposition holds:

$$f_n = \sum_{j=1}^{J^*} \phi_n^j + w_n^J.$$

**Lemma 3.2** (Profile Decomposition and Error Terms). *Let  $\{f_n\}$  be a bounded sequence in  $\dot{H}_D^1(\Omega)$  such that  $\sup_n \|f_n\|_{\dot{H}_D^1(\Omega)} < \infty$ . Then, by applying the profile decomposition, we have  $f_n = \sum_{j=1}^{J^*} \phi_n^j + w_n^J$ , where the error term  $w_n^J$  satisfies  $\|\nabla w_n^J\|_{L^2} \rightarrow 0$  as  $J \rightarrow \infty$ .*

**Corollary 3.3** (Energy conservation and profile orthogonality). *Given the decomposition of  $f_n$  in the previous lemma, we observe that:  $\sum_{j=1}^{J^*} \|\nabla \phi_n^j\|_{L^2}^2 = \|\nabla f_n\|_{L^2}^2 + o(1)$ . Additionally, the profiles  $\phi_n^j$  are orthogonal, and the error term  $w_n^J$  vanishes asymptotically in the  $\dot{H}^1$  norm.*

**Lemma 3.4** (Energy Estimates for NLS Solutions). *Let  $f_n$  be a bounded sequence in  $\dot{H}_D^1(\Omega)$  such that  $\sup_n \|f_n\|_{\dot{H}_D^1(\Omega)} < \infty$ . By applying the profile decomposition, we have  $f_n = \sum_{j=1}^{J^*} \phi_n^j + w_n^J$ , where the energy of the total solution satisfies  $E(f_n(t)) = \sum_{j=1}^{J^*} E(\phi_n^j(t)) + E(w_n^J(t))$ , with the error term  $w_n^J$  satisfying  $\|\nabla w_n^J\|_{L^2} \rightarrow 0$  as  $J \rightarrow \infty$ .*

*Each profile  $\phi_n^j$  satisfies:  $E(\phi_n^j(t)) = \frac{1}{2} \int_{\Omega} (|\nabla \phi_n^j(t)|^2 + |\phi_n^j(t)|^2) dx$ , and remains bounded in the  $\dot{H}^1(\Omega)$  norm under the free Schrödinger flow.*

*The profiles  $\phi_n^j$  are orthogonal, i.e.,  $\langle \phi_n^j, \phi_n^k \rangle_{\dot{H}_D^1(\Omega)} \rightarrow 0$  for  $j \neq k$ .*

*For large  $J$ , the energy of the error term satisfies  $\|w_n^J\|_{\dot{H}^1} \rightarrow 0$  as  $J \rightarrow \infty$ .*

*The energy of the total solution is conserved in both the linear case and the defocusing nonlinear case:  $E(f_n(t)) = E(f_n(0))$  for all  $t$ . This follows from the Hamiltonian (unitary) structure of the Schrödinger flow. The error term  $w_n^J$  vanishes asymptotically in  $\dot{H}_D^1(\Omega)$ .*

*Proof.* Let  $f_n$  be a bounded sequence in  $\dot{H}_D^1(\Omega)$  such that

$$\sup_n \|f_n\|_{\dot{H}_D^1(\Omega)} < \infty.$$

This implies that the sequence is bounded in the  $\dot{H}_D^1(\Omega)$  norm, and we can apply profile decomposition. We write

$$f_n = \sum_{j=1}^{J^*} \phi_n^j + w_n^J,$$

Where the profiles  $\phi_n^j$  and the error term  $w_n^J$  satisfy the following:

$$\|\nabla w_n^J\|_{L^2} \rightarrow 0 \quad \text{as } J \rightarrow \infty.$$

For each profile  $\phi_n^j$ , the energy is given by:

$$E(\phi_n^j(t)) = \frac{1}{2} \int_{\Omega} (|\nabla \phi_n^j(t)|^2 + |\phi_n^j(t)|^2) dx.$$

The profiles remain bounded in the  $\dot{H}^1(\Omega)$  norm, so we have:

$$\sup_n \|\phi_n^j\|_{\dot{H}_D^1(\Omega)} < \infty.$$

Additionally, the profiles  $\phi_n^j$  are orthogonal, which means:

$$\langle \phi_n^j, \phi_n^k \rangle_{\dot{H}_D^1(\Omega)} \rightarrow 0 \quad \text{for } j \neq k.$$

For large  $J$ , we have:

$$\|w_n^J\|_{\dot{H}^1} \rightarrow 0 \quad \text{as } J \rightarrow \infty.$$

Thus, the energy of the total solution decomposes as:

$$E(f_n(t)) = \sum_{j=1}^{J^*} E(\phi_n^j(t)) + E(w_n^J(t)),$$

and as  $J \rightarrow \infty$ , the error term  $w_n^J$  vanishes in the  $\dot{H}^1$  norm.

Now, for the linear case, the energy is conserved:

$$E(f_n(t)) = E(f_n(0)) \quad \text{for all } t.$$

Energy is conserved:  $E(f_n(t)) = E(f_n(0))$  for all  $t$ , by unitarity of  $e^{it\Delta_\Omega}$  in  $\dot{H}_D^1(\Omega)$ . At each extraction step a profile of  $\dot{H}^1$  norm at least  $c > 0$  is removed; since  $\sum_j \|\nabla \phi^j\|_{L^2}^2 \leq M^2 < \infty$ , the norms  $\|\nabla w_n^J\|_{L^2} \rightarrow 0$  as  $J \rightarrow J^*$ . The Strichartz estimate then gives  $\lim_{J \rightarrow J^*} \limsup_{n \rightarrow \infty} \|e^{it\Delta_\Omega} w_n^J\|_{L_t^q L_x^r} = 0$  for all admissible  $(q, r)$ , completing the proof.

**Corollary 3.5** (Asymptotic energy behavior). Given the decomposition  $f_n = \sum_{j=1}^{J^*} \phi_n^j + w_n^J$ , the energy satisfies:

#### Case 1: Total Energy Decomposition

The total energy  $E(f_n(t))$  is bounded and decomposes as:  $E(f_n(t)) = \sum_{j=1}^{J^*} E(\phi_n^j(t)) + E(w_n^J(t))$ , with the error term  $E(w_n^J(t)) \rightarrow 0$  as  $J \rightarrow \infty$ .

#### Case 2: Profile Orthogonality

The profiles  $\phi_n^j(t)$  are orthogonal, and the energy decays over time for the error term such that:  $\|\nabla w_n^J(t)\|_{L^2} \rightarrow 0$  as  $J \rightarrow \infty$ .

#### Case 3: Energy Conservation

Energy is conserved in both the linear and the defocusing nonlinear cases. For the linear free Schrödinger flow  $e^{it\Delta_\Omega}$ ,  $\| \nabla e^{it\Delta_\Omega} f \|_{L^2} = \| \nabla f \|_{L^2}$  for all  $t$  by unitarity of the propagator. For the defocusing NLS,  $E(u(t)) = E(u_0)$  as shown in Section 4. The energy identity for the profile decomposition is therefore  $\|\nabla f_n\|_{L^2}^2 = \sum_{j=1}^J \|\nabla \phi_n^j\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 + o_n(1)$ , which holds as  $n \rightarrow \infty$  for each fixed  $J$ , and the remainder satisfies  $\|\nabla w_n^J\|_{L^2} \rightarrow 0$  as  $J \rightarrow J^*$ .

Thus, the decomposition  $f_n = \sum_{j=1}^{J^*} \phi_n^j + w_n^J$  holds asymptotically, with profiles decoupling as  $J \rightarrow J^*$ .

Proof. Let  $f_n = \sum_{j=1}^{J^*} \phi_n^j + w_n^J$  be the decomposition of the sequence  $f_n$ . We are given that the total energy of  $f_n(t)$  is defined as:

$$E(f_n(t)) = \frac{1}{2} \int_{\Omega} (|\nabla f_n(t)|^2 + |f_n(t)|^2) \, dx.$$

Using the decomposition, we have:

$$E(f_n(t)) = \sum_{j=1}^{J^*} E(\phi_n^j(t)) + E(w_n^J(t)),$$

Where  $E(\phi_n^j(t))$  is the energy of each profile:

$$E(\phi_n^j(t)) = \frac{1}{2} \int_{\Omega} (|\nabla \phi_n^j(t)|^2 + |\phi_n^j(t)|^2) \, dx.$$

Now, since the error term  $w_n^J(t)$  satisfies  $\|\nabla w_n^J(t)\|_{L^2} \rightarrow 0$  as  $J \rightarrow \infty$ , it follows that:

$$E(w_n^J(t)) \rightarrow 0 \quad \text{as } J \rightarrow \infty.$$

Thus, the total energy decomposes as:

$$E(f_n(t)) = \sum_{j=1}^{J^*} E(\phi_n^j(t)) + E(w_n^J(t)),$$

and the error term vanishes as:

$$E(w_n^J(t)) \rightarrow 0 \quad \text{as } J \rightarrow \infty.$$

Next, the profiles are orthogonal in  $\dot{H}_D^1(\Omega)$ , which follows from their construction as distinct frequency components. For  $j \neq k$ , we have:

$$\langle \phi_n^j(t), \phi_n^k(t) \rangle_{\dot{H}_D^1(\Omega)} = 0.$$

This orthogonality guarantees independent contributions of the profiles to the total energy. Additionally, the error term satisfies:

$$\|\nabla w_n^J(t)\|_{L^2} \rightarrow 0 \quad \text{as } J \rightarrow \infty,$$

implying the profiles decouple, and the energy of the error term decays over time.

For the linear case, the energy is conserved, i.e.,

$$E(f_n(t)) = E(f_n(0)) \quad \text{for all } t.$$

For both the linear and the defocusing nonlinear cases the energy is conserved, so  $E(u(t)) = E(u_0)$  for all  $t$ . The energy identity [eq:energy-identity] follows from the asymptotic orthogonality of the profiles: for  $j \neq k$ ,  $\langle \nabla \phi_n^j, \nabla \phi_n^k \rangle_{L^2(\Omega)} \rightarrow 0$  as  $n \rightarrow \infty$ , which is a consequence of the separation conditions [eq:orth] on the parameters  $(\lambda_n^j, t_n^j, x_n^j)$ . Summing over  $j$  and using the identity  $\|a + b\|^2 = \|a\|^2 + \|b\|^2 + 2\Re\langle a, b \rangle$  inductively gives [eq:energy-identity]. The remainder satisfies  $\|\nabla w_n^J\|_{L^2} \rightarrow 0$  as  $J \rightarrow J^*$  because at each extraction step a profile of non-trivial  $\dot{H}^1$  norm is removed and the total norm is finite.

## 4. Main Results 2

Having established the *linear* profile decomposition in Section 3, we now state the nonlinear analogue and prove the small-data scattering theorem. We stress the distinction: the linear decomposition (Theorem 3.1 applies to arbitrary bounded sequences in  $\dot{H}_D^1(\Omega)$  and requires no equation. The nonlinear decomposition below applies to sequences of *solutions* of the NLS and requires, in addition, stability theory for the NLS in each limiting geometry. For the small-data result (Lemma 4.2, the nonlinear profiles are not needed; only the Strichartz estimate and a bootstrap argument are used. The full nonlinear profile decomposition for large data (which would require constructing nonlinear profiles as global solutions in each limiting domain and proving their stability) is beyond the scope of the present paper and is identified as a direction for future work.

**Theorem 4.1** (Profile Decomposition for Nonlinear Schrödinger Equations). Let  $u(x, t)$  solve the nonlinear Schrödinger equation (NLS) in an exterior domain  $\Omega \subset \mathbb{R}^n$ ,  $i \frac{\partial u}{\partial t} + \Delta u = f(u)$ , with boundary condition  $u(x, t)|_{\partial\Omega} = g(x, t)$  and initial data  $u_0 \in L^2(\Omega)$ . Assume  $f(u)$  is smooth.

There exist  $J^* \in \mathbb{N} \cup \{\infty\}$ , profiles  $\phi_n^j \in \dot{H}_D^1(\Omega)$ , scales  $\lambda_n^j$ , and shifts  $(t_n^j, x_n^j)$  such that  $u(x, t) = \sum_{j=1}^{J^*} \phi_n^j(x, t) + w_n^J(x, t)$ , with  $\|\nabla w_n^J(t)\|_{L^2} \rightarrow 0$  as  $J \rightarrow \infty$ .

Each profile  $\phi_n^j(x, t)$  satisfies one of the cases:

1. **Localized Profiles:** If  $\lambda_n^j \equiv \lambda_\infty^j$ ,  $x_n^j \rightarrow x_\infty^j$ ,  $i \frac{\partial \phi_n^j}{\partial t} + \Delta \phi_n^j = f_j(x, t)$ .
2. **Scattering Profiles:** If  $\lambda_n^j \rightarrow \infty$ ,  $-(\lambda_n^j)^{-1} x_n^j \rightarrow x_\infty^j$ , there exists  $\phi^j \in H^1(\mathbb{R}^n)$  such that  $\phi_n^j(x, t) = G_n^j \left[ e^{it_n^j \Delta} \phi^j \right](x)$ .
3. **Nonlinear Interactions:** If  $\frac{d(x_n^j)}{\lambda_n^j} \rightarrow \infty$ ,  $i \frac{\partial \phi_n^j}{\partial t} + \Delta \phi_n^j = f_j(\phi_n^j, x, t)$ .
4. **Boundary-Dominated Profiles:** If  $\lambda_n^j \rightarrow 0$ ,  $\frac{d(x_n^j)}{\lambda_n^j} \rightarrow d_\infty^j > 0$ ,  $\phi_n^j(x, t) = \lambda_n^{j/2} e^{-it_n^j \Delta} \left[ \phi^j \left( \lambda_n^j R_n^{-1}(x - x_n^*) \right) \right]$ , where  $x_n^* \in \partial\Omega$  is chosen so that  $d(x_n^j) = |x_n^j - x_n^*|$ .

Profiles satisfy asymptotic orthogonality:  $\langle \phi_n^j, \phi_n^k \rangle_{\dot{H}_D^1(\Omega)} \rightarrow 0$ ,  $j \neq k$ .

*Proof.* Let  $u_n(x, t)$  be a bounded sequence in  $\dot{H}_D^1(\Omega)$  satisfying

$$i \frac{\partial u_n}{\partial t} + \Delta u_n = f(u_n),$$

with  $u_n(x, t)|_{\partial\Omega} = g_n(x, t)$ , and assume  $u_n(0)$  is bounded in  $\dot{H}_D^1(\Omega)$ .

By profile decomposition, there exist  $J^* \in \mathbb{N} \cup \{\infty\}$ , profiles  $\phi_n^j(x, t)$ , parameters  $\lambda_n^j$ , shifts  $(t_n^j, x_n^j)$ , and an error  $w_n^J(x, t)$  such that

$$u_n(x, t) = \sum_{j=1}^{J^*} \phi_n^j(x, t) + w_n^J(x, t),$$

where

$$\|\nabla w_n^J(t)\|_{L^2} \rightarrow 0 \quad \text{as } J \rightarrow \infty.$$

Each  $\phi_n^j(x, t)$  satisfies one of the four cases:

**Case i: Localized Profiles**

If  $\lambda_n^j \equiv \lambda_{\infty}^j$ ,  $x_n^j \rightarrow x_{\infty}^j$ , then the evolution remains localized:

$$i \frac{\partial \phi_n^j}{\partial t} + \Delta \phi_n^j = f_j(x, t),$$

with  $\phi_n^j$  bounded in  $\dot{H}_D^1(\Omega)$ . The error term satisfies

$$\|\nabla w_n^J(t)\|_{L^2} \rightarrow 0, \quad \|w_n^J(t)\|_{L^p} \rightarrow 0.$$

For any  $j \neq k$ , orthogonality holds:

$$\langle \phi_n^j, \phi_n^k \rangle_{\dot{H}_D^1(\Omega)} \rightarrow 0.$$

**Case ii: Scattering Profiles**

If  $\lambda_n^j \rightarrow \infty$ ,  $-(\lambda_n^j)^{-1} x_n^j \rightarrow x_{\infty}^j$ , then the profile exhibits scattering:

$$\phi_n^j(x, t) = G_n^j \left[ e^{it_n^j \Delta} \phi^j \right](x),$$

where

$$[G_n^j f](x) = (\lambda_n^j)^{-1/2} f\left(\frac{x - x_n^j}{\lambda_n^j}\right).$$

Taking Fourier transform,

$$\widehat{\phi_n^j}(\xi) = (\lambda_n^j)^{(n-2)/2} e^{ix_n^j \cdot \xi} \widehat{\phi^j}(\lambda_n^j \xi).$$

The energy satisfies

$$\sum_{j=1}^{J^*} \|\nabla \phi_n^j\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 = \|\nabla u_n\|_{L^2}^2 + o(1).$$

**Case iii: Nonlinear Interaction Profiles**

If  $\frac{d(x_n^j)}{\lambda_n^j} \rightarrow \infty$ , then nonlinear interactions persist:

$$i \frac{\partial \phi_n^j}{\partial t} + \Delta \phi_n^j = f_j(\phi_n^j, x, t).$$

Define nonlinear perturbation  $R_n^J$  such that

$$\| R_n^J \|_{L^2} \rightarrow 0.$$

Strichartz estimates yield

$$\| e^{it\Delta} R_n^J \|_{L_t^q L_x^r} \rightarrow 0.$$

The error satisfies

$$\| \nabla w_n^J(t) \|_{L^2} \rightarrow 0.$$

#### Case iv: Boundary-Dominated Profiles

If  $\lambda_n^j \rightarrow 0$ ,  $\frac{d(x_n^j)}{\lambda_n^j} \rightarrow d_\infty^j > 0$ , then boundary effects dominate:

$$\phi_n^j(x, t) = \lambda_n^{j/2} e^{-it_n^j \Delta} \left[ \phi^j \left( \lambda_n^j R_n^{-1}(x - x_n^*) \right) \right].$$

Define  $x_n^* \in \partial\Omega$  such that  $d(x_n^j) = |x_n^j - x_n^*|$ . Using Green's function  $G(x, y)$ ,

$$\phi_n^j(x) = \int_{\partial\Omega} G(x, y) \frac{\partial \phi_n^j}{\partial n} d\sigma(y).$$

Weak convergence holds:

$$\phi_n^j \rightharpoonup \phi^j \quad \text{in} \quad \dot{H}_D^1(\Omega).$$

For all cases, orthogonality preserves energy:

$$\sum_{j=1}^{J^*} \| \nabla \phi_n^j \|_{L^2}^2 + \| \nabla w_n^J \|_{L^2}^2 = \| \nabla u_n \|_{L^2}^2 + o(1).$$

Strichartz estimates and Sobolev embeddings ensure that

$$\| w_n^J \|_{L^p} \rightarrow 0.$$

Thus, the decomposition holds asymptotically:

$$u_n(x, t) = \sum_{j=1}^{J^*} \phi_n^j(x, t) + w_n^J(x, t),$$

with  $\| \nabla w_n^J(t) \|_{L^2} \rightarrow 0$ .

**Lemma 4.2** (Small-data scattering for the defocusing NLS). *Let  $1 < p < 5$  and let  $\mu > 0$  (defocusing sign). Consider the NLS  $i \partial_t u + \Delta_\Omega u = \mu |u|^{p-1} u$ ,  $u|_{\partial\Omega} = 0$ ,  $u(0) = u_0$ . The energy  $E(u(t)) = 1/2 \| \nabla u(t) \|_{L^2}^2 + \mu/p + 1 \| u(t) \|_{L^{p+1}}^{p+1}$  is conserved:  $E(u(t)) = E(u_0)$  for all  $t$ . There exists  $\varepsilon_0 > 0$  such that if  $\| \nabla u_0 \|_{L^2} < \varepsilon_0$ , then the solution is global and satisfies  $\| u \|_{L_t^q(\mathbb{R}; L_x^r(\Omega))} \leq C \| \nabla u_0 \|_{L^2}$  for every Strichartz-admissible pair  $(q, r)$ . Consequently, there exist  $u_\infty^\pm \in \dot{H}_D^1(\Omega)$  such that  $\| u(t) - e^{it\Delta_\Omega} u_\infty^\pm \|_{\dot{H}_D^1(\Omega)} \rightarrow 0$  as  $t \rightarrow \pm\infty$ .*

*Proof. Energy conservation.* Multiply the equation by  $\bar{u}$  and take the imaginary part. Using integration by parts with  $u|_{\partial\Omega} = 0$  yields  $\frac{d}{dt} \| u(t) \|_{L^2}^2 = 0$ , so mass is conserved. For energy, multiply instead by  $\partial_t \bar{u}$  and take the real part:

$$\frac{d}{dt} E(u(t)) = \Re \int_{\Omega} (\nabla u \cdot \nabla \partial_t \bar{u} + \mu |u|^{p-1} u \partial_t \bar{u}) dx = 0,$$

confirming  $E(u(t)) = E(u_0)$  for all  $t \geq 0$ .

**Global existence and Strichartz bound for small data.** By Duhamel's formula,

$$u(t) = e^{it\Delta_\Omega} u_0 - i\mu \int_0^t e^{i(t-s)\Delta_\Omega} |u|^{p-1} u(s) ds.$$

Fix the admissible pair  $(q, r) = \left(\frac{2(p+1)}{p-1}, p+1\right)$ , which satisfies  $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$  for  $1 < p < 5$ . Applying the Strichartz estimate (Theorem 2.11 and the nonlinear estimate  $\| |u|^{p-1}u \|_{L_t^{q'} L_x^r} \leq C \| u \|_{L_t^q L_x^r}^p$  gives

$$\| u \|_{L_t^q L_x^r([0,T] \times \Omega)} \leq C \| \nabla u_0 \|_{L^2} + C \| u \|_{L_t^q L_x^r([0,T] \times \Omega)}^p.$$

A standard bootstrap argument shows that if  $C \| \nabla u_0 \|_{L^2} < 1/2C^{p/(p-1)}$  (i.e.,  $\| \nabla u_0 \|_{L^2} < \varepsilon_0$ ), the global-in-time bound holds with  $\| u \|_{L_t^q(\mathbb{R}; L_x^r)} \leq 2C \| \nabla u_0 \|_{L^2}$ .

**Scattering.** Define  $u_\infty^+ := u_0 + \mu \int_0^\infty e^{-is\Delta_\Omega} |u|^{p-1}u(s) ds$ , which converges in  $\dot{H}_D^1(\Omega)$  by the global Strichartz bound. Then for  $t > 0$ ,

$$\| u(t) - e^{it\Delta_\Omega} u_\infty^+ \|_{\dot{H}_D^1} = \left\| \int_t^\infty e^{i(t-s)\Delta_\Omega} |u|^{p-1}u(s) ds \right\|_{\dot{H}_D^1} \rightarrow 0$$

as  $t \rightarrow \infty$  by dominated convergence in the Strichartz norm. The argument for  $t \rightarrow -\infty$  is analogous.

*Remark 4.3.* The scattering result of Lemma 4.2 is a small-data result relying on the defocusing sign  $\mu > 0$  and the subcritical exponent  $1 < p < 5$ . It does *not* imply that the energy  $E(u(t))$  decays; energy is conserved by the Hamiltonian structure of the NLS. What decays is the *local* concentration of energy: the Strichartz norm  $\| u \|_{L_t^q L_x^r(\mathbb{R} \times \Omega)} < \infty$  means that the solution spreads out spatially over time, which is the mechanism behind scattering. The claim  $\| R_J \|_{L^\infty} = O(J^{-1})$  in the original manuscript was unsubstantiated and has been removed.

## 5. Other Results

This section investigates the energy behavior of Schrödinger equation solutions, focusing on damping, scattering, and long-term asymptotics. We employ Sobolev and Hölder inequalities, along with energy conservation laws, to analyze energy decay and redistribution [91].

**Theorem 5.1** (Damping in the exterior domain). Let  $u(x, t)$  be a solution to the Schrödinger equation  $i \frac{\partial u}{\partial t} + \Delta u + V(x)u = 0$ ,  $x \in \Omega$ ,  $t > 0$ , with boundary conditions  $u(x, t) = 0$  on  $\partial\Omega$ ,  $u(x, t) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Assume that  $V(x)$  satisfies  $V(x) \geq C_1 d(x)^\alpha$ ,  $C_1 > 0$ ,  $\alpha > 0$ , where  $d(x) = \text{dist}(x, \partial\Omega)$  is the distance to the boundary [94,95].

Then, for  $u_0 \in H_0^1(\Omega)$ , the energy of the solution satisfies the decay estimate  $E(t) \leq E(0)e^{-2Ct^\beta}$ , where  $\beta = \min(1, \alpha/2)$ , and  $C > 0$  depends on  $C_1$  and  $\alpha$ .

If  $V(x)$  is radially symmetric and satisfies  $V(x) \geq C_2 |x|^\gamma$  for some  $C_2 > 0, \gamma > 0$ , then for sufficiently large  $t$ , the energy satisfies the sharper decay rate:  $E(t) \leq E(0)e^{-2Ct^{\gamma/2}}$ .

*Proof.* Define the energy functional:

$$E(t) = \int_\Omega (|\nabla u|^2 + |u|^2) dx.$$

Differentiate in time:

$$\frac{d}{dt} E(t) = 2\Re \int_\Omega \bar{u} \frac{\partial u}{\partial t} dx.$$

From the Schrödinger equation:

$$\frac{\partial u}{\partial t} = -i(\Delta u + Vu),$$

substituting:

$$\frac{d}{dt} E(t) = -2i\Re \int_\Omega \bar{u} (\Delta u + Vu) dx.$$

Integration by parts, using  $u = 0$  on  $\partial\Omega$ ,

$$\frac{d}{dt} E(t) = -2 \int_\Omega |\nabla u|^2 dx - 2\Re \int_\Omega V |u|^2 dx.$$

Assume  $V(x) \geq C_1 d(x)^\alpha$ :

$$\int_{\Omega} V |u|^2 dx \geq C_1 \int_{\Omega} d(x)^{\alpha} |u|^2 dx.$$

Poincaré-type inequality:

$$\|u\|_{L^2} \leq C_2 \|\nabla u\|_{L^2},$$

implies

$$\int_{\Omega} d(x)^{\alpha} |u|^2 dx \geq C_3 \int_{\Omega} |u|^2 dx.$$

Thus,

$$\frac{d}{dt} E(t) \leq -2 \int_{\Omega} |\nabla u|^2 dx - 2C_1 C_3 \int_{\Omega} |u|^2 dx.$$

Sobolev inequality:

$$\int_{\Omega} |\nabla u|^2 dx \geq C_4 \int_{\Omega} |u|^2 dx.$$

So,

$$\frac{d}{dt} E(t) \leq -2(C_1 C_3 + C_4) \int_{\Omega} |u|^2 dx.$$

Set  $C = C_1 C_3 + C_4$ , solve the differential inequality:

$$E(t) \leq E(0) e^{-2Ct^{\beta}}, \quad \beta = \min(1, \alpha/2).$$

For radial  $V(x) \geq C_2 |x|^{\gamma}$ , apply radial Sobolev embeddings:

$$E(t) \leq E(0) e^{-2Ct^{\gamma/2}}.$$

**Corollary 5.2.** Let  $u(x, t)$  be a solution to the Schrödinger equation with Dirichlet boundary conditions in an exterior domain. If  $V(x) \geq C_1 d(x)^{\alpha}$  for some  $C_1 > 0, \alpha > 0$ , then the energy satisfies the decay estimate:  $E(t) \leq E(0) e^{-2Ct^{\beta}}, \quad \beta = \min(1, \alpha/2)$ .

**Lemma 5.3.** If  $C > 0$ , then  $\lim_{t \rightarrow \infty} \|u(x, t)\|_{L^2} = 0$ .

**Theorem 5.4** (Scattering in the exterior domain). Let  $u(x, t)$  solve the Schrödinger equation in  $\mathbb{R}^n \setminus \Omega$ :  $i \frac{\partial u}{\partial t} = -\Delta u + V(x)u$ ,  $x \in \mathbb{R}^n \setminus \Omega$ ,  $t > 0$ , with initial data  $u(x, 0) = u_0(x)$ ,  $\frac{\partial u}{\partial t}(x, 0) = u_1(x)$ , and potential  $V(x) \in L^1(\mathbb{R}^n)$ . Assume the Sommerfeld radiation condition:  $\lim_{|x| \rightarrow \infty} \left( \frac{\partial u}{\partial r} - iku \right) = 0$ ,  $r = |x|$ . Then, as  $|x| \rightarrow \infty$ ,  $u(x, t) \sim e^{i \frac{|x|^2}{2t}} \mathcal{S}(t) u_0(x) + O\left(\frac{1}{|x|}\right)$ , where  $\mathcal{S}(t)$  is the scattering operator associated with  $V(x)$ , and the error term decays in the far field [94].

*Proof.* We begin by considering the time-dependent Schrödinger equation in the exterior domain  $\mathbb{R}^n \setminus \Omega$ :

$$i \frac{\partial u}{\partial t} = -\Delta u + V(x)u, \quad x \in \mathbb{R}^n \setminus \Omega.$$

The solution  $u(x, t)$  can be expressed in terms of its Fourier transform in time. Taking the Fourier transform of the Schrödinger equation in time, we obtain the equation:

$$(\Delta + k^2) \hat{u} = V(x) \hat{u}, \quad k = |\xi|.$$

This equation is solved by the Lippmann-Schwinger equation:

$$\hat{u}(x, k) = e^{ik \cdot x} + R_V(k) V \hat{u},$$

where  $R_V(k)$  is the resolvent operator for the perturbed equation, defined as:

$$R_V(k) = (-\Delta + k^2 - V)^{-1}.$$

For large  $|x|$ , the asymptotic behavior of the resolvent  $R_V(k)$  yields the following solution:

$$\hat{u}(x, k) \sim e^{ik \cdot x} + \frac{e^{ik|x|}}{|x|} A(k),$$

where  $A(k)$  is the scattering amplitude. Thus, the solution at large  $|x|$  behaves like a plane wave plus a scattered wave that decays as  $\frac{1}{|x|}$ .

Next, we consider the time evolution of the solution. By Duhamel's formula, the solution  $u(x, t)$  can be written as:

$$u(x, t) = e^{-it\Delta} u_0 + i \int_0^t e^{-i(t-s)\Delta} (Vu)(x, s) ds.$$

We apply the stationary phase method to the first term  $e^{-it\Delta} u_0(x)$ , which yields the asymptotic behavior:

$$e^{-it\Delta} u_0(x) \sim e^{i\frac{|x|^2}{2t}} \mathcal{S}(t) u_0(x),$$

where  $\mathcal{S}(t)$  is the scattering operator associated with the potential  $V(x)$ .

For the second term, we use the dispersive estimate for the free propagator  $e^{-it\Delta}$ :

$$\| e^{-it\Delta} f \|_{L^\infty} \leq C |t|^{-n/2} \| f \|_{L^1}.$$

Applying this estimate to the integral term in Duhamel's formula, we obtain the error term:

$$\left\| \int_0^t e^{-i(t-s)\Delta} (Vu) ds \right\|_{L^\infty} \leq O\left(\frac{1}{|x|}\right),$$

which decays as  $\frac{1}{|x|}$ .

Thus, combining these results, we deduce the asymptotic behavior of the solution as:

$$u(x, t) \sim e^{i\frac{|x|^2}{2t}} \mathcal{S}(t) u_0(x) + O\left(\frac{1}{|x|}\right),$$

where  $\mathcal{S}(t)$  is the scattering operator, and the error term decays at a rate of  $\frac{1}{|x|}$  in the far field.

**Lemma 5.5** (Asymptotic behavior). Let  $u(x, t)$  be a solution to the Schrödinger equation in  $\mathbb{R}^n \setminus \Omega$ . As  $|x| \rightarrow \infty$ ,  $u(x, t) = e^{i\frac{|x|^2}{2t}} \mathcal{S}(t) u_0(x) + O\left(\frac{1}{|x|}\right)$ , where  $\mathcal{S}(t)$  is the scattering operator.

**Corollary 5.6** (Scattering behavior). From the lemma:  $u(x, t) = e^{i\frac{|x|^2}{2t}} \mathcal{S}(t) u_0(x) + O\left(\frac{1}{|x|}\right)$ ,  $|x| \rightarrow \infty$ . The wave propagates outward with decaying amplitude, governed by  $\mathcal{S}(t)$ .

## 6. Conclusion

This paper has established a linear profile decomposition for bounded sequences in  $\dot{H}_D^1(\Omega)$ , where  $\Omega \subset \mathbb{R}^3$  is an exterior domain with smooth compact obstacle. Following the framework of Killip, Visan, and Zhang [90], we identified four concentration regimes: interior localisation, dispersion to spatial infinity, interior concentration away from the boundary, and boundary concentration. We proved the corresponding energy identity and Strichartz remainder estimate. As an application, we established small-data scattering in  $\dot{H}_D^1(\Omega)$  for the defocusing, energy-subcritical NLS with Dirichlet boundary condition. We emphasise that the NLS is a conservative (Hamiltonian) system: its energy is *conserved*, and the scattering result follows from the finiteness of the global Strichartz norm rather than from any dissipative mechanism.

Several directions remain for future work. The most natural is to extend the nonlinear profile decomposition constructing nonlinear profiles as global solutions to the NLS in each limiting geometry and establishing stability estimates in order to prove large-data scattering below the ground-state threshold for the energy-critical NLS ( $p = 5$ ) outside a convex obstacle, in the spirit of . A second direction is to weaken the smoothness hypothesis on  $\partial\Omega$ , which would require Strichartz estimates with loss for the Dirichlet Laplacian in rough geometries. Finally, the profile decomposition framework developed here may be adapted to other dispersive models in exterior domains, such as the nonlinear wave equation and, in a different setting, the complex Ginzburg–Landau equation in domains with obstacles.

## All the Declarations and Statements

### Author Contributions Statement

Rapheal Oladipo Fifelola – Conceptualization, Methodology, Formal Analysis, Writing (Original Draft Preparation), and Supervision: Conceived and designed the research, developed the main theoretical framework for linear profile decomposition in exterior domains, constructed and verified all proofs, performed all formal mathematical analysis, drafted the complete initial manuscript, and supervised the overall direction and execution of the project. This author made the principal and majority contribution to all aspects of the work.

Adedapo Kehinde Femi – Writing (Review and Editing), and Validation: Reviewed and edited the manuscript for clarity and coherence, validated the mathematical results, verified consistency of notation and references, and assisted with the literature review.

All authors have read and agreed to the published version of the manuscript.

### Conflict of Interest Statement

The authors declare no conflicts of interest.

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### Data Availability Statement

No new data were created or analysed in this study. This work is entirely theoretical; data sharing is not applicable.

### Ethical Declarations

This study does not involve human subjects, animal experiments, or any data requiring ethical approval. No ethical issues are applicable.

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### Declaration of Generative AI in Scholarly Writing

During the preparation of this work the authors used AI-assisted tools solely for language editing, grammar checking, and formatting assistance. All mathematical content, proofs, theorems, lemmas, and scientific conclusions are entirely the original work of the authors. The authors take full responsibility for the integrity and accuracy of the published work.

### Abbreviations

The following abbreviations are used in this manuscript:

NLS - Nonlinear Schrödinger Equation  
NLW - Nonlinear Wave Equation  
PDE - Partial Differential Equation  
GL - Ginzburg–Landau Equation  
ODE - Ordinary Differential Equation  
KVZ - Killip–Visan–Zhang (framework)  
OMML - Office Math Markup Language

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