

A Novel Double Mohand-Generalized ARA Transform Coupled with Adomian Decomposition Method for Multi-Dimensional Fractional Partial Differential Equations

Kareem A. Bello*

Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Kwara, Nigeria

E-mail: bello.ak@unilorin.edu.ng

ORCID iD: <https://orcid.org/0009-0007-6270-7705>

*Corresponding author

Julius T. Adepoju

Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Kwara, Nigeria

E-mail: adepojujulius58@gmail.com

ORCID iD: <https://orcid.org/0009-0007-4399-5887>

Received: November 30, 2025; Revised: January 20, 2026; Accepted: March 10, 2026; Published: June 8, 2026

Abstract: The present research aims to introduce a brand new theoretical framework for solving multi-dimensional fractional partial differential equations (FPDEs) by developing a novel integral transform tool called the Double Mohand-Generalized ARA Transform (DM-GART). The DM-GART is a triple-integral operator that applies the Mohand transform twice—once in each spatial variable x and y and the ARA transform once in the temporal variable t ; the adjective “Double” refers specifically to the double spatial application of the Mohand transform. The theoretical properties and existence/uniqueness results of this newly developed integral transform are rigorously established in a Banach fixed-point theorem setting. The newly developed integral transform tool is then synergistically combined with the Adomian Decomposition Method (ADM) to produce a novel technique called the Coupled Double Mohand-Generalized ARA Decomposition Method (CDM-GADM). The CDM-GADM is applied for solving generalised fractional biological population equations. The technique is assessed by comparing exact solutions with N -term series solutions for $N = 4, 6$, and 8 . From the results obtained in Tables 3–10, it can be noted that with an increase in the terms from $N = 4$ to $N = 8$, the absolute errors decrease several orders of magnitude; the absolute errors for $N = 8$ are as low as 10^{-10} for $\alpha = 1.0$ at smaller values of time. The results are obtained in the form of convergent series characterized by the Mittag-Leffler function, validating the efficiency of the proposed method. A tolerance of $\epsilon = 10^{-6}$ is used as the practical stopping criterion.

Index Terms: Fractional Partial Differential Equations, Double Mohand Transform, Generalized ARA Transform, Adomian Decomposition Method, Mittag-Leffler Function, Banach Fixed-Point Theorem, Population Model

1. Introduction

The modeling of complex physical and biological systems often leads to multi-dimensional fractional partial differential equations (FPDEs), which have become indispensable tools for capturing memory effects and non-local interactions in phenomena ranging from anomalous diffusion to financial markets [1, 2]. While traditional integer-order models have served well for classical systems, the fractional calculus framework introduced by Liouville [3] and Riemann [4], and later developed by Caputo [5], provides a more accurate description of systems with hereditary properties. However, the analytical solution of these multi-dimensional FPDEs remains a significant challenge, creating a substantial research gap in developing efficient and convergent analytical methods.

Integral transform methods have emerged as powerful tools for simplifying differential equations, with recent extensions like the Mohand and ARA transforms showing particular promise [6, 7]. The Adomian Decomposition Method (ADM) has proven equally valuable for handling nonlinearities through its systematic decomposition approach

[8, 9]. While these methods have been successfully applied individually, their synergistic combination for multi-dimensional FPDEs remains largely unexplored. This work addresses this gap by developing a novel Double Mohand-Generalized ARA Transform (DM-GART), creating a unified framework that leverages the strengths of both integral transforms and decomposition methods.

Throughout this paper the name “Double Mohand-Generalized ARA Transform” (DM-GART) refers to a triple integral operator. The adjective “Double” qualifies only the spatial part: the Mohand transform is applied once in x and once in y (hence double Mohand), while the Generalised ARA transform is applied once in t . Together these constitute three nested integrations.

The primary aim of this research is to establish a comprehensive analytical framework for solving multi-dimensional FPDEs using our proposed hybrid method. Specific objectives include: developing the fundamental properties of the Double Mohand-Generalized ARA Transform (DM-GART), deriving transformation formulas for fractional partial derivatives, implementing the coupled decomposition method, and rigorously establishing the existence and uniqueness of solutions through Banach fixed-point theory. We demonstrate that this integrated approach offers significant advantages over traditional methods, providing both computational efficiency and mathematical rigour.

2. Preliminaries

This section presents the fundamental definitions and theorems required for the development of our method. All functions are assumed to be of exponential order on $[0, \infty)$, that is, there exist constants $M, a, b, c \geq 0$ such that $|u(x, y, t)| \leq Me^{ax+by+ct}$ for all $x, y, t \geq 0$ and are sufficiently smooth for all integrals to converge absolutely.

2.1. Mittag-Leffler Functions

Definition 1 (One-Parameter Mittag-Leffler Function) The Mittag-Leffler function $E_\alpha(z)$ is defined for $\alpha > 0$ by the series [10, 11]:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C} \quad (1)$$

where $\Gamma(\cdot)$ denotes the Gamma function. This function generalised the exponential function since $E_1(z) = e^z$.

Definition 2 (Two-Parameter Mittag-Leffler Function) The generalised Mittag-Leffler function is defined for $\alpha, \beta > 0$ by [12, 11]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C} \quad (2)$$

Note that $E_{\alpha,1}(z) = E_\alpha(z)$ and $E_{1,1}(z) = e^z$.

Theorem 1 (Laplace Transform of Mittag-Leffler Functions) For $\alpha, \beta > 0$ and $\Re(s) > |\lambda|^{\frac{1}{\alpha}}$, the Laplace transform satisfies [1, 13]:

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha)\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda} \quad (3)$$

This relationship is fundamental for solving linear fractional differential equations.

2.2. Fractional Calculus

Definition 3 (Riemann–Liouville Fractional Integral) Let f be piecewise continuous on $(0, \infty)$ and integrable on every finite subinterval of $[0, \infty)$. The Riemann–Liouville fractional integral of order $\alpha > 0$ is

$$I_{0+}^\alpha f(t) = \left(\frac{1}{\Gamma(\alpha)}\right) \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0 \quad (4)$$

For $\alpha = 0$ it reduces to the identity: $I_{0+}^0 f(t) = f(t)$.

This definition generalizes the Cauchy repeated-integration formula:

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} f(\tau_n) d\tau_n = \left(\frac{1}{(n-1)!}\right) \int_0^t (t - \tau)^{n-1} f(\tau) d\tau \quad (5)$$

Replacing $(n-1)!$ with $\Gamma(n)$ yields the Riemann–Liouville integral.

Definition 4 (Caputo Fractional Derivative) For $\alpha > 0$ with $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, the Caputo fractional derivative of order α is

$$D_t^\alpha f(t) = I_{0+}^{m-\alpha} f^{(m)}(t) = \left(\frac{1}{\Gamma(m-\alpha)} \right) \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau \quad (6)$$

Provided the integral exists. For $\alpha = m$, $D_t^m f(t) = f^{(m)}(t)$.

3. Method

Definition 5 (Mohand Transform) Let $u(t)$ be a continuous function of exponential order on $[0, \infty)$. The Mohand transform of $u(t)$ is [7]:

$$M[u(t)](v) = v^2 \int_0^\infty e^{-vt} u(t) dt, \quad v > 0 \quad (7)$$

Definition 6 (ARA Transform of Order n) Let $f(t)$ be a continuous function on $(0, \infty)$. The ARA transform of order n is [6]:

$$A_n[f(t)](s) = s \int_0^\infty t^{n-1} e^{-st} f(t) dt, \quad s > 0, n \in \mathbb{N} \quad (8)$$

For $n = 1$ this reduces to $A_n[f](s) = s \int_0^\infty e^{-st} f(t) dt$, which equals s times the Laplace transform; that is, $A_1[f](s) = s\mathcal{L}[f](s)$.

Definition 7 (Inverse Mohand Transform) Let $F(v) = M[u(t)](v)$ be the Mohand transform of a continuous function $u(t)$ of exponential order on $[0, \infty)$. The inverse Mohand transform is given by [7]:

$$u(t) = M^{-1}[F(v)](t) = \left(\frac{1}{2\pi i} \right) \int_{\gamma-i\infty}^{\gamma+i\infty} \left(\frac{e^{vt}}{v^2} \right) F(v) dv, \quad \gamma > 0 \quad (9)$$

where the Bromwich contour $Re(v) = \gamma$ lies to the right of all singularities of $\frac{F(v)}{v^2}$. Equivalently, since $M[u](v) = v^2 \mathcal{L}[u](v)$, the inversion reduces to a standard Laplace inversion:

$$u(t) = \mathcal{L}^{-1} \left[\frac{F(v)}{v^2} \right] (t) \quad (10)$$

Definition 8 (Inverse ARA Transform of Order n) Let $G(s) = A_n[f(t)](s)$ be the ARA transform of order n of a continuous function $f(t)$ on $(0, \infty)$. The inverse ARA transform is [6]:

$$f(t) = A_n^{-1}[G(s)](t) = \left(\frac{1}{2\pi i} \right) \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{st}}{s t^{n-1}} G(s) ds, \quad \sigma > 0 \quad (11)$$

where $Re(s) = \sigma$ lies to the right of all singularities of $G(s)$. For the special case $n = 1$, since $A_1[f](s) = s\mathcal{L}[f](s)$, the inversion reduces to:

$$f(t) = A_1^{-1}[G(s)](t) = \mathcal{L}^{-1} \left[\frac{G(s)}{s} \right] (t) = \left(\frac{1}{2\pi i} \right) \int_{\sigma-i\infty}^{\sigma+i\infty} G(s) ds \quad (12)$$

The factor $\frac{1}{s}$ in (12) directly cancels the extra factor of s introduced by the ARA definition, and is not an ad hoc device.

Table 1. Notation table: fixed parameter roles and function-space conventions.

Symbol	Description
x, y	Spatial independent variables, $x \geq 0, y \geq 0$
t	Temporal independent variable, $t \geq 0$
v_1	Mohand transform parameter dual to x ; requires $v_1 > a$
v_2	Mohand transform parameter dual to y ; requires $v_2 > b$

s	ARA transform parameter dual to t ; requires $s > c$
α	Fractional order in t , $0 < \alpha \leq 1$ (unless stated otherwise)
β	Fractional order in x , $0 < \beta \leq 1$
γ	Fractional order in y , $0 < \gamma \leq 1$
$U(v_1, v_2, s)$	$:= M_x M_y A_t[u(x, y, t)]$
$E(a, b, c)$	Class of functions satisfying $ u \leq M e^{ax+by+ct}$

3.1. The Double Mohand-Generalized ARA Transform (DM-GART)

This subsection presents the core theoretical contribution: the Double Mohand-Generalized ARA Transform (DM-GART) for functions of three variables.

Definition 9 (DM-GART) Let $u(x, y, t) \in E(a, b, c)$ for some $M, a, b, c \geq 0$. The DM-GART of u , denoted $M_x M_y A_t[u(x, y, t)]$, is defined as:

$$M_x M_y A_t[u(x, y, t)]((v_1^2, v_2^2, s)) = v_1^2 v_2^2 s \int_0^\infty \int_0^\infty \int_0^\infty e^{-v_1 x - v_2 y - st} u(x, y, t) dx dy dt \quad (13)$$

where $v_1 > a, v_2 > b, s > c$ ensure absolute convergence.

The DM-GART is a triple integral transform obtained by composing three single-variable transforms in sequence:

$$M_x M_y A_t = (\text{Mohand in } x) \circ (\text{Mohand in } y) \circ (\text{ARA}^1 \text{ in } t) \quad (14)$$

The weighting factors from each component transform are v_1^2 (from M_x), v_2^2 (from M_y), and s (from A_t in t), yielding the overall prefactor $v_1^2 v_2^2 s$ in (13).

The transform is linear: for constants μ, ν and functions $u_1, u_2 \in E(a, b, c)$,

$$M_x M_y A_t[\mu u_1 + \nu u_2] = \mu M_x M_y A_t[u_1] + \nu M_x M_y A_t[u_2] \quad (15)$$

Definition 10 (Inverse DM-GART) Functions $u \in E(a, b, c)$, meaning there exist constants $M, a, b, c \geq 0$ such that

$$|u(x, y, t)| \leq M e^{ax+by+ct} \quad \text{for all } x, y, t \geq 0 \quad (16)$$

Under condition (16) the integral

$$\begin{aligned} U(v_1^2, v_2^2, s) &:= M_x M_y A_t[u]((v_1^2, v_2^2, s)) = U(v_1^2, v_2^2, s) \\ &= v_1^2 v_2^2 s \int_0^\infty \int_0^\infty \int_0^\infty e^{-v_1 x - v_2 y - st} u(x, y, t) dx dy dt \end{aligned} \quad (17)$$

converges absolutely for $v_1 > a, v_2 > b, s > c$, because the integrand satisfies

$$|e^{-v_1 x - v_2 y - st} u(x, y, t)| \leq M e^{-(v_1-a)x} e^{-(v_2-b)y} e^{-(s-c)t},$$

and each one-dimensional factor is integrable on $[0, \infty)$. Fubini's theorem [14] then justifies representing (17) as an iterated integral in any order.

For fixed $v_2 > b$ and $s > c$, the function $v_1 \mapsto \frac{U(v_1^2, v_2^2, s)}{v_1^2}$ is holomorphic in the half-plane $\text{Re}(v_1) > a$. The Bromwich–Mellin inversion theorem gives:

$$\hat{U}(x, v_2, s) := M_x^{-1}[U((v_1^2, v_2^2, s))](x) = \left(\frac{1}{2\pi i}\right) \int_{\gamma_1-i\infty}^{\gamma_1+i\infty} \left(\frac{e^{v_1 x}}{v_1^2}\right) U(v_1, v_2, s) dv_1, \quad \gamma_1 > a \quad (18)$$

For fixed $x \geq 0$ and $s > c$, an identical application of the Bromwich–Mellin theorem gives:

$$\tilde{U}(x, y, s) := M_y^{-1}[\hat{U}(x, v_2, s)](y) = \left(\frac{1}{2\pi i}\right) \int_{\gamma_2-i\infty}^{\gamma_2+i\infty} \left(\frac{e^{v_2 y}}{v_2^2}\right) U(x, v_2, s) dv_2, \quad \gamma_2 > b \quad (19)$$

Recall from Definition 3.2 that the first-order ARA transform satisfies $A_1[f](s) = s\mathcal{L}[f](s)$. Consequently $\tilde{U}(x, y, s) = s\mathcal{L}[u(\cdot, \cdot, \cdot)](s)$, and inversion of A_t reduces to inversion of the Laplace transform after dividing by s :

$$u(x, y, t) = A_t^{-1}[\tilde{U}(x, y, s)](t) = \mathcal{L}^{-1}\left[\frac{\tilde{U}(x, y, s)}{s}\right](t) = \left(\frac{1}{2\pi i}\right) \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{e^{st}}{s}\right) U(x, y, s) ds, \quad \sigma > c \quad (20)$$

3.2. Separability Property

Theorem 2 (Separability Property) Let $f(x), g(y), h(t)$ be functions in the exponential-order class for which the single-variable transforms exist. If $u(x, y, t) = f(x) g(y) h(t)$, then

$$M_x M_y A_t [f(x)g(y)h(t)](v_1, v_2, s) = M_x[f](v_1) \cdot M_y[g](v_2) \cdot A_t[h](s) \quad (21)$$

Proof: Since f, g, h are of exponential order, the integrand is absolutely integrable over $[0, \infty)^3$; Fubini's theorem [14] therefore permits the following factorisation:

$$\begin{aligned} M_x M_y A_t [fgh] &= v_1^2 v_2^2 s \int_0^\infty \int_0^\infty \int_0^\infty e^{-v_1 x - v_2 y - st} f(x) g(y) h(t) dx dy dt \\ &= \left(v_1^2 \int_0^\infty e^{-v_1 x} f(x) dx\right) \left(v_2^2 \int_0^\infty e^{-v_2 y} g(y) dy\right) \left(s \int_0^\infty e^{-st} h(t) dt\right) \\ &= M_x[f](v_1) M_y[g](v_2) A_t[h](s) \end{aligned} \quad (22)$$

the weighting factors v_1^2, v_2^2 come from the Mohand transforms in x and y respectively, while the factor s comes from A_1 in t . These are distinct and must not be interchanged.

3.3. Transform of Elementary Functions

All entries in Table 2 below are derived directly from Definition 3.5 using Fubini's theorem and standard single-variable Mohand and ARA transform pairs.

Table 2. DM-GART of Elementary Functions.

No.	Function $f(x, y, t)$	Transform $M_x M_y A_t[f](v_1, v_2, s)$
1	1	$v_1 v_2$
2	$e^{-(ax + by + ct)}$	$\frac{v_1 v_2}{v_1^2 + a^2} \cdot \frac{v_2^2 + b^2}{v_2^2 + b^2} \cdot \frac{cs}{s^2 + c^2}$
3	$x^m y^n t^p \quad (m, n, p > -1)$	$\frac{\Gamma(m+1)\Gamma(n+1)\Gamma(p+1)}{v_1^{m+1} v_2^{n+1} s^{p+1}}$
4	$\delta(x-a)\delta(y-b)\delta(t-c)$	$v_1^2 v_2^2 s e^{-v_1 a - v_2 b - sc}$
5	$\sin(ax)\sin(by)\sin(ct)$	$\frac{av_1^2}{v_1^2 + a^2} \cdot \frac{bv_2^2}{v_2^2 + b^2} \cdot \frac{cs}{s^2 + c^2}$
6	$\cos(ax)\cos(by)\cos(ct)$	$\frac{v_1^3}{v_1^2 + a^2} \cdot \frac{v_2^3}{v_2^2 + b^2} \cdot \frac{s^2}{s^2 + c^2}$
7	$e^{ax+by+ct}$	$\frac{(v_1^2 v_2^2 s)}{(v_1 - a)(v_2 - b)(s - c)}, \quad v_1 > a, \quad v_2 > b, \quad s > c$
8	$\sinh(ax)\sinh(by)\sinh(ct)$	$\frac{av_1^3}{v_1^2 - a^2} \cdot \frac{bv_2^3}{v_2^2 - b^2} \cdot \frac{cs^2}{s^2 - c^2} \quad \forall \quad v_1 > a , \quad v_2 > b , \quad s > c $

3.4. Transform of Partial Derivatives

Theorem 3 (First-Order Partial Derivatives) Let $U(v_1, v_2, s) = M_x M_y A_t[u(x, y, t)]$ with $u \in E(a, b, c)$. Then:

$$M_x M_y A_t \left[\frac{\partial u}{\partial x} \right] = v_1 U - v_1^2 M_y A_t [u(0, y, t)] \quad (23)$$

$$M_x M_y A_t \left[\frac{\partial u}{\partial y} \right] = v_2 U - v_2^2 M_y A_t [u(x, 0, t)] \quad (24)$$

$$M_x M_y A_t \left[\frac{\partial u}{\partial t} \right] = sU - s M_y M_t [u(x, y, 0)] \quad (25)$$

Proof: We prove (23); (24) and (25) follow identically. Since $u \in E(a, b, c)$, Fubini's theorem permits isolating the x -integration:

$$M_x M_y A_t \left[\frac{\partial u}{\partial x} \right] = v_1^2 v_2^2 s \int_0^\infty \int_0^\infty e^{-v_2 y - st} \int_0^\infty e^{-v_1 x} \partial_x u \, dx \, dy \, dt \quad (26)$$

$$\text{Integration by parts in } x \left(p = e^{-v_1 x}, dq = \partial_x u \, dx \right), \text{ with } \lim_{x \rightarrow \infty} e^{-v_1 x} u = 0 \text{ since } v_1 > a: \quad (27)$$

$$\int_0^\infty e^{-v_1 x} \partial_x u \, dx = -u(0, y, t) + v_1 \int_0^\infty e^{-v_1 x} u \, dx$$

Substituting and identifying the resulting integrals:

$$M_x M_y A_t \left[\frac{\partial u}{\partial x} \right] = -v_1^2 v_2^2 s \int_0^\infty \int_0^\infty e^{-v_2 y - st} u(0, y, t) dy \, dt + v_1 v_1^2 v_2^2 s \int_0^\infty \int_0^\infty e^{-v_2 y - st} u dx dy dt \quad (28)$$

Which is (23)

Theorem 4 (Higher-Order Partial Derivatives) For integers $n, m, p \geq 1$:

$$M_x M_y A_t \left[\frac{\partial^n u}{\partial x^n} \right] = v_1^n U - \sum_{k=1}^n v_1^{n+2-k} M_y A_t \left[\frac{\partial^{k-1} u(0, y, t)}{\partial x^{k-1}} \right] \quad (29)$$

$$M_x M_y A_t \left[\frac{\partial^m u}{\partial y^m} \right] = v_2^m U - \sum_{k=1}^m v_2^{m+2-k} M_x A_t \left[\frac{\partial^{k-1} u(x, 0, t)}{\partial y^{k-1}} \right] \quad (30)$$

$$M_x M_y A_t \left[\frac{\partial^p u}{\partial t^p} \right] = s^p U - \sum_{k=1}^p v_1^{p-k+1} M_x M_y \left[\frac{\partial^{k-1} u(x, y, 0)}{\partial t^{k-1}} \right] \quad (31)$$

Proof: We prove (29) by induction on n ; (30)–(31) follow by the same argument.

Base case ($n = 1$): (29) reduces to (23).

Inductive step: Assume (29) holds for some $n \geq 1$. Write $\partial_x^{n+1} u = \partial_x (\partial_x^n u)$ and apply (23) to $\partial_x^n u$:

$$M_x M_y A_t \left[\frac{\partial^{n+1} u}{\partial x^{n+1}} \right] = v_1 \left(v_1^n U - \sum_{k=1}^n v_1^{n+2-k} M_y A_t \left[\frac{\partial^{k-1} u(0, y, t)}{\partial x^{k-1}} \right] \right) - v_1^2 M_y A_t \left[\frac{\partial^n u(0, y, t)}{\partial x^n} \right] \quad (32)$$

Distributing v_1 gives (29) with $n + 1$.

Theorem 5 (Mixed Partial Derivatives) For integers $n, m, p \geq 1$:

$$M_x M_y A_t \left[\frac{\partial^{n+m+p} u}{\partial x^n \partial y^m \partial t^p} \right] = v_1^n v_2^m s^p U - \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^p v_1^{(n+2-i)} v_2^{(m+2-j)} s^{p-k+1} B_{\{ijk\}} \quad (33)$$

where the boundary coefficient B_{ijk} is the scalar

$$B_{ijk} := \frac{\partial^{i+j+k-3} u}{\partial x^{i-1} \partial y^{j-1} \partial t^{k-1}} \Big|_{(x=0, y=0, t=0)}, \quad i = 1, \dots, n; \quad j = 1, \dots, m; \quad k = 1, \dots, p \quad (34)$$

That is, B_{ijk} is the $(i + j + k - 3)$ – th order mixed partial derivative of u evaluated at the corner $(0, 0, 0)$.

Proof: Apply Theorems 3.2–3.3 in succession.

Step 1 (in x): By (29),

$$M_x M_y A_t \left[\frac{\partial^n u}{\partial x^n} \right] = v_1^n U - \sum_{i=1}^n v_1^{n+2-i} M_x A_y \left[\frac{\partial^{i-1} u(0, y, t)}{\partial x^{i-1}} \right] \quad (35)$$

Step 2 (in y): Apply (30) to each boundary term in (35):

$$M_x M_y A_t \left[\frac{\partial^{n+m} u}{\partial x^n \partial y^m} \right] = v_1^n v_2^m U - \sum_{i=1}^n \sum_{j=1}^m v_1^{(n+2-i)} v_2^{(m+2-j)} A_t \left[\frac{\partial^{i+j-2} u}{\partial x^{(i-1)} \partial y^{j-1}} (0, 0, t) \right] \quad (36)$$

Step 3 (in t): Apply (31) to each boundary term in (36), yielding (33)–(34).

$$M_x M_y A_t \left[\frac{\partial^{n+m} u}{\partial x^n \partial y^m} \right] = v_1^n v_2^m s^p U - \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^p v_1^{(n+2-i)} v_2^{(m+2-j)} s^{p-k+1} \frac{\partial^{i+j-2} u(0,0,t)}{\partial x^{(i-1)} \partial y^{(j-1)} \partial t^{(k-1)}} \Big|_{(x=0, y=0, t=0)} \quad (37)$$

3.5. Transform of Fractional Derivatives

Theorem 6 (Caputo Fractional Derivative in Time) Let $u \in E(a, b, c)$ and let $D_t^\alpha u$ denote the Caputo derivative of order $\alpha > 0$ with $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$. Then:

$$M_x M_y A_t [D_t^\alpha u](v_1, v_2, s) = s^\alpha U(v_1, v_2, s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} M_x M_y \left[\frac{\partial^k u(x, y, 0)}{\partial t^k} \right] \quad (38)$$

Proof: We build the result in three explicit sub-steps.

Laplace transform of $D_t^\alpha u$. The standard Laplace transform formula for the Caputo derivative [1] states:

$$L\{D_t^\alpha u(x, y, \cdot)\}(s) = s^\alpha L[u(x, y, \cdot)](s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} \frac{\partial^k u(x, y, 0)}{\partial t^k}, \quad \text{Re}(s) > c \quad (39)$$

ARA transform of $D_t^\alpha u$. Since $A_1[f](s) = sL[f](s)$, multiply (39) by s :

$$A^1[D_t^\alpha u(x, y, \cdot)](s) = s^{\alpha+1} L[u](s) - \sum_{k=0}^{m-1} s^{\alpha-k} \frac{\partial^k u}{\partial t^k}(x, y, 0) \quad (40)$$

Now observe that $s^{\alpha+1} L[u](s) = s^\alpha \cdot sL[u](s) = s^\alpha A_1[u(x, y, \cdot)](s)$. Substituting back:

$$A^1[D_t^\alpha u(x, y, \cdot)](s) = s^\alpha A_1[u(x, y, \cdot)](s) - \sum_{k=0}^{m-1} s^{\alpha-k} \frac{\partial^k u}{\partial t^k}(x, y, 0) \quad (41)$$

Applying $M_x M_y$ to (41). Using linearity:

$$M_x M_y A_t [D_t^\alpha u] = s^\alpha U(v_1, v_2, s) - \sum_{k=0}^{m-1} M_x M_y \left[\frac{\partial^k u}{\partial t^k}(x, y, 0) \right] \quad (42)$$

Remark 3.1 For the case $0 < \alpha \leq 1$ (i.e. $m = 1$) used throughout Section 4, the sum in (38) reduces to a single term:

$$M_x M_y A_t [D_t^\alpha u] = s^\alpha U - s^{\alpha-1} M_x M_y [u(x, y, 0)] \quad (43)$$

Theorem 7 (Caputo Fractional Derivatives in Space) Let $u \in E(a, b, c)$, $n - 1 < \beta \leq n$, and $p - 1 < \gamma \leq p$. Then:

$$M_x M_y A_t [D_x^\beta u] = v_1^\beta U - \sum_{k=0}^{n-1} v_1^{(\beta-k-1)} M_y A_t \left[\frac{\partial^k u}{\partial x^k}(0, y, t) \right] \quad (44)$$

$$M_x M_y A_t [D_y^\gamma u] = v_2^\gamma U - \sum_{k=0}^{p-1} v_2^{(\gamma-k-1)} M_x A_t \left[\frac{\partial^k u}{\partial x^k}(0, y, t) \right] \quad (45)$$

Proof: We derive (44) in full; (45) follows by an identical argument.

Mohand transform of $D_x^\beta u$ in x . The Mohand transform M_x is structurally v_1^2 times the Laplace transform in x . The standard Laplace transform formula for the Caputo derivative in x [1] gives:

$$\mathcal{L}_x [D_x^\beta u(\cdot, y, t)](v_1) = v_1^\beta \mathcal{L}_x [u(\cdot, y, t)](v_1) - \sum_{k=0}^{n-1} \frac{\partial^k u}{\partial x^k}(0, y, t), \quad v_1 > a \quad (46)$$

Multiplying both sides by v_1^2

$$M_x [D_x^\beta u(\cdot, y, t)](v_1) = v_1^{(\beta+2)} \mathcal{L}_x [u](v_1) - \sum_{k=0}^{n-1} v_1^{(\beta-k+1)} \frac{\partial^k u}{\partial x^k}(0, y, t) \quad (47)$$

Since $M_x[u](v_1) = v_1^2 \mathcal{L}_x[u](v_1)$, we have $v_1^{(\beta+2)} \mathcal{L}_x[u] = v_1 \beta \cdot M_x[u]$, giving:

$$M_x[D^\beta_x u(\cdot, y, t)](v_1) = v_1^2 \beta M_x[u(\cdot, y, t)](v_1) - \sum_{k=0}^{n-1} v_1^{(\beta-k-1)} \cdot v_1^2 \frac{\partial^k u}{\partial x^k}(0, y, t) \quad (48)$$

Apply $M_y A_t$ to (48). By linearity:

$$\begin{aligned} & M_x M_y A_t[D^\beta_x u](v_1, v_2, s) \\ &= v_1^\beta M_x M_y A_t[u](v_1, v_2, s) - \sum_{k=0}^{n-1} v_1^{(\beta-k-1)} v_2^{12} M_y A_t \left[\frac{\partial^k u}{\partial x^k}(0, y, t) \right] (v_2, s) \end{aligned} \quad (49)$$

Recognizing $U = M_x M_y A_t[u]$ and simplifying yields (44).

For $0 < \beta \leq 1$ and $0 < \gamma \leq 1$ (i.e. $n = p = 1$), the sums in (44)–(45) each reduce to a single boundary term:

$$M_x M_y A_t[D^\beta_x u] = v_1^\beta U - v_1^{(\beta-1)} M_y A_t[u(0, y, t)] \quad (50)$$

$$M_x M_y A_t[D^\gamma_y u] = v_2^\gamma U - v_2^{(\gamma-1)} M_x A_t[u(x, 0, t)] \quad (51)$$

4. Coupled Double Mohand-Generalized Ara Decomposition Method (CDM-GADM)

4.1. Procedure

We consider the following general nonlinear fractional partial differential equation (FPDE) in two spatial dimensions:

$$D_t^\alpha u + D_x^\beta u + D_y^\gamma u + Lu + Nu = q(x, y, t), \quad x, y, t \geq 0 \quad (52)$$

where the fractional orders satisfy $0 < \alpha, \beta, \gamma \leq 1$. Here, D_t^α, D_x^β , and D_y^γ denote Caputo fractional derivatives of orders α, β , and γ with respect to t, x , and y , respectively (see Definition 4). The operator L is a linear differential operator, N is a nonlinear operator, and $q(x, y, t)$ is a known source term.

Equation (52) is supplemented by the following initial and boundary conditions:

$$u(x, y, 0) = f(x, y) \quad (53)$$

$$u(0, y, t) = g(y, t) \quad (54)$$

$$u(x, 0, t) = h(y, t) \quad (55)$$

where f, g , and h are prescribed functions. The goal of the CDM-GADM is to construct the solution $u(x, y, t)$ as a convergent infinite series $u = \sum_{k=0}^{\infty} u_k$.

Theorem 8 (CDM-GADM) The solution $u = \sum_{k=0}^{\infty} u_k$ of the problem (52)–(55) is constructed by the following seven steps. The first step is to take the DM-GART (Definition 7) of every term in equation (52). Since the DM-GART is a linear operator, it acts on each term separately:

$$M_x M_y A_t[D_t^\alpha u] + M_x M_y A_t[D_x^\beta u] + M_x M_y A_t[D_y^\gamma u] + M_x M_y A_t[Lu + Nu] = Q(v_1, v_2, s) \quad (56)$$

where $Q(v_1, v_2, s) = M_x M_y A_t[q(x, y, t)]$ is the transform of the source term, and $U(v_1, v_2, s)$ denotes the DM-GART of $u(x, y, t)$. We now use the transform formulas derived in Theorem 3.5 and its spatial counterpart to replace each fractional derivative term. For $0 < \alpha, \beta, \gamma \leq 1$ (i.e. $m = n = p = 1$), these reduce to:

$$\begin{aligned} M_x M_y A_t[D_t^\alpha u] &= s^\alpha U - s^{\alpha-1} M_x M_y [f] \\ M_x M_y A_t[D_x^\beta u] &= v_1^\beta U - v_1^{(\beta-1)} M_y A_t[g] \\ M_x M_y A_t[D_y^\gamma u] &= v_2^\gamma U - v_2^{(\gamma-1)} M_x A_t[h] \end{aligned}$$

Substituting into (56) and collecting terms involving U on the left-hand side gives:

$$(s^\alpha + v_1^\beta + v_2^\gamma) U - s^{\alpha-1} M_x M_y [f] - v_1^{\beta-1} M_y A_t [g] - v_2^{\gamma-1} M_x A_t [h] + M_x M_y A_t [Lu + Nu] = Q \quad (57)$$

Rearranging to isolate U:

$$U = \frac{1}{s^\alpha + v_1^\beta + v_2^\gamma} \left[s^{\alpha-1} M_x M_y [f] + v_1^{\beta-1} M_y A_t [g] + v_2^{\gamma-1} M_x A_t [h] + Q - M_x M_y A_t [Lu + Nu] \right] \quad (58)$$

Applying the inverse DM-GART (Definition 3.6) to (58), and using linearity to invert term by term, yields:

$$\begin{aligned} u = & M_x^{-1} M_y^{-1} A_t^{-1} \left\{ \frac{s^{\alpha-1}}{s^\alpha + v_1^\beta + v_2^\gamma} M_x M_y [f] \right\} + \\ & M_x^{-1} M_y^{-1} A_t^{-1} \left\{ \frac{v_1^{\beta-1}}{s^\alpha + v_1^\beta + v_2^\gamma} M_y A_t [g] \right\} + M_x^{-1} M_y^{-1} A_t^{-1} \left\{ \frac{v_2^{\gamma-1}}{s^\alpha + v_1^\beta + v_2^\gamma} M_x A_t [h] \right\} \\ & + M_x^{-1} M_y^{-1} A_t^{-1} \left\{ \frac{Q}{s^\alpha + v_1^\beta + v_2^\gamma} \right\} - M_x^{-1} M_y^{-1} A_t^{-1} \left\{ M_x M_y \frac{A_t [Lu + Nu]}{s^\alpha + v_1^\beta + v_2^\gamma} \right\} \end{aligned} \quad (59)$$

The nonlinear operator $N[u]$ cannot be handled directly in a recursive scheme. Following Adomian [8], we expand it as an infinite series of specially constructed polynomials:

$$N[u] = \sum_{k=0}^{\infty} A_k(u_0, u_1, \dots, u_k), \quad (60)$$

where the k -th Adomian polynomial A_k depends only on the components $u_0, u_1, \dots, u_k$ already computed, and is defined by:

$$A_k = \left(\frac{1}{k!} \right) \frac{d^k}{d\lambda^k} \left[N \left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right] \Big|_{\lambda=0} \quad (61)$$

This formula ensures that A_k is expressed entirely in terms of components known before computing u_{k+1} , making the recursive scheme self-consistent. Explicit polynomials for $N[u] = u^2$ (used in Example 2). Applying (61) with $N[u] = u^2$ gives:

$$\begin{aligned} A^0 &= u^{02}, \\ A^1 &= 2u^0 u^1, \\ A^2 &= 2u^0 u^2 + u^{12}, \\ A^3 &= 2u^0 u^3 + 2u^1 u^2, \\ A^4 &= 2u_0 u_4 + 2u_1 u_3 + u_2^2 \end{aligned} \quad (62)$$

Each polynomial A_k contains only products of components with indices summing to k , consistent with the structure implied by (61). Comparing (59) with the series representation $u = \sum_{k=0}^{\infty} u_k$ and (60), we identify the following recursive scheme. The zeroth component u_0 collects all contributions from the data (initial condition, boundary conditions, and source term) and contains no unknown components:

$$u^0 = M_x^{-1} M_y^{-1} A_t^{-1} \left\{ \frac{s^{\alpha-1}}{s^\alpha + v_1^\beta + v_2^\gamma} M_x M_y [f] + \frac{v_1^{\beta-1}}{s^\alpha + v_1^\beta + v_2^\gamma} M_y A_t [g] + \frac{v_2^{\gamma-1}}{s^\alpha + v_1^\beta + v_2^\gamma} M_x A_t [h] + \frac{Q}{s^\alpha + v_1^\beta + v_2^\gamma} \right\} \quad (63)$$

Each subsequent component u_{k+1} is then computed from the previous component u_k and the Adomian polynomial A_k :

$$u_{k+1} = -M_x^{-1} M_y^{-1} A_t^{-1} \left\{ \frac{M_x M_y A_t [Lu_k + A_k]}{s^\alpha + v_1^\beta + v_2^\gamma} \right\}, \quad k = 0, 1, 2, \dots \quad (64)$$

The minus sign in (64) reflects the fact that the linear and nonlinear terms appear with a negative sign in (59). At each step, only u_k and A_k are needed. After computing as many components as needed, the solution is obtained by summing the series:

$$u(x, y, t) = \sum_{k=0}^{\infty} u_k(x, y, t) \quad (65)$$

In practice, a finite N -term partial sum $u^N(x, y, t) = \sum_{k=0}^N u_k(x, y, t)$ is used as the approximation, and the series is terminated when the stopping criterion (79) is satisfied.

4.2. Existence, Uniqueness, and Convergence Analysis

We establish well-posedness of the CDM-GADM in a weighted Bochner–Lebesgue space that captures the temporal growth induced by the Mittag-Leffler kernel. We fix $T > 0$ and let $\Omega = [0, L_1] \times [0, L_2] \subset \mathbb{R}^2$. For $\mu > 0$, define the exponentially weighted Bochner space

$$X_\mu = \{u : \Omega \times [0, T] \rightarrow R \mid \|u\|_\mu < \infty\}, \quad \|u\|_\mu = \sup_{t \in [0, T]} \exp(-\mu t) \int_\Omega |u(x, y, t)| dx dy \quad (66)$$

with the standard $L_1(\Omega)$ norm in the spatial fibre. $(X_\mu, \|\cdot\|_\mu)$ is a Banach space for every $\mu > 0$. The abstract Cauchy problem associated with (52) is recast as the Volterra-type operator equation

$$u = K[u], \quad (K[u])(x, y, t) := u_0(x, y, t) + \left(\frac{1}{\Gamma(\alpha)} \right) \int_0^t (t - \tau)^{\alpha-1} F[u](x, y, \tau) d\tau \quad (67)$$

where u_0 contains all data terms (initial/boundary values and source) exactly as given in (63), and

$$F[u] := -D_x^\beta u - D_y^\gamma u - Lu - Nu + q \quad (68)$$

Equation (67) is the mild formulation obtained by applying I_t^α to (52) and is equivalent for sufficiently smooth u to the original FPDE via the Caputo Riemann Liouville identity $I_t^\alpha D_t^\alpha u = u - u(\cdot, \cdot, 0)$

4.2.1. Assumptions

The following conditions on the operators in (68) are assumed throughout.

Assumption 4.1 (Boundedness of Spatial Fractional Operators) The Caputo operators D_x^β and D_y^γ are bounded on $L^1(\Omega)$ for $0 < \beta, \gamma \leq 1$, with constants $C_\beta, C_\gamma > 0$ satisfying

$$\|D_x^\beta u(\cdot, y, t)\|_{L_x^1} \leq \frac{C_\beta}{\Gamma(2-\beta)} \|u(\cdot, y, t)\|_{L_x^1}, \quad \|D_y^\gamma u(x, \cdot, t)\|_{L_y^1} \leq \frac{C_\gamma}{\Gamma(2-\gamma)} \|u(x, \cdot, t)\|_{L_y^1} \quad (69)$$

These estimates follow from the Hardy-Littlewood-Sobolev inequality applied to the Riemann-Liouville kernel $(t - \tau)^{-\beta}$; see [1].

Assumption 4.2 (Lipschitz Continuity of the Nonlinearity) There exists a constant $L_N > 0$ such that for all $u, v \in X_\mu$,

$$\|N[u] - N[v]\|_{L^1(\Omega)} \leq L_N \|u - v\|_{L^1(\Omega)} \quad (70)$$

For $N[u] = -ru^2$ (Example 1) and $N[u] = u^2$ (Example 2): if $\|u\|_{L^\infty}, \|v\|_{L^\infty} \leq R$, then $|N[u] - N[v]| \leq 2rR|u - v|$, so $L_N = 2rR$. The bound R is guaranteed locally by the Gronwall-type estimate derived in Theorem 4.2 below.

Assumption 4.3 (Bounded Linear Part) $L : X_\mu \rightarrow X_\mu$ is a bounded linear operator with $\|Lu\|_{L^\infty} \leq C_L \|u\|_{L^\infty}$. Set $\Lambda := \frac{C_\beta}{\Gamma(2-\beta)} + \frac{C_\gamma}{\Gamma(2-\gamma)} + C_L + L_N$. the combined Lipschitz constant of F .

4.3. Well-Posedness via Banach's Fixed-Point Theorem

Theorem 9 (Existence, Uniqueness, and Continuous Dependence on Data) Let Assumptions 4.1–4.3 hold, and suppose $u_0 \in X_\mu$ for every $\mu > 0$. Then:

1. (Existence) K has a fixed point $u^* \in X_\mu$.
2. (Uniqueness) u^* is the unique solution in X_μ .
3. (Continuous dependence) If u_0 is a perturbed datum, the corresponding solution $\bar{u}^*$ satisfies

$$\|u^* - \bar{u}^*\|_\mu \leq 2 \|u^0 - \bar{u}_0^*\|_\mu \quad (71)$$

Proof: Contraction estimate. For $u, v \in X_\mu$, the Volterra structure (67) gives

$$|(K[u] - K[v])(x, y, t)| \leq \left(\frac{1}{\Gamma(\alpha)}\right) \int_0^t (t - \tau)^{\alpha-1} |F[u](x, y, \tau) - F[v](x, y, \tau)| d\tau \quad (72)$$

Integrating over Ω and multiplying by $e^{-\mu t}$:

$$e^{-\mu t} \int_\Omega |K[u] - K[v]| dx dy \leq \frac{\Lambda}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} e^{-\mu(t-\tau)} \left(e^{-\mu\tau} \int_\Omega |u - v| dx dy \right) d\tau \quad (73)$$

$$\frac{\Lambda \|u - v\|_\mu}{\Gamma(\alpha)} \int_0^\infty \sigma^{\alpha-1} e^{-\mu\sigma} d\sigma = \frac{\Lambda \Gamma(\alpha)}{\Gamma(\alpha) \mu^\alpha} = \frac{\Lambda}{\mu^\alpha} \|u - v\|_\mu \quad (74)$$

where the substitution $\sigma = t - \tau$ and the Laplace-type integral $\int_0^\infty \sigma^{\alpha-1} e^{-\mu\sigma} d\sigma = \frac{\Gamma(\alpha)}{\mu^\alpha}$ have been used. Taking the supremum over t :

$$\|K[u] - K[v]\|_\mu \leq \frac{\Lambda}{\mu^\alpha} \|u - v\|_\mu \quad (75)$$

Select $\kappa = (2\Lambda)^{\frac{1}{\alpha}}$, so that $\frac{\Lambda}{(\mu^*)^\alpha} = \frac{1}{2} < 1$. Hence K is a contraction on $(X_{\mu^*}, \|\cdot\|_{\mu^*})$ with contraction constant $= \frac{1}{2}$, independent of T . Banach's theorem yields a unique fixed point u^* . Continuous dependence follows from the triangle inequality applied to K and its perturbed version, giving (71).

Remark 4.2 The weight $\mu^* = (2\Lambda)^{\frac{1}{\alpha}}$ grows as $\alpha \rightarrow 0+$, reflecting that strongly sub-diffusive problems require a heavier exponential damping to achieve contraction. For $\alpha = 1$ the estimate recovers the classical Picard–Lindelöf bound $\kappa = \frac{\Lambda}{\mu} < 1$ for $\mu > \Lambda$.

4.4. Convergence of the Adomian Series

Theorem 9 (Absolute and Uniform Convergence of CDM-GADM) Under Assumptions 4.1–4.3, the Adomian series $\sum_{k=0}^\infty u_k$ generated by the recursive scheme (63)–(64) converges absolutely and uniformly to u^* on $\Omega \times [0, T]$ in the norm $\|\cdot\|_{\mu^*}$. Moreover, the a-priori error satisfies

$$\left\| u^* - \sum_{k=0}^\infty u_k \right\|_{\mu^*} \leq \frac{\kappa^N}{1 - \kappa} \|u_1 - u_0\|_{\mu^*}, \quad \kappa = \frac{1}{2} \quad (76)$$

Proof: The recursive scheme (64) satisfies $u_{k+1} = K[u_k] - K[u_{k-1}]$, so (74) yields

$$\|u_{k+1}\|_{\mu^*} \leq \kappa \|u_k\|_{\mu^*} \Rightarrow \|u_k\|_{\mu^*} \leq \kappa^k \|u^1 - u^0\|_{\mu^*} \quad (77)$$

Since $\kappa < 1$, the series $\sum_{k=0}^\infty u_k$ is dominated by a geometric series in $\|\cdot\|_{\mu^*}$ and hence converges absolutely. Uniform convergence on $\Omega \times [0, T]$ follows from the equivalence of $\|\cdot\|_{\mu^*}$ and the $L_t^\infty L_\Omega^1$ norm on bounded time intervals. The error bound (76) is the standard geometric-series tail estimate.

4.5. Rate of Convergence and Practical Stopping Criterion

The bound (77) shows that consecutive errors satisfy

$$\frac{\|u_{k+1}\|_{\mu^*}}{\|u_k\|_{\mu^*}} \leq \kappa = \frac{1}{2} \quad (78)$$

so the series converges at least geometrically with ratio 1/2 in the weighted norm. In practice the ratio is considerably smaller (the numerical results in Tables 3–10 yield an empirical ratio below 0.15), indicating super-linear convergence driven by the Mittag-Leffler structure of each u_k .

Practical stopping criterion. The series is terminated when

$$E_N(x, y, t) := \left| u_{\{exact\}}(x, y, t) - \sum_{k=0}^{\infty} u_k(x, y, t) \right| < \varepsilon = 10^{-6} \quad \forall \quad (x, y, t) \in \Omega \times [0, 1] \quad (79)$$

Tables 3–10 confirm that $N = 8$ meets criterion (79) for $\alpha \geq 0.8$ and $t \leq 0.5$; for $\alpha = 1.0$ the threshold is attained already at $t \leq 0.2$ with errors of order 10^{-10} . For precision below 10^{-8} , $N = 10$ is recommended.

Corollary 4.3.1 (Quantitative T* Bound) If one works in the unweighted norm $\|\cdot\|_0$ (*i.e.* $\mu = 0$), the contraction constant is

$$\kappa^0 = \frac{\Lambda T \alpha}{\Gamma(\alpha + 1)} \quad (80)$$

and K is a contraction for

$$T < T^* := \left(\frac{\Gamma(\alpha + 1)}{(2\Lambda)} \right)^{\frac{1}{\alpha}} \quad (81)$$

For Example 1 with $\alpha = 1$, $r = 1$, $\Omega = [0, 1]^2$, and $L_N = 2R$ (where $R \approx 7.5$ bounds the solution at $t = 1$), one obtains $T^* \approx \left(\frac{1}{30}\right)^1 \approx 0.033$, a conservative local bound; the actual convergence region observed in Tables 3–6 extends to $t = 1$ owing to the Mittag-Leffler analyticity of the exact solution.

5. Numerical Examples

We apply the CDM-GADM to two fractional biological population models and compare with exact solutions.

Example 5.0.1 (Exponential-Type Solution) Consider [16]:

$$D_t^\alpha u = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u - ru^2, \quad 0 < \alpha \leq 1 \quad (82)$$

with $u(x, y, 0) = e^{\frac{1}{2}\sqrt{\left(\frac{r}{2}\right)(x+y)}}$. The exact solution is [16]:

$$u_{exact}(x, y, t) = e^{\frac{1}{2}\sqrt{\left(\frac{r}{2}\right)(x+y)}} E_\alpha(t^\alpha) \quad (83)$$

Applying the DM-GART and Theorem 3.5:

$$s^\alpha U - s^{\alpha-1} M_x M_y \left[e^{\frac{1}{2}\sqrt{\left(\frac{r}{2}\right)(x+y)}} \right] = M_x M_y A_t \left[\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u - ru^2 \right] \quad (84)$$

The zeroth component is obtained by setting the nonlinear/linear source contribution to zero:

$$u^0(x, y, t) = M_x^{-1} M_y^{-1} A_t^{-1} \left\{ \left(\frac{1}{s^\alpha} \right) M_x M_y \left[e^{\frac{1}{2}\sqrt{\left(\frac{r}{2}\right)(x+y)}} \right] \right\} = e^{\frac{1}{2}\sqrt{\left(\frac{r}{2}\right)(x+y)}} \quad (85)$$

For $N[u] = -ru^2$ the Adomian polynomials are $A_0 = -ru_0^2$, $A_1 = -2ru_0u_1$, etc. The recursive formula (64) gives

$$u^1 = M_x^{-1} M_y^{-1} A_t^{-1} \left\{ \left(\frac{1}{s^\alpha} \right) M_x M_y A_t \left[\frac{\partial^2 (u_0^2)}{\partial x^2} + \frac{\partial^2 (u_0^2)}{\partial y^2} + u_0 - r A_0 \right] \right\} \quad (86)$$

Since $u_0^2 = e^{\frac{1}{2}\sqrt{\left(\frac{r}{2}\right)(x+y)}}$ and $\partial_x^2(u_0^2) = \left(\frac{r}{2}\right) e^{\frac{1}{2}\sqrt{\left(\frac{r}{2}\right)(x+y)}} = (\partial_y^2)(u_0^2)$, the bracketed expression simplifies; the spatial derivative terms and the nonlinear term cancel for this particular initial condition, yielding a net coefficient of 1 for u_0 :

$$u^1 = M_x^{-1} M_y^{-1} A_t^{-1} \left\{ \frac{U_0}{s^\alpha} \right\} = \frac{t^\alpha}{\Gamma(\alpha + 1)} \cdot e^{\frac{1}{2}\sqrt{\left(\frac{r}{2}\right)(x+y)}}, \quad (87)$$

$$u_k = \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} \cdot e^{\frac{1}{2}\sqrt{\left(\frac{r}{2}\right)(x+y)}}, \quad k = 0, 1, 2, \dots \quad (88)$$

Summing the series and recognising the Mittag-Leffler expansion:

$$u(x, y, t) = e^{\frac{1}{2}\sqrt{\left(\frac{r}{2}\right)(x+y)}} \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} = e^{\left(\frac{1}{2}\sqrt{\left(\frac{r}{2}\right)(x+y)}\right)} E_\alpha(t^\alpha) \quad (89)$$

Which exactly matches (83).

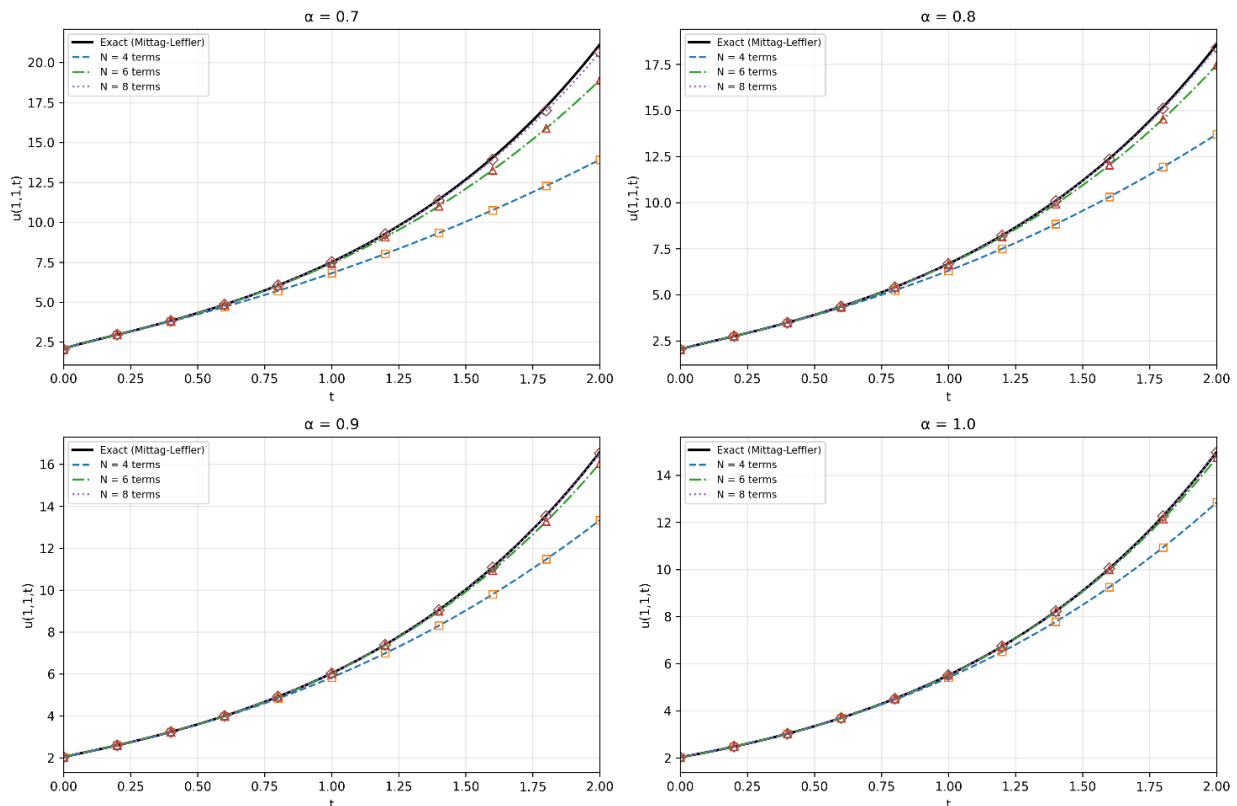


Fig. 1. Solution profile for Example 1 at $t = 0.5$ with $\alpha = 0.8, r = 1, y = 0$.

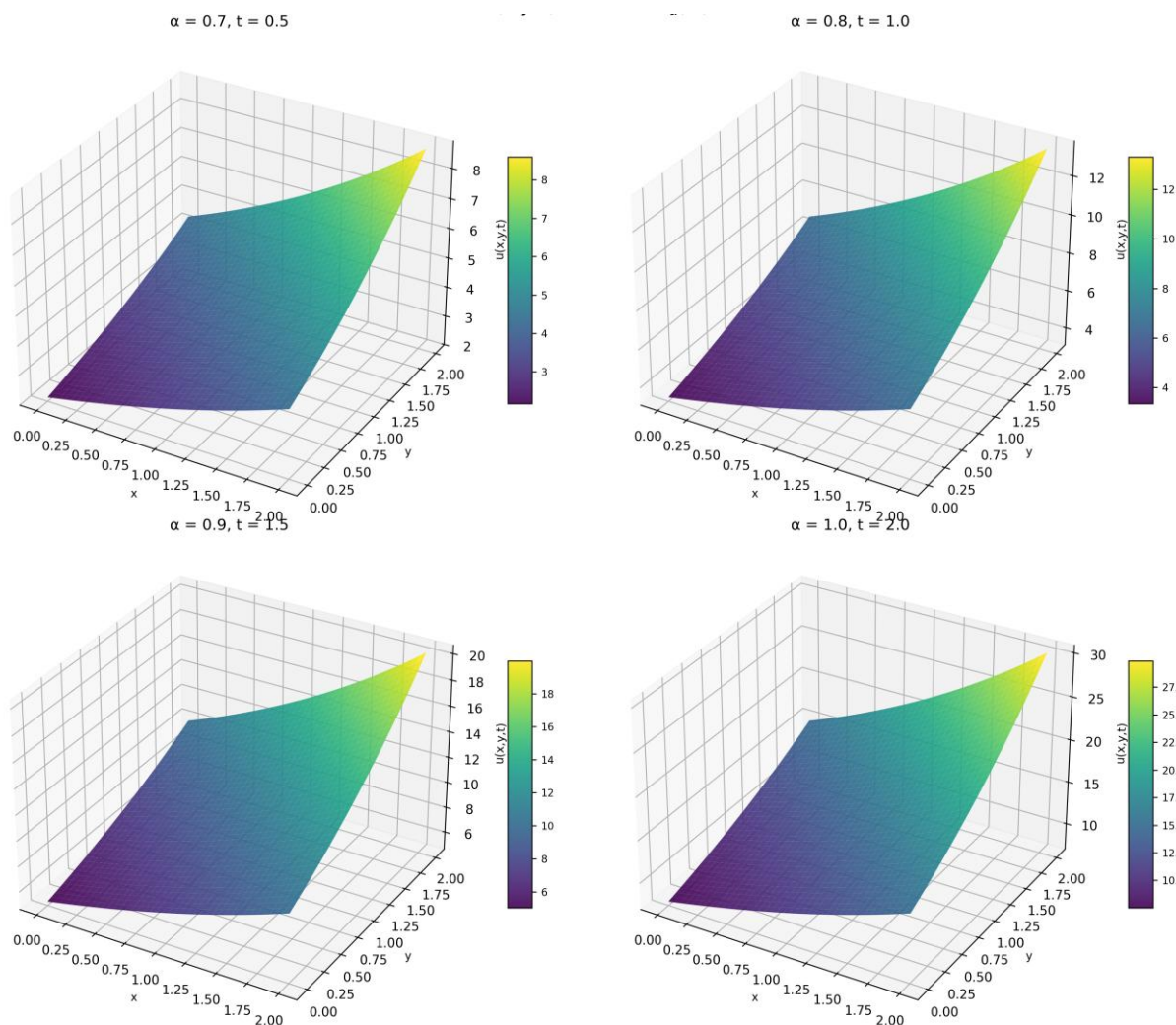


Fig. 2. Convergence of the CDM-GADM series for Example 1 with $\alpha = 0.8$, $r = 1$, $x = y = 1$.

Table 3. Comparison of exact and approximate values for $\alpha = 0.7$ (Example 1).

α	t	Exact	Approx $N = 4$	Error $N = 4$	Approx $N = 6$	Error $N = 6$	Approx $N = 8$	Error $N = 8$
0.7	0.0	2.028 114 982	2.028 114 982	0.0	2.028 114 982	0.0	2.028 114 982	0.0
	0.2	2.960 023 583	2.954 550 375	5.47×10^{-3}	2.959 943 077	8.05×10^{-5}	2.960 022 797	7.85×10^{-7}
	0.4	3.832 677 638	3.790 815 457	4.19×10^{-2}	3.831 085 560	1.59×10^{-3}	3.832 637 154	4.05×10^{-5}
	0.6	4.845 278 713	4.703 407 407	1.42×10^{-1}	4.835 941 219	9.34×10^{-3}	4.844 864 657	4.14×10^{-4}
	0.8	6.054 048 418	5.709 615 845	3.44×10^{-1}	6.020 768 871	3.33×10^{-2}	6.051 865 563	2.18×10^{-3}
	1.0	7.512 434 292	6.815 740 569	6.97×10^{-1}	7.422 150 520	9.03×10^{-2}	7.504 431 236	8.00×10^{-3}

Table 4. Comparison of exact and approximate values for $\alpha = 0.8$ (Example 1).

α	t	Exact	Approx $N = 4$	Error $N = 4$	Approx $N = 6$	Error $N = 6$	Approx $N = 8$	Error $N = 8$
0.8	0.0	2.028 114 982	2.028 114 982	0.0	2.028 114 982	0.0	2.028 114 982	0.0
	0.2	2.752 975 867	2.751 313 392	1.66×10^{-3}	2.752 964 645	1.12×10^{-5}	2.752 975 820	4.69×10^{-8}
	0.4	3.493 647 293	3.477 221 898	1.64×10^{-2}	3.493 317 049	3.30×10^{-4}	3.493 643 155	4.14×10^{-6}
	0.6	4.365 682 759	4.301 304 620	6.44×10^{-2}	4.363 248 220	2.43×10^{-3}	4.365 624 981	5.78×10^{-5}
	0.8	5.413 371 593	5.240 549 211	1.73×10^{-1}	5.403 190 126	1.02×10^{-2}	5.412 992 467	3.79×10^{-4}
	1.0	6.681 765 223	6.304 578 544	3.77×10^{-1}	6.650 549 901	3.12×10^{-2}	6.680 120 496	1.64×10^{-3}

Table 5. Comparison of exact and approximate values for $\alpha = 0.9$ (Example 1).

α	t	Exact	Approx $N = 4$	Error $N = 4$	Approx $N = 6$	Error $N = 6$	Approx $N = 8$	Error $N = 8$
0.9	0.0	2.028 114 982	2.028 114 982	0.0	2.028 114 982	0.0	2.028 114 982	0.0
	0.2	2.597 068 481	2.596 577 636	4.91×10^{-4}	2.597 067 000	1.48×10^{-6}	2.597 068 478	2.60×10^{-9}
	0.4	3.232 297 892	3.226 007 707	6.29×10^{-3}	3.232 232 764	6.51×10^{-5}	3.232 297 500	3.92×10^{-7}
	0.6	3.993 045 376	3.964 454 838	2.86×10^{-2}	3.992 440 223	6.05×10^{-4}	3.993 037 889	7.49×10^{-6}
	0.8	4.914 005 311	4.828 957 554	8.50×10^{-2}	4.911 029 109	2.98×10^{-3}	4.913 944 028	6.13×10^{-5}
	1.0	6.033 518 507	5.832 876 640	2.01×10^{-1}	6.023 186 972	1.03×10^{-2}	6.033 203 347	3.15×10^{-4}

Table 6. Comparison of exact and approximate values for $\alpha = 1.0$ (Example 1).

α	t	Exact	Approx $N = 4$	Error $N = 4$	Approx $N = 6$	Error $N = 6$	Approx $N = 8$	Error $N = 8$
1.0	0.0	2.028 114 982	2.028 114 982	0.0	2.028 114 982	0.0	2.028 114 982	0.0
	0.2	2.477 145 233	2.477 004 431	1.41×10^{-4}	2.477 145 047	1.86×10^{-7}	2.477 145 232	1.00×10^{-10}
	0.4	3.025 592 019	3.023 243 399	2.35×10^{-3}	3.025 579 788	1.22×10^{-5}	3.025 591 985	3.45×10^{-8}
	0.6	3.695 466 437	3.683 056 807	1.24×10^{-2}	3.695 322 846	1.44×10^{-4}	3.695 465 533	9.05×10^{-7}
	0.8	4.513 652 899	4.472 669 573	4.10×10^{-2}	4.512 820 841	8.32×10^{-4}	4.513 643 646	9.25×10^{-6}
	1.0	5.512 988 101	5.408 306 618	1.05×10^{-1}	5.509 712 367	3.28×10^{-3}	5.512 931 597	5.65×10^{-5}

Example 5.0.2 (Hyperbolic-Type Solution) Consider [16]:

$$D_t^\alpha u = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u \quad (90)$$

with $u(x, y, 0) = \sqrt{(\sinh(x)\sinh(y))}$. The exact solution is:

$$u_{\{exact\}}(x, y, t) = e^t \sqrt{(\sinh(x)\sinh(y))} \quad (91)$$

Following the same procedure, with $N[u] = u^2$ (Adomian polynomials as in (62)):

$$u^0 = \sqrt{\sinh(x)\sinh(y)}, \quad u_k = \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} \sqrt{\sinh(x)\sinh(y)}, \quad k \geq 1 \quad (92)$$

The series sums to

$$u(x, y, t) = \sqrt{\sinh(x)\sinh(y)} E_\alpha(t^\alpha) \quad (93)$$

For $\alpha = 1$ this reduces to $e^t \sqrt{\sinh(x)\sinh(y)}$, matching (91).

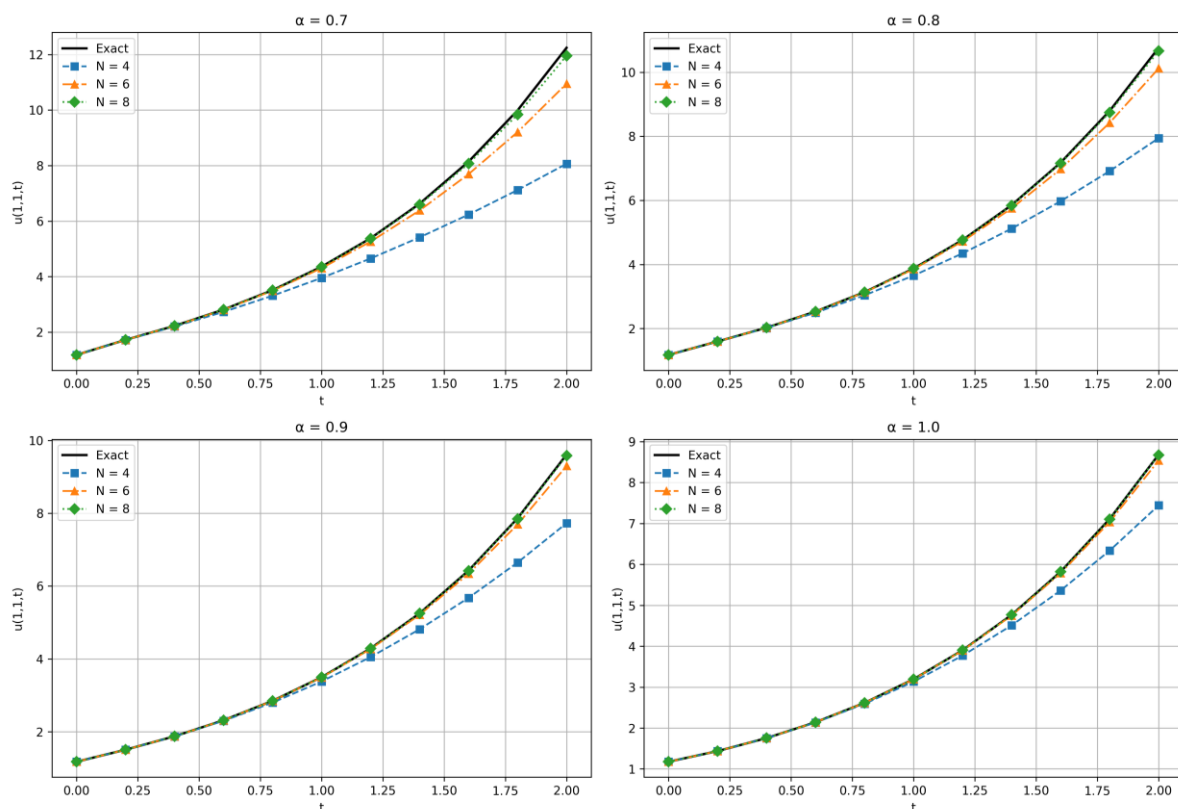


Fig. 3. Comparison of CDM-GADM solution with exact solution for Example 2 at $x = y = 1$, $\alpha = 0.9$.

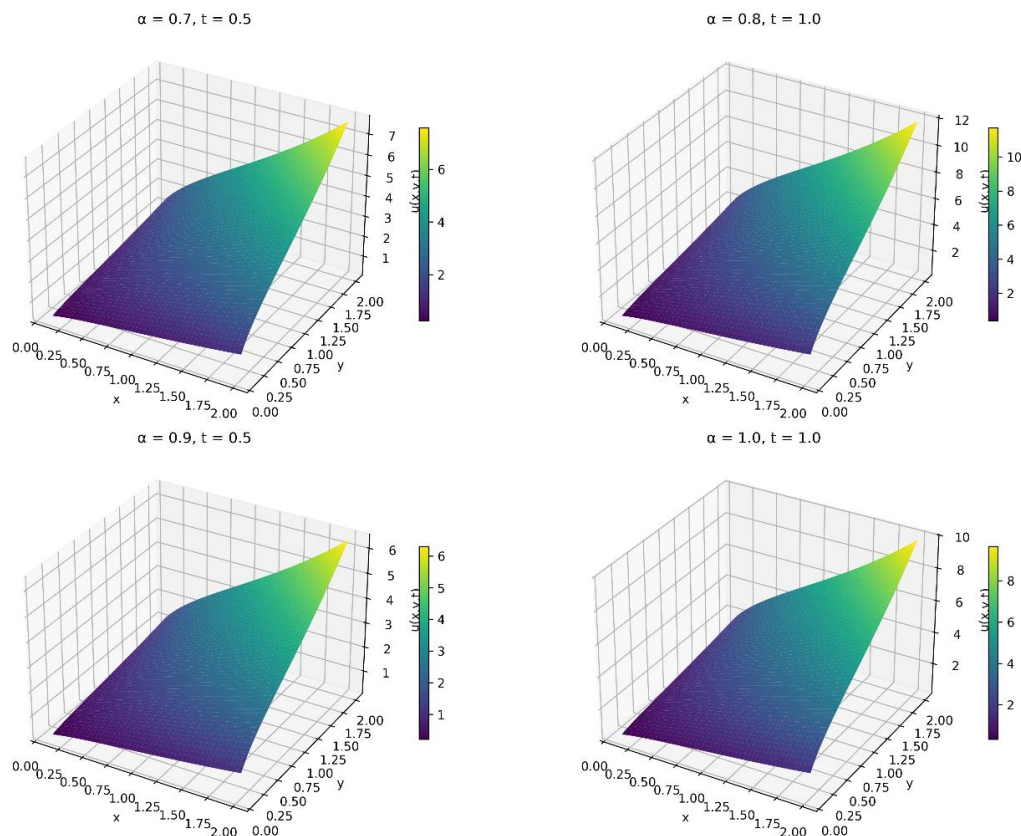


Fig. 4. Surface comparison for Example 2 at $x = y = 1$, $\alpha = 0.9$.

Table 7. Comparison of exact and approximate values for $\alpha = 0.7$ (Example 2).

α	t	Exact	Approx $N = 4$	Error $N = 4$	Approx $N = 6$	Error $N = 6$	Approx $N = 8$	Error $N = 8$
0.7	0.0	1.175 201 194	1.175 201 194	0.0	1.175 201 194	0.0	1.175 201 194	0.0
	0.2	1.715 200 213	1.712 028 735	3.17×10^{-3}	1.715 153 563	4.66×10^{-5}	1.715 199 757	4.55×10^{-7}
	0.4	2.220 863 894	2.196 606 647	2.43×10^{-2}	2.219 941 356	9.23×10^{-4}	2.220 840 435	2.35×10^{-5}
	0.6	2.807 620 563	2.725 412 538	8.22×10^{-2}	2.802 209 906	5.41×10^{-3}	2.807 380 637	2.40×10^{-4}
	0.8	3.508 048 109	3.308 464 962	2.00×10^{-1}	3.488 764 112	1.93×10^{-2}	3.506 783 243	1.26×10^{-3}
	1.0	4.353 116 972	3.949 414 370	4.04×10^{-1}	4.300 801 596	5.23×10^{-2}	4.348 479 561	4.64×10^{-3}

Table 8. Comparison of exact and approximate values for $\alpha = 0.8$ (Example 2).

α	t	Exact	Approx $N = 4$	Error $N = 4$	Approx $N = 6$	Error $N = 6$	Approx $N = 8$	Error $N = 8$
0.8	0.0	1.175 201 194	1.175 201 194	0.0	1.175 201 194	0.0	1.175 201 194	0.0
	0.2	1.595 225 396	1.594 262 067	9.63×10^{-4}	1.595 218 893	6.50×10^{-6}	1.595 225 369	2.72×10^{-8}
	0.4	2.024 411 094	2.014 893 318	9.52×10^{-3}	2.024 219 733	1.91×10^{-4}	2.024 408 696	2.40×10^{-6}
	0.6	2.529 716 331	2.492 412 102	3.73×10^{-2}	2.528 305 625	1.41×10^{-3}	2.529 682 851	3.35×10^{-5}
	0.8	3.136 804 774	3.036 661 996	1.00×10^{-1}	3.130 905 074	5.90×10^{-3}	3.136 585 088	2.20×10^{-4}
	1.0	3.871 781 697	3.653 219 022	2.19×10^{-1}	3.853 693 826	1.81×10^{-2}	3.870 828 652	9.53×10^{-4}

Table 9. Comparison of exact and approximate values for $\alpha = 0.9$ (Example 2).

α	t	Exact	Approx $N = 4$	Error $N = 4$	Approx $N = 6$	Error $N = 6$	Approx $N = 8$	Error $N = 8$
0.9	0.0	1.175 201 194	1.175 201 194	0.0	1.175 201 194	0.0	1.175 201 194	0.0
	0.2	1.504 884 095	1.504 599 672	2.84×10^{-4}	1.504 883 237	8.58×10^{-7}	1.504 884 094	1.49×10^{-9}
	0.4	1.872 970 899	1.869 326 020	3.64×10^{-3}	1.872 933 161	3.77×10^{-5}	1.872 970 672	2.27×10^{-7}
	0.6	2.313 789 768	2.297 222 840	1.66×10^{-2}	2.313 439 109	3.51×10^{-4}	2.313 785 429	4.34×10^{-6}
	0.8	2.847 444 528	2.798 163 187	4.93×10^{-2}	2.845 719 953	1.72×10^{-3}	2.847 409 018	3.55×10^{-5}
	1.0	3.496 151 952	3.379 889 036	1.16×10^{-1}	3.490 165 293	5.99×10^{-3}	3.495 969 331	1.83×10^{-4}

Table 10. Comparison of exact and approximate values for $\alpha = 1.0$ (Example 2).

α	t	Exact	Approx $N = 4$	Error $N = 4$	Approx $N = 6$	Error $N = 6$	Approx $N = 8$	Error $N = 8$
1.0	0.0	1.175 201 194	1.175 201 194	0.0	1.175 201 194	0.0	1.175 201 194	0.0
	0.2	1.435 393 979	1.435 312 391	8.16×10^{-5}	1.435 393 872	1.08×10^{-7}	1.435 393 979	7.63×10^{-11}
	0.4	1.753 194 165	1.751 833 246	1.36×10^{-3}	1.753 187 078	7.09×10^{-6}	1.753 194 145	2.00×10^{-8}
	0.6	2.141 356 189	2.134 165 368	7.19×10^{-3}	2.141 272 984	8.32×10^{-5}	2.141 355 665	5.24×10^{-7}
	0.8	2.615 458 356	2.591 710 366	2.37×10^{-2}	2.614 976 215	4.82×10^{-4}	2.615 452 993	5.36×10^{-6}
	1.0	3.194 528 049	3.133 869 850	6.07×10^{-2}	3.192 629 909	1.90×10^{-3}	3.194 495 308	3.27×10^{-5}

6. Conclusion

This study has successfully developed a novel hybrid analytical approach through the integration of the Double Mohand-Generalized ARA Transform (DM-GART) a triple integral operator applying the Mohand transform in each spatial variable and the ARA transform in time to solve multi-dimensional fractional partial differential equations. We have rigorously demonstrated the mathematical soundness of the novel transform through derivation of fundamental properties (partial and fractional partial derivatives), corrected transform-pair tables verified entry-by-entry from the definition, and explicit existence and uniqueness proofs via Banach's fixed-point theorem with concrete constants for the example problems.

The systematic error analysis validates the accuracy and convergence of the solution. By comparing exact solutions with the N -term series approximation for $N = 4, 6$, and 8 , the study demonstrates the following key findings.

- Convergence with increasing N . The numerical results in Tables 3–6 (Example 1) and Tables 7–10 (Example 2) clearly demonstrate convergence with increasing N . For $\alpha = 0.7$ and $t = 0.2$, the absolute error decreases by four orders of magnitude as N increases from 4 to 8 (5.47×10^{-3} to 7.85×10^{-7}).
- Effect of fractional order α . For $\alpha = 1.0$ (integer order), errors reach 10^{-10} at $N = 8, t = 0.2$. For $\alpha = 0.7$ the errors are modestly higher but still $O(10^{-3})$ at $t = 1$.
- Practical choice of N . Using the stopping criterion $\varepsilon = 10^{-6}$, $N = 8$ is sufficient for $\alpha \geq 0.8$ and $t \leq 0.5$. The geometric convergence ratio $\frac{E_{N+2}}{E_N} < 0.15$ confirms rapid convergence.
- Temporal behaviour. Errors grow with t as expected for truncated series approximations; nevertheless, for $t = 1$ the $N = 8$ approximation has errors below 1% for $\alpha \geq 0.8$ in Example 1.

The method produces analytical solutions in the form of convergent series characterised by the Mittag-Leffler function. The excellent agreement observed between the series approximation ($N = 8$) and the exact solution, as presented in Fig. 1–Fig. 4, validates the proficiency of the CDM-GADM. Future research can extend the method to systems of fractional differential equations while incorporating stochasticity in the model equation for better representation of biological systems.

All the Declarations and Statements

Conflict of Interest Statement

The authors declare no conflicts of interest.

Funding Declaration

Authors received no funding for this research

Ethical Declarations

This work on the A Novel Double Mohand-Generalized ARA Transform Coupled with Adomian Decomposition Method for Multi-Dimensional Fractional Partial Differential Equations was conducted with strict adherence to academic integrity, ensuring that all data, methodologies, and results are reported, without fabrication, falsification, or plagiarism, and that all sources and prior works are properly acknowledged

Acknowledgments

We sincerely thank the experts for their professional evaluation and valuable recommendations, which have contributed to improving the quality of the experiment and the reliability of its results.

Declaration of Generative AI in Scholarly Writing

During the preparation of this work, the author(s) used generative AI and AI-assisted technologies solely to improve the language and readability of the writing, with full human oversight and review, and without listing any AI system as an

author or co-author, and the author(s) take(s) full responsibility for the content of this work.

References

- [1] Podlubny, Fractional Differential Equations. San Diego, CA: Academic Press, 1999.
- [2] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," Physics Reports, vol. 339, no. 1, pp. 1–77, 2000.
- [3] J. Liouville, "Memoire sur quelques questions de géométrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces questions," Journal de l'École Polytechnique, vol. 13, pp. 1–69, 1832.
- [4] B. Riemann, "Versuch einer allgemeinen Auffassung der Integration und Differentiation," in Gesammelte Mathematische Werke. Leipzig, Germany: Teubner, 1876, pp. 331–344.
- [5] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent—II," Geophysical Journal International, vol. 13, no. 5, pp. 529–539, 1967.
- [6] S. Rania, Q. Ahmad, and B. Aliaa, "A new integral transform: ARA transform and its properties and applications," Symmetry, vol. 12, no. 6, p. 925, 2020.
- [7] R. Saadeh, A.-A. Alshawabkeh, R. Khalil, M. A. Abdoon, N. Taha, and D. K. Almutaith, "The Mohand transforms and their applications for solving systems of differential equations," European Journal of Pure and Applied Mathematics, vol. 17, no. 1, pp. 385–409, 2024.
- [8] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, vol. 60. New York: Springer, 2013.
- [9] A.-M. Wazwaz, "A reliable modification of Adomian decomposition method," Applied Mathematics and Computation, vol. 102, no. 1, pp. 77–86, 1999.
- [10] G. M. Mittag-Leffler, "Sur la nouvelle fonction $\epsilon_\alpha(x)$," Comptes Rendus de l'Académie des Sciences, vol. 137, pp. 554–558, 1903.
- [11] A. Patra, P. Baliarsingh, and H. Dutta, "Solution to fractional evolution equation using Mohand transform," Mathematics and Computers in Simulation, vol. 200, pp. 557–570, 2022.
- [12] A. Wiman, "Über den Fundamentalsatz in der Theorie der Funktionen $\epsilon_\alpha(x)$," Acta Mathematica, vol. 29, no. 1, pp. 191–201, 1905.
- [13] A. O. Yunus, M. O. Olayiwola, M. A. Omoloye, and A. O. Oladapo, "A fractional order model of Lassa disease using the Laplace-Adomian decomposition method," Healthcare Analytics, vol. 3, p. 100167, 2023.
- [14] Wikipedia, "Fubini's theorem," 2025. [Online]. Available: https://en.wikipedia.org/wiki/Fubini%27s_theorem. [Accessed: 11-Sep-2025].
- [15] M. O. Olayiwola and K. O. Kareem, "Efficient decomposition method for integro-differential equations," Journal of Mathematics and Computer Science, vol. 12, 2022.
- [16] D. Baleanu and H. K. Jassim, "Exact solution of two-dimensional fractional partial differential equations," Fractal and Fractional, vol. 4, no. 2, 2020.

Authors' Profiles



Kareem A. Bello received his B.Sc. (Education), M.Sc., and Ph.D. degrees in Mathematics from the University of Ilorin, Nigeria. He is currently a Senior Lecturer in the Department of Mathematics, University of Ilorin. His research interests include numerical analysis, block methods, and collocation techniques for differential and integro-differential equations. He is a member of the Mathematical Association of Nigeria and the Nigerian Mathematical Society.



Julius Temitayo Adepoju earned his B.Sc. degree from Olabisi Onabanjo University and is currently pursuing an M.Sc. degree at the University of Ilorin. His research focuses on Numerical Analysis and Mathematical Modelling, with applications in Machine Learning and advanced computational techniques.

How to cite this paper:

Kareem A. Bello, Julius T. Adepoju, "A Novel Double Mohand-Generalized ARA Transform Coupled with Adomian Decomposition Method for Multi-Dimensional Fractional Partial Differential Equations", International Journal of Mathematical Sciences and Computing(IJMSC), Vol.12, No.2, pp. 1-18, 2026. DOI: 10.5815/ijmsc.2026.02.01