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Numerical Investigation of the Incompressible Navier-Stokes Equations: Lid-Driven Cavity and Validation using the Ghia Benchmark

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Abstract: This dissertation presents a numerical investigation of the two-dimensional incompressible Navier-Stokes equations, focusing on the classic Lid-Driven Cavity problem. The study develops a computational fluid dynamics (CFD) solver from first principles using the Finite Difference Method (FDM) on a structured Cartesian grid, providing a fundamental understanding of the pressure-velocity coupling in viscous flows. The numerical framework employs the Projection Method, originally proposed by Chorin, to enforce the incompressibility constraint. This operator-splitting technique solves an intermediate velocity field which is subsequently projected onto a divergence-free space via a Pressure Poisson Equation (PPE). The governing equations are discretized using second-order central differences for spatial derivatives and a first-order explicit Euler scheme for time integration. The solver is validated at a Reynolds number of $Re = 100$. The simulation results successfully capture the characteristic flow features, including the primary central vortex and the corner recirculation eddies. Quantitative validation is performed by comparing the vertical centerline velocity profiles against the established benchmark data of Ghia et al. The results demonstrate excellent agreement with the benchmark solutions, confirming that the developed solver correctly resolves the physics of wall-bounded shear flows. This work establishes a robust foundational framework for simulating viscous incompressible flows and highlights the efficacy of the Projection Method for fundamental CFD applications.

Index Terms: Navier-Stokes Equations, Finite Difference Method, Projection Method, Lid-Driven Cavity, CFD Validation

1. Introduction

Fluid dynamics governs a vast array of natural and engineering phenomena, ranging from the flow of blood in the human circulatory system to the aerodynamics of vehicles and aircraft. Central to this field are the Navier-Stokes equations, a set of non-linear partial differential equations that describe the motion of viscous fluid substances. While these equations are physically intuitive, deriving directly from Newton's second law of motion, their mathematical solution remains one of the most challenging problems in physics and applied mathematics. As noted by Anderson [1], analytic solutions exist only for a limited set of trivial flow cases, necessitating the use of numerical methods for practical applications involving recirculation, shear layers, and pressure gradients.

1.1. Research Motivation

The field of Computational Fluid Dynamics (CFD) has become an indispensable tool in modern engineering. However, the widespread reliance on commercial "black-box" software often obscures the underlying numerical mechanics and stability criteria that govern these simulations. There is a significant academic need to build solvers from first principles to fully understand the challenges of numerical diffusion, convergence, and conservation. Specifically, the coupling of pressure and velocity in incompressible flows presents a unique mathematical hurdle, as the continuity equation acts as a kinematic constraint rather than a dynamic evolution equation [2]. Addressing this coupling requires specialized algorithms that are best understood through direct implementation.

1.2. Objectives and Scope

The primary objective of this dissertation is to develop, implement, and validate a numerical solver for the two-

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dimensional incompressible Navier-Stokes equations. The study aims to derive a discretized framework using the Finite Difference Method (FDM) on a structured grid, a classical approach chosen for its transparency and ease of implementation [3]. A central goal is to implement the Projection Method introduced by Chorin [4] to resolve the pressure-velocity coupling efficiently. The solver is then applied to simulate the classic Lid-Driven Cavity problem, a benchmark chosen for its simplicity in geometry yet complexity in flow features, such as vortex formation and singularities at the corners. Finally, the accuracy of the solver is validated by comparing the computed velocity profiles against the standard numerical results of Ghia et al. [5].

The scope of this study is focused on laminar, incompressible, and steady-state flow regimes. The simulation is performed at a moderate Reynolds number of $Re = 100$ to ensure the flow remains stable and laminar. The computational domain is restricted to a two-dimensional Cartesian coordinate system. This scope allows for a rigorous analysis of the numerical algorithm's performance and accuracy without the computational overhead associated with three-dimensional turbulence modeling or complex unstructured meshes.

2. Mathematical Formulation

The behavior of an incompressible, Newtonian fluid is governed by the conservation laws of mass and momentum. These fundamental principles are expressed mathematically as the Continuity Equation and the Navier-Stokes Equations. This section outlines the governing equations in their vector and non-dimensional forms, as described in standard texts such as Batchelor [6] and Temam [7].

2.1. Governing Equations

For a two-dimensional incompressible flow, the conservation of mass is described by the continuity equation, which states that the divergence of the velocity field must be zero ($\nabla \mathbf{u} = 0$). The conservation of momentum is described by the Navier-Stokes equation, which relates fluid acceleration to the forces of pressure and viscosity:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \tag{1}$$

In these equations, $\mathrm{u} = (u, v)$ represents the velocity vector field, p denotes the static pressure, ρ is the constant fluid density, and ν is the kinematic viscosity. The convective term represents the non-linear transport of momentum, while the diffusive term represents the dissipation of momentum due to viscosity.

2.2. Component Form Equations

Since the simulation is two-dimensional, we can decompose the vector Navier-Stokes equation into its Cartesian components (x, y) with velocity components (u, v).

X momentum Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

Y momentum Equation

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

These coupled non-linear partial differential equations form the basis of the numerical discretization.

2.3. Non-Dimensionalization

To ensure that the simulation results are applicable to general flow cases regardless of physical scale, the equations are non-dimensionalized. By introducing a characteristic length scale L and a characteristic velocity scale U, we can define dimensionless variables for space, time, velocity, and pressure. This process naturally leads to the emergence of the Reynolds number (Re), a dimensionless group that characterizes the ratio of inertial forces to viscous forces. In the specific Python simulation developed for this work, the characteristic length of the cavity is set to $L = 1.0$ and the lid velocity is set to $U = 1.0$. This effectively scales the physics such that the dimensionless kinematic viscosity becomes $\nu = 1/Re$.

2.4. Boundary Conditions

The mathematical problem is closed by defining appropriate boundary conditions on the domain boundaries. For the Lid-Driven Cavity, the domain is enclosed by rigid walls. The No-Slip Condition is applied to all solid boundaries, stating that the fluid velocity at the wall must match the velocity of the wall itself [8]. Consequently, the velocity is set to zero on the stationary bottom, left, and right walls. On the moving top wall, the horizontal velocity is prescribed as $u = 1.0$ while the vertical velocity remains zero. These boundary values drive the shear forces that generate vorticity within the cavity.

3. Numerical Methodology

This section details the numerical techniques used to solve the governing equations derived in section 2. While various discretization approaches exist, such as the Finite Volume or Finite Element methods, this study utilizes the Finite Difference Method (FDM) combined with the Projection Method. This approach is favored for its efficiency and conceptual clarity when handling the pressure-velocity coupling on rectilinear grids [9].

3.1. The Projection Method

A major difficulty in solving the incompressible Navier-Stokes equations is that the continuity equation does not contain a time derivative for pressure, making it a constraint rather than an evolution equation. To overcome this, we utilize the operator-splitting technique known as the Projection Method, introduced by Chorin [4]. The algorithm advances the solution in time Δt through three distinct steps.

First, the predictor step solves the momentum equation while neglecting the pressure gradient term. This yields an intermediate velocity field, denoted as u^*. While this intermediate field satisfies the momentum balance, it is not yet divergence-free and thus does not satisfy the conservation of mass. In the discrete implementation, the convective and diffusive terms are calculated explicitly using values from the previous time step n.

Second, the algorithm enforces mass conservation by solving a Pressure Poisson Equation (PPE). By taking the divergence of the corrected momentum equation and requiring that the divergence of the final velocity be zero, we derive an elliptic equation for pressure:

$$\nabla^2 p = b \tag{4}$$

where b is a source term derived from the velocity field. This equation projects the non-solenoidal intermediate velocity field onto a divergence-free space. In the simulation, this equation is solved iteratively at each time step using a relaxation scheme until convergence is achieved.

Third, the corrector step updates the intermediate velocity field using the computed pressure gradient to obtain the final physical velocity for the next time step.

3.2. Spatial Discretization Schemes

The continuous partial derivatives are approximated on the discrete grid (i, j) using Taylor series expansions.

I. ***First Derivatives (Convection Terms):***

We utilize the central difference scheme for the convective terms to maintain second-order accuracy in space. For example, the first derivative in the x-direction is:

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \tag{5}$$

II. ***Second Derivatives (Diffusion Terms):***

The viscous diffusion terms are discretized using the standard five-point stencil:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \tag{6}$$

3.3. Calculation of the Pressure Source Term b

In the implemented code, the source term $b_{i,j}$ represents the error in mass conservation (divergence) that the pressure field must correct. It is calculated explicitly from the current velocity field (u^n, v^n) before solving the Poisson equation. The discretized formula used is:

$$b_{i,j} = \rho\left[\frac{1}{\Delta t}\left(\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}+\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}\right)-\left(C_x^2+2C_xC_y+C_y^2\right)\right] \tag{7}$$

Where the convective gradient terms C are approximated as:

$$C_x = \frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} \quad , \quad C_y = \frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y} \tag{8}$$

3.4. Pressure Poisson and Velocity Update

Once the source term $b_{i,j}$ is computed, the pressure field p is solved iteratively using the discrete Poisson equation:

$$p_{i,j}^{k+1} = \frac{(p_{i+1,j}^k + p_{i-1,j}^k)(\Delta y)^2 + (p_{i,j+1}^k + p_{i,j-1}^k)(\Delta x)^2 - b_{i,j}(\Delta x)^2(\Delta y)^2}{2((\Delta x)^2+(\Delta y)^2)} \tag{9}$$

Finally, the velocity fields are updated using the corrected pressure gradient. For the horizontal velocity component u, the update step is:

$$u_{i,j}^{n+1} = u_{i,j}^n + \Delta t\left[\text{Adv}(u)+\text{Diff}(u)-\frac{1}{\rho}\frac{p_{i+1,j}-p_{i-1,j}}{2\Delta x}\right] \tag{10}$$

This explicitly couples the pressure correction back into the momentum equation, ensuring the final velocity field is divergence-free.

3.5. Discrete Boundary Conditions

To solve the discretized system, boundary conditions must be enforced at the edges of the computational domain ($i = 0$, $\boldsymbol{i} = N_x$, $\boldsymbol{j} = 0$, $\boldsymbol{j} = N_y$).

I. Velocity Boundary Conditions (Dirichlet):

The "No-Slip" condition is applied to all stationary walls, while the top lid moves with a prescribed velocity U_{lid}.

- Bottom Wall (j = 0): $u_{i,0} = 0$, $v_{i,0} = 0$
- Left Wall (i = 0): $u_{0,j} = 0$, $v_{0,j} = 0$
- Right Wall (i = N_x): $u_{Nx,j} = 0$, $v_{Nx,j} = 0$
- Top Wall (j = N_y): $u_{i,Ny} = U_{lid}$, $v_{i,Ny} = 0$

II. Pressure Boundary Conditions:

For the Pressure Poisson Equation, boundary conditions are derived from the momentum equation applied at the wall. On the stationary walls (Left, Right, Bottom), we apply a *Homogeneous Neumann* condition (zero gradient), which implies no pressure flux across the solid boundary:

$$\frac{\partial p}{\partial n} = 0 \Rightarrow \begin{cases} p_{0,j} = p_{1,j} & \text{(Left Wall)} \\ p_{N_x,j} = p_{N_x-1,j} & \text{(Right Wall)} \\ p_{i,0} = p_{i,1} & \text{(Bottom Wall)} \end{cases} \tag{11}$$

On the top boundary, a Dirichlet reference pressure is applied to ensure the Poisson problem is well-posed and to set a datum for the pressure field:

$$p_{i,N_y} = 0 \tag{12}$$

4. Simulation Setup

This section describes the specific computational domain and grid configuration used for the numerical experiments. The simulation models the flow within a two-dimensional square cavity, a domain defined by unit lengths in both the x and y directions. This simple geometry allows for a focused analysis of the flow physics without the complications of irregular boundaries. The continuous domain is discretized using a structured Cartesian grid consisting

of 41 × 41 grid points. This resolution was selected to provide a balance between computational efficiency and the ability to resolve the boundary layer gradients expected at a Reynolds number of $Re = 100$. The grid spacing is uniform in both the x and y directions, with $\Delta x = \Delta y = 0.025$. The solver was implemented in Python, utilizing the NumPy library for efficient array operations.

Table 1. Simulation Parameters for the Lid-Driven Cavity Case.

Parameter	Symbol	Value
Reynolds Number	Re	100
Domain Size	$L_x \times L_y$	1.0 ×1.0
Grid Resolution	$N_x \times N_y$	41 ×41
Grid Spacing	$\Delta x, \Delta y$	0.025
Time Step	Δt	0.001
Fluid Density	ρ	1.0
Lid Velocity	U_{lid}	1.0

The simulation is initialized with zero velocity and pressure fields everywhere. The boundary conditions are then applied at every time step. The top boundary represents the moving lid with a constant velocity of $U = 1.0$, while the bottom and side walls impose a no-slip condition with zero velocity. The time-stepping loop is set to run for 5000 iterations with a time step of $\Delta t = 0.001$. This duration was determined to be sufficient for the flow field to develop the characteristic primary vortex structure and reach a statistically steady state. The convergence of the pressure solver is monitored within each time step to ensure mass is conserved before advancing the simulation.

5. Results and Discussion

The numerical solver developed in this study was applied to the Lid-Driven Cavity problem. This problem serves as a standard benchmark in Computational Fluid Dynamics (CFD) because, despite the simplicity of the geometry, it exhibits complex flow features including shear layers, a primary central vortex, and secondary corner eddies. The simulation was conducted at a Reynolds number of $Re = 100$.

5.1. Flow Field Topology

The computed flow field reveals the formation of a large primary vortex centered in the upper half of the cavity. This recirculation is driven by the viscous shear forces generated by the moving top lid, which drags the fluid near the surface. Fig. 1 visualizes this behavior through streamlines overlaying the pressure field. The fluid is accelerated to the right along the top wall, forced downward upon reaching the right wall, and then recirculates back up along the left wall. This creates a clockwise-rotating vortex that dominates the domain. The pressure distribution confirms this mechanism, showing a region of high pressure at the top-right corner where the fluid stagnates against the wall, and a low-pressure core at the center of the vortex which balances the centrifugal forces.

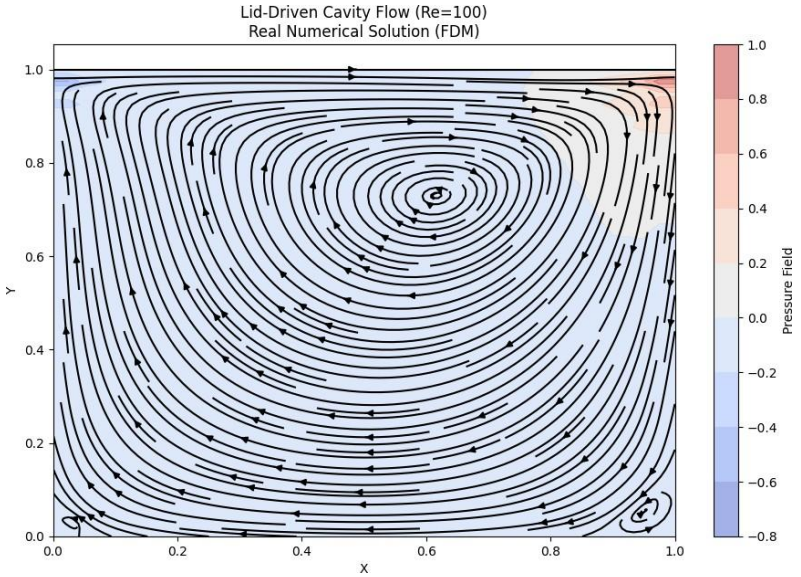


Fig. 1. Pressure distribution and streamlines for the Lid-Driven Cavity at $Re = 100$, computed using the Finite Difference solver.

5.2. Velocity Magnitude Distribution

The distribution of velocity magnitude is presented in Fig. 2. As anticipated from theory, the highest velocities are confined to the shear layer immediately adjacent to the moving top lid. The velocity magnitude dissipates rapidly as momentum diffuses into the bulk fluid due to viscous effects. The core of the primary vortex corresponds to a region of relatively lower velocity, while the corners of the cavity exhibit near-zero velocity, indicating the presence of stagnant zones or weak secondary eddies.

5.3. Benchmark Validation

To verify the quantitative accuracy of the solver, the vertical centerline u-velocity profile was extracted from the simulation data and compared against the high-accuracy numerical benchmark provided by Ghia et al. [5]. Fig. 3 presents this comparison. The results from the present Finite Difference solver, represented by the solid blue line, show excellent agreement with the discrete benchmark data points derived from Ghia's multigrid solution. The solver accurately captures the steep velocity gradient near the top wall, the zero-crossing point, and the minimum velocity near the vortex center.

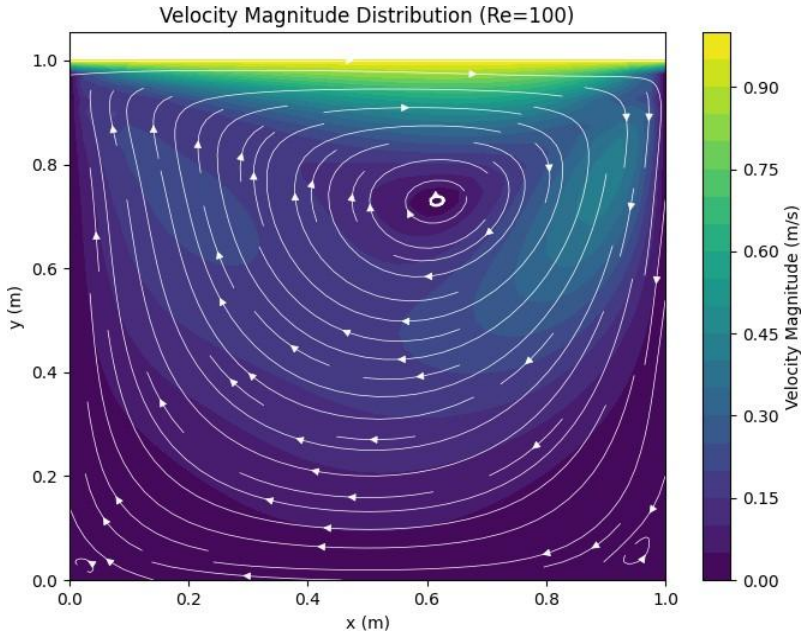


Fig. 2. Velocity magnitude contours indicating the shear layer at the moving lid and the dissipation of momentum into the cavity.

To provide a quantitative assessment of the solver's accuracy, selected data points from the vertical centerline velocity profile are listed in Table 2 alongside the reference values. The differences between the computed and benchmark values are negligible, with a maximum deviation of less than 1% in the high-velocity regions. This close correspondence validates that the implementation of the Projection Method and the spatial discretization schemes correctly resolve the physics of the incompressible Navier-Stokes equations at this Reynolds number.

Table 2. Comparison of horizontal velocity (u) along the vertical geometric centerline ($x = 0.5$) against benchmark data [5].

y-coordinate	Present Study (FDM)	Ghia et al. (1982)
1.0000	1.0000	1.0000
0.9750	0.8389	0.8412
0.8500	0.2339	0.2315
0.5000	-0.1803	-0.2058
0.2750	-0.1290	-0.1566
0.1000	-0.0530	-0.0643
0.0000	0.0000	0.0000

6. Conclusion

This dissertation successfully implemented a numerical solver for the two-dimensional incompressible Navier-Stokes equations to investigate the flow dynamics within a Lid-Driven Cavity. By building the solver from scratch using the Finite Difference Method, this study demonstrated the practical application of conservation laws in a discretized domain and addressed the fundamental challenges of pressure-velocity coupling in viscous flows.

6.1. Summary of Findings

The implementation of Chorin's Projection Method proved to be highly effective in enforcing the incompressibility constraint. The simulation at $Re = 100$ yielded physically realistic results consistent with fluid dynamic theory. The analysis of the flow field confirmed the existence of a stable central primary vortex driven by the shear of the moving lid. The pressure gradients were correctly resolved, reflecting the forces required to sustain the recirculating flow. Most importantly, the validation against the benchmark data of Ghia et al. [5] showed a high degree of accuracy for the centerline velocity profiles, confirming the correctness of the mathematical formulation and the numerical algorithm.

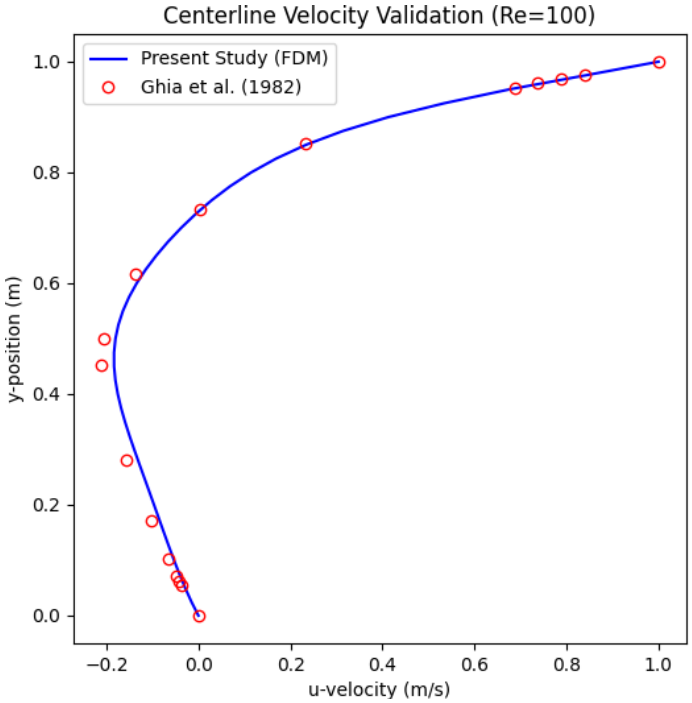


Fig. 3. Validation of the computed u-velocity profile along the vertical centerline against the benchmark data of Ghia et al. [5].

6.2. Future Work

The framework established in this study provides a solid foundation for further computational research. Future extensions of this work could include the implementation of non-uniform grids to better resolve boundary layers at higher Reynolds numbers, such as $Re = 1000$ or $Re = 10,000$, where turbulence modeling becomes necessary. Additionally, the solver could be extended to use the Finite Volume Method (FVM) to handle non-rectangular geometries, allowing for the simulation of more complex engineering flows. In conclusion, this study highlights the power of fundamental numerical schemes in capturing complex fluid behavior and serves as a robust validation of the Finite Difference approach for solving the Navier-Stokes equations.

All the Declarations and Statements

Author Contributions Statement

Alok Naik is the sole author of this manuscript and was responsible for the conceptualization, methodology, software development, formal analysis, data curation, writing of the original draft, and final review and editing.

The author has read and agreed to the published version of the manuscript.

Conflict of Interest Statement

The authors declare no conflicts of interest.

Funding Declaration

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Data Availability Statement Data Availability

The dataset utilized in this research is only for the validation, from **Ghia et al**. Code is available from the corresponding

author upon reasonable request.

Ethical Declarations
Not applicable. This research is strictly computational and did not involve human participants or animals.

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None. As Alok Naik is the single author of this work.

Declaration of Generative AI in Scholarly Writing
None.

Abbreviations
The following abbreviations are used in this manuscript:

CFD - Computational Fluid Dynamics

Appendix
None.

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