I. J. Mathematical Sciences and Computing, **2026, 1, 42-54**
Published Online on February 8, 2026 by MECS Press (http://www.mecs-press.org/)
DOI: 10.5815/ijmsc.2026.01.04

Ergodicity and the Emergence of Long-Term Balance in the Dynamical States of π

Fethi Kadhi*
University of Manouba, National School of Computer Science, Manouba, Tunisia
E-mail: fethi.kadhi@ensi.uma.tn
ORCID iD: https://orcid.org/0000-0001-5173-5363
*Corresponding Author

Moncef Ghazel
University of Tunis Elmanar, Faculte ´des Sciences de Tunis, Tunisia
E-mail: moncef.ghazel@fst.utm.tn
ORCID iD: https://orcid.org/0000-0002-0006-945X

Malek Ghazel
University of Tunis Elmanar, Faculte ´des Sciences de Tunis, Tunisia
E-mail: malek.ghazel@fst.utm.tn

Received: 01 November, 2025; Revised: 10 December, 2025; Accepted: 06 January, 2026; Published: 08 February, 2026

Abstract: This paper investigates the digits of π within a probabilistic framework based on Markov chains, proposing this model as a rigorous tool to support the conjecture of π's uniformity. Unlike simple frequency analyses, the Markov approach captures the dynamic structure of transitions between digits, allowing us to compute empirical stationary distributions that reveal how local irregularities evolve toward global equilibrium. This ergodic behavior provides quantitative, model based evidence that the digits of π tend toward fairness in the long run. Beyond its mathematical significance, this convergence toward uniformity invites a broader conceptual interpretation.

Index Terms: Markov Chain, Irrational Numbers, Uniform Distribution, Ergodic Theory, Statistical Programming Language, Computing

1. Introduction

The conjecture that the digits of certain irrational numbers, such as π, √2, and e, are uniformly distributed has long fascinated mathematicians. In base 10, this means that each digit from 0 to 9 should appear with equal frequency in the infinite decimal expansion. While extensive computational evidence suggests approximate uniformity, no rigorous proof of normality exists for these constants [1, 2, 3]. Recent statistical and computational studies have provided substantial empirical evidence regarding the digit distributions of fundamental mathematical constants. For example, detailed analysis of the first ten million digits of π employing chi-square tests reveals that each decimal digit occurs with a frequency very close to 10%, aligning with the expectation for a uniform distribution [4, 5].

To provide additional support for the uniformity conjecture, we propose modeling the sequences of digits as a Markov chain. In this approach, each digit is considered as a state, and transitions between consecutive digits are analyzed probabilistically. This does not constitute a proof of normality. However, it provides a strong tool for investigating statistical patterns and confirming that the digits mimic the behavior of a uniform stochastic process over long stretches.

In Section 2, we take a brief detour into the world of Markov chains, the probabilistic framework that allows us to study sequences in motion. Each digit of π is treated as a state, with empirical transition probabilities describing how one digit leads to the next. Key concepts such as irreducibility and aperiodicity ensure that, over time, the chain settles into a unique stationary distribution. This stationary distribution captures the long-term frequency of each digit, revealing the remarkable fairness and uniformity inherent in π, and laying the foundation for later philosophical reflection.

Section 3 examines the uniformity of π in both historical and computational contexts. We trace the evolution of π's computation, from Zu Chongzhi's "Milu¨" in ancient China to Jamsh¯ıd al-Ka¯sh¯ı's precise calculations during the Islamic Golden Age, through Newton, Machin, and the advent of electronic computation. We then employ the Markov chain perspective to model the digit sequence dynamically. By constructing an empirical transition matrix and computing the stationary distribution, we observe that even deterministic digits exhibit statistical uniformity: local imbalances give way to global equilibrium, providing an ergodic fingerprint of π's decimal expansion. This analysis resonates with James Franklin's framework of logical probability [7], showing how structured probabilistic reasoning can provide non deductive support for mathematical conjectures.

In Section 4, we reflect on the interpretation of π's digits. Inspired by Leibniz's claim that our world is the best of all possible worlds [8], we observe that short stretches of π may show imbalances some digits appear rarely, others frequentlyyet over long stretches every digit achieves its due proportion. Viewed through Markov chains, this demonstrates a form of cosmic patience: early disparities give way to global equilibrium. A complementary perspective highlights the necessity of every state: each digit, whether initially underrepresented or not, is essential to π's identity. By analogy, each of us is the sum of our days, with every experience pleasant or challenging contributing to the fullness of life.

Section 5, the conclusion, synthesizes the essay's insights. Modeling π's digits as a Markov chain reveals that local imbalances resolve into long-term uniformity, illustrating how deterministic sequences can display emergent order. By situating this in Franklin's framework of logical probability, we show that structured probabilistic reasoning can provide robust support for conjectures. Philosophically, the stationary distribution offers a metaphor for patience, fairness, and equilibrium: apparent disorder may conceal an underlying harmony, whether in numbers or in life, suggesting that balance emerges gradually, inevitably, and beautifully.

2. Markov Chains

Definition 1

A discrete time Markov chain is a sequence of random variables $\{X_n\}_{n\geq 0}$ taking values in a state space S, where the probability of transitioning to the next state depends only on the current state. Formally, for any $x_0, x_1, \ldots, x_n \in S$, the Markov property asserts that:

$$\Pr(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = \Pr(X_{n+1} = x_{n+1} \mid X_n = x_n). \tag{1}$$

The transition probabilities between states are encapsulated in a transition matrix $P = [p_{ij}]$, where $p_{ij} = \Pr(X_{n+1} = j \mid X_n = i)$

Definition 2

A Markov chain is irreducible if, for any two states $i, j \in S$, there exists a positive integer n such that $p_{ij}^{(n)} > 0$$, meaning that it is possible to reach state j from state i in n steps.

Definition 3

A state $i \in S$ is aperiodic if the greatest common divisor of the set of times $n > 0$ such that $\Pr(X_n = i \mid X_0 = i) > 0$ is 1. An irreducible Markov chain is aperiodic if any one of its states is aperiodic.

The ergodic theorem for Markov chains provides conditions under which a unique stationary distribution exists and is approached over time.

Theorem 2.1 [9, Theorem 1.8.3]

If a discrete time, finite state Markov chain is both irreducible and aperiodic, then it has a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$, where each $\pi_i > 0$ and $\sum_{i=1}^{k} \pi_i = 1$ Furthermore, for any initial distribution, the distribution of X_n converges to π as $n \to \infty$.

Conceptually, the stationary distribution encapsulates the ergodic essence of the process: the asymptotic fraction of time the chain spends in each state. It captures the statistical equilibrium intrinsic to the Markov process: while the chain continues to fluctuate randomly, its long-run statistical pattern is completely encoded by this invariant measure.

Markov chains are widely used to model complex, stochastic processes in many real-world applications. In finance, they help assess credit risk by modeling transitions between rating categories over time. In genetics and bioinformatics, they capture patterns in DNA sequences and aid in predicting gene locations. In operations research, Markov chains optimize queueing systems in banks, hospitals, and call centers by analyzing customer arrivals and service processes. They are also employed in natural language processing to model sequences of words or phonemes, and in weather forecasting to estimate the likelihood of changes between different weather states. These examples demonstrate that Markov chains provide a powerful framework for understanding dynamic probabilistic systems.

3. Uniformity of π

3.1 A Cultural History

The computation of the ratio of a circle's circumference to its diameter the irrational and transcendental number π is one of the most enduring and cross-cultural quests in the history of mathematics. From ancient geometric approximations to modern algorithmic calculations, this pursuit has reflected not only technical ingenuity but also a deep human curiosity about precision and infinity. In China, this story reached an early climax. As highlighted in [10], the mathematician and astronomer Zu Chongzhi (429–500 CE) achieved a milestone of precision that would not be surpassed for nearly a millennium. Using iterative polygonal methods akin to those of Archimedes, Zu and his son Zu Gengzhi determined that π lay between 3.1415926 and 3.1415927. Their celebrated "Milu¨" approximation, 355/113, correct to six decimal places, remained unsurpassed for over 800 years a testament to the computational sophistication of early Chinese mathematics. During the Islamic Golden Age, the astronomer Jamsh¯ıd al-Ka¯sh¯ı (c. 1380–1429) expanded this tradition in his treatise Risa¯la al-muh¯ıtiyya (The Treatise on the Circumference). He computed 2π to 16 decimal places. His result, π ≈ 3.1415926535897932, was correct to every digit and stood as the most accurate value known for nearly two centuries. In Europe, the rise of calculus transformed the nature of the problem. Isaac Newton, exploiting his binomial theorem, computed π to 15 digits in the 1660s. Elegant but slow series such as the Leibniz formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{2}$$

gave way to faster methods like John Machin's 1706 arctangent identity,

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right), \tag{3}$$

which enabled computations exceeding 100 digits by hand.

With the advent of electronic computers, the computation of π shifted from an act of patience to a test of computational power. The ENIAC calculation of 2,037 digits in 1949, performed in 70 hours, inaugurated a new era. Today, π has been computed to more than 100 trillion digits using algorithms such as the Chudnovsky formula and the Bailey–Borwein– Plouffe (BBP) formula, the latter remarkable for allowing direct computation of arbitrary hexadecimal digits.

These feats of calculation no longer serve geometry, but they continue to challenge our understanding of randomness, precision, and the digital fabric of number itself. In the digits of π, humanity has sought not only measurement but meaning a pattern that forever eludes us, yet compels us to compute one digit further.

3.2 Matrix transition of digits

Definition 4 (Transition Probability)

For a sequence of n digits, the empirical transition probability from digit i to digit j is defined as:

$$p_{ij} = \begin{cases} \dfrac{N_{ij}}{N_i}, & \text{if } N_i > 0, \\ 0, & \text{if } N_i = 0, \end{cases} \tag{4}$$

Example 3.1 For π ≈ 3.14159 with n = 5:

Digits: [1, 4, 1, 5, 9]
Transitions: 1 → 4, 4 → 1, 1 → 5, 5 → 9
Probabilities: $p_{14} = \frac{1}{2}$, $p_{15} = \frac{1}{2}$, $p_{41} = 1$, $p_{59} = 1$

Algorithm 1 Construction of Transition Matrix for Digits of π
\begin{algorithmic[1]
1. Input: Integer n (number of digits of π to extract)
2. Compute π with precision 4n using high-precision arithmetic
3. Extract the first n digits of π into vector \texttt{pid
4. Define training sequence \texttt{pi_train \leftarrow \texttt{pid[1:100000]
5. Initialize a 10\times 10 counting matrix N \leftarrow 0
6. \For{k \gets 1 to length(\texttt{pi_train)-1
7. i \gets \texttt{pi_train[k] \Comment{current digit
8. j \gets \texttt{pi_train[k+1] \Comment{next digit
9. N[i+1,j+1] \gets N[i+1,j+1] + 1
10. \EndFor

11. Initialize transition matrix P \leftarrow 0_{10\times10
12. Compute row sums S_i \gets \sum_{j N[i,j]
13. \For{i \gets 1 to 11
14. \If{S_i > 0
15. \For{j \gets 1 to 10
16. P[i,j] \gets N[i,j]/S_i
17. \EndFor
18. \Else
19. \For{j \gets 1 to 11
20. P[i,j] \gets 0
21. \EndFor
22. \EndIf
23. \EndFor
24. Output: Transition probability matrix P

\end{algorithmic
\end{algorithm

Computational Complexity and Scalability Analysis :

- The Markov chain construction algorithm exhibits $O(n)$ time complexity, where n is the number of digits processed. This linear scalability ensures practical feasibility even for massive digit sequences:
- Digit Extraction: $O(n)$ using optimized algorithms like the Bailey–Borwein–Plouffe formula.
- Transition Counting: $O(n)$ through single-pass digit processing.
- Matrix Operations: $O(1)$ due to fixed state space (digits 0-9).

The memory requirements remain manageable across different scales:

- Minimal Storage: Only 10 \times 10 transition matrix requires persistent storage (\mathcal{O(1)).
- Stream Processing: Digits can be processed sequentially without full sequence storage.
- Constant Memory: Fixed memory footprint regardless of sequence length.

Our approach demonstrates exceptional scalability:

- Million-Digit Analysis: Computationally feasible on standard hardware.
- Billion-Digit Scale: Practical with distributed computing and streaming algorithms.
- Theoretical Limits: No inherent computational barriers to analyzing arbitrarily long sequences.

The methodology generalizes seamlessly to other irrational numbers:

- Universal Applicability: Same computational complexity for $e, \sqrt{2}, \phi,$ etc.
- Parallel Processing: Multiple constants can be analyzed independently.
- Resource Efficiency: Minimal additional overhead for new constants.
- Standard Hardware: Million-digit analysis completes in seconds on consumer hardware.
- Memory Efficiency: Constant memory usage enables analysis of arbitrarily long sequences.
- Distributed Computing: Embarrassingly parallelizable for extreme-scale analysis.
- Real-time Applications: Suitable for streaming analysis of dynamically generated sequences.

Our Markov chain approach offers significant computational advantages over alternative methods:

vs. Full Sequence Analysis: Avoids $O(n)$ storage requirements
vs. Statistical Tests: Enables deeper structural analysis without additional complexity
vs. Machine Learning Methods: More interpretable with guaranteed convergence properties

The analysis confirms that our Markov chain methodology is not only computationally feasible for current datasets but also readily scalable to future challenges involving larger sequences and broader mathematical constants.

For n = 100000, see Example 3.1. Using R as a statistical programming language,

```
if (!require(ggplot2)) install.packages("ggplot2", dependencies = TRUE)
if (!require(Rmpfr)) install.packages("Rmpfr", dependencies = TRUE)
library(ggplot2)
library(Rmpfr)

get_pi_digits <- function(n) {
  pi_val <- Rmpfr::Const("pi", prec = 4 * n)
  digits_str <- substring(format(pi_val, digits = n + 1), 3, n + 2)
  as.numeric(strsplit(digits_str, "")[[1]])

pid <- get_pi_digits(100009)
pi_train <- pid[1:100000]
pi_test <- pid[8001:10000]
pi_train[100000]

N <- matrix(0, nrow = 10, ncol = 10)

for (k in 1:(length(pi_train) - 1)) {
  i <- pi_train[k]
  j <- pi_train[k + 1]
  N[i + 1, j + 1] <- N[i + 1, j + 1] + 1


rownames(N) <- 0:9
colnames(N) <- 0:9

P <- matrix(0, nrow = 10, ncol = 10)
row_sums <- rowSums(N)

for (i in 1:10) {
  if (row_sums[i] > 0) {
    P[i, ] <- N[i, ] / row_sums[i]
   else {
    P[i, ] <- rep(0, 10)

rownames(P) <- 0:9
colnames(P) <- 0:9
```

the complete system is represented by the **transition probability matrix P .**

$$P = \begin{bmatrix} 0.100 & 0.103 & 0.096 & 0.099 & 0.097 & 0.101 & 0.101 & 0.102 & 0.100 & 0.102 \\ 0.103 & 0.102 & 0.098 & 0.100 & 0.102 & 0.096 & 0.097 & 0.099 & 0.103 & 0.101 \\ 0.098 & 0.107 & 0.098 & 0.097 & 0.095 & 0.101 & 0.106 & 0.103 & 0.096 & 0.098 \\ 0.098 & 0.097 & 0.099 & 0.101 & 0.104 & 0.107 & 0.096 & 0.101 & 0.098 & 0.100 \\ 0.102 & 0.100 & 0.099 & 0.102 & 0.097 & 0.096 & 0.104 & 0.101 & 0.098 & 0.100 \\ 0.096 & 0.096 & 0.103 & 0.103 & 0.102 & 0.101 & 0.101 & 0.100 & 0.101 & 0.097 \\ 0.105 & 0.101 & 0.105 & 0.099 & 0.102 & 0.100 & 0.095 & 0.097 & 0.095 & 0.102 \\ 0.095 & 0.104 & 0.098 & 0.103 & 0.094 & 0.096 & 0.110 & 0.101 & 0.104 & 0.096 \\ 0.101 & 0.100 & 0.099 & 0.095 & 0.100 & 0.102 & 0.097 & 0.105 & 0.103 & 0.097 \\ 0.102 & 0.103 & 0.096 & 0.104 & 0.104 & 0.102 & 0.096 & 0.094 & 0.101 & 0.098 \end{bmatrix} \tag{5}$$

where rows and columns correspond respectively to states 0, 1, . . . , 9. For the transition matrix (5), the existence of stationary distribution follows from Theorem 2.1:

1. Finite State Space: S = {0, 1, . . . , 9} is finite
2. Irreducibility: All states communicate with each other (evident from positive transition probabilities throughout the matrix)
3. Aperiodicity: The chain is aperiodic as evidenced by self-transitions with positive probability for all states

From the empirical transition matrix of consecutive digits of π, we can compute the stationary distribution by solving $\pi P = \pi$. This vector represents the equilibrium frequencies of digits the proportions that would emerge if the observed digit-to-digit transitions governed an idealized stochastic process. Using the markovchain package in R, This vector represents the equilibrium frequencies of digits the proportions that would emerge if the observed digit-to-digit transitions governed an idealized stochastic process. Using the markovchain package in R,

```
library ( markovchain )
state_names <- as. character (0:9)
rspi <- new(" markovchain ", states = state_names , transitionMatrix = P)
round( steadyStates (rspi), 3)
```

we obtain:

$$\pi = \begin{bmatrix} 0.100 & 0.101 & 0.099 & 0.100 & 0.100 & 0.100 & 0.100 & 0.100 & 0.100 & 0.099 \end{bmatrix}. \tag{6}$$

Each component of π lies strikingly close to 0.1, the value expected under perfect uniformity. This near-flat equilib-rium is not imposed by any probabilistic assumption it emerges from the deterministic sequence of π itself. The stationary distribution thus captures a subtler form of uniformity than raw frequency counts: it measures balance not only among digits, but among their transitions.

3.3. Chi-Squared Test for Uniformity

The stationary distribution from analyzing 100,000 digits of π, rounded to 5 digits is:

$$\pi = \begin{bmatrix} 0.09999 & 0.10136 & 0.09908 & 0.10025 & 0.09971 & 0.10026 & 0.10029 & 0.10025 & 0.09978 & 0.09902 \end{bmatrix}$$

The standard statistical hypothesis testing framework:

- Null Hypothesis (H0): The digits follow a uniform distribution (pi = 0.1 for all i = 0, . . . , 9)
- Alternative Hypothesis (H1): The digits do not follow a uniform distribution (at least one pi , 0.1)

First, we perform a one-sample t-test to compare the observed stationary distribution π against the theoretical uniform distribution where each digit proportion equals 0.1.

```
# Observed stationary distribution
pi_ observed <- c(0.09999 , 0.10136 , 0.09908 , 0.10025 , 0.09971 ,
0.10026 , 0.10029 , 0.10025 , 0.09978 , 0.09902)
# One -sample t-test against theoretical mean of 0.1
t_test_result <- t.test(pi_observed , mu = 0.1)
# Additional descriptive statistics
mean_ deviation <- mean(abs(pi_ observed - 0.1))
```

```
max_ deviation <- max(abs(pi_ observed - 0.1))
cat("T-test results :\n")
cat("T- statistic :", t_test_ result statistic , "\n")# -0.00470494
cat("P-value:", t_test_ result p.value , "\n")# 0.9963487
cat("Mean absolute  deviation :", mean_deviation , "\n")# 0.000483
cat(" Maximum absolute deviation :", max_deviation , "\n")# 0.00136
```

The one-sample t-test comparing the observed stationary distribution against the theoretical uniform distribution yields an exceptionally high p-value of 0.9963487. Rejecting the null hypothesis of uniformity would require accepting a 99.63% risk of error, far exceeding the conventional 5% threshold used in scientific research. This overwhelming statistical evidence provides resounding support for the uniformity conjecture of π's digits. Now, we perform a chi-squared goodness of-fit test to evaluate whether this distribution significantly deviates from uniformity.

R Code for Chi-Squared Test:

```
# Chi - squared test for n=100 ,000 digits ( corrected distribution )
pi_ stationary _100k <- c(0.09999 , 0.10136 , 0.09908 , 0.10025 , 0.09971 ,
0.10026 , 0.10029 , 0.10025 , 0.09978 , 0.09902)
n_100k <- 100000
expected <- rep (0.1 , 10)

# Method 1: Direct calculation
chi_sq_stat_100k <- n_100k * sum ((pi_ stationary _100k - expected )ˆ2 / expected )
df <- 9
p_value_100k <- 1 - pchisq (chi_sq_stat_100k, df)

cat("Chi - squared statistic (n=100 ,000):", chi_sq_stat_100k, "\n")#4.0657
cat(" Degrees of freedom :", df , "\n")#9
cat("P-value:", p_value_100k, "\n")# 0.9070353
cat(" Critical value(alpha =0.05) :", qchisq (0.95 , df), "\n")# 16.91898
```

The chi-squared test yields the following results:
• Chi-squared statistic: 4.0657
• Degrees of freedom: 9
• P-value: 0.907

Key observations:
• The test fails to reject the uniform distribution hypothesis ($p = 0.907 > 0.05$)
• The chi-squared statistic (4.067) is well below the critical value of 16.92
• This stationary distribution shows remarkable consistency with the uniform distribution

This analysis demonstrates that the Markov chain model provides strong evidence supporting the uniformity conjecture for π's digits.

Traditional analyses of π treat the sequence as a static collection of digits, testing whether each symbol appears equally often. The Markov-chain approach instead views the digits dynamically, as a system of interrelated states whose interactions generate statistical equilibrium. In this sense, the stationary distribution serves as an ergodic fingerprint a structural invariant encoding how local transitions sustain global uniformity. That this invariant aligns almost perfectly with the uniform vector (0.1, 0.1, . . . , 0.1) is compelling evidence that the decimal expansion of π behaves, at least statistically, as a perfectly mixed system. Ergodic theory provides the conceptual bridge linking this empirical observation to deeper principles of order and randomness. In an irreducible and aperiodic Markov chain, every state communicates with every other, and the system forgets its initial condition. Over time, the chain settles into a unique stationary distribution a mathematical embodiment of balance through motion. When applied to the digits of π, this principle acquires a poetic resonance. Each digit's appearance depends deterministically on arithmetic law, yet their collective behavior mirrors that of a random process approaching equilibrium. The digits seem to "forget" their origins, converging toward a statistical symmetry that no human design enforced. In this, π offers a striking metaphor for ergodic order: the calm of uniformity arising from the turbulence of local variation.

This perspective extends the classical conjecture of π's normality. Rather than viewing uniformity as a mere counting property, the ergodic viewpoint reveals it as an emergent balance an invariant of transition and recurrence. Philosophically, it unites two mathematical aesthetics: the precision of determinism and the fluidity of chance. In the stationary distribution of π, we glimpse a harmony where necessity imitates randomness, and where the infinite digits of a circle's ratio whisper the deeper music of equilibrium. To better understand why the stationary distribution of a Markov chain provides stronger evidence than simple statistics, it is helpful to consider Google's PageRank algorithm as an analogy. In PageRank, the web is modeled as a Markov chain: each web page is a state, and hyperlinks define the probability of transitioning from one page to another. The stationary distribution of this chain represents the long-term visit probabilities of each page, effectively ranking pages according to their connectivity and importance. This approach is superior to simple counting statistics because it captures the global structure of the system rather than just local frequencies. We applied this analogy to the digits of π.

We constructed a Markov chain where each state corresponds to a digit 0, 1, . . . , 9, and transitions are defined based on observed sequences of digits in π. Since the chain is sufficiently ergodic (irreducible and aperiodic), it has a unique stationary distribution. Observing that this stationary distribution is approximately uniform across all digits implies that, in the long term, each digit is equally likely, independent of initial conditions or local clustering. In this context, all states of π (i.e., digits 0–9) have the same stationary probability, analogous to equally ranked pages in a perfectly balanced web. This is stronger than simply counting the frequency of digits because the stationary distribution accounts for all possible sequences and transitions, not just isolated occurrences. Therefore, convergence to a uniform stationary distribution provides a robust, global statistical argument supporting the practical uniformity of the digits of π. Unlike a simple frequency test, it inherently considers the dynamics of digit sequences and their long-term behavior, making the evidence more conclusive. Other approaches, such as higher-order Markov models, hidden Markov models, or stochastic processes with memory, could also be explored, but they often require more parameters and computational complexity. Therefore, the Markov chain offers a balance between analytical tractability, interpretability, and the ability to provide meaningful statistical evidence of approximate uniformity, while leaving the door open for future work using alternative probabilistic frameworks.

3.4. Generalization to Irrational Numbers

In the same spirit of functional programming, as partially discussed in [6], one can define a sequence of R functions that allows this study to be generalized to other irrational numbers:

```
library (Rmpfr)
library ( markovchain )

# --- Robust digits () function ---
digits <- function (x, n) {
# Convert input to mpfr with sufficient precision
x_mpfr <- mpfr(x, prec = 4 * n)

# Convert to string without scientific notation
x_char <- format (x_mpfr , scientific = FALSE)

# Remove integer part and decimal point
x_char <- sub("^[0 -9]+\\.", "", x_char)

# Pad with zeros if string is shorter than n
if (nchar(x_char) < n) {
x_char <- paste0 (x_char , paste(rep("0", n - nchar(x_char)), collapse =""))
}

# Extract exactly n digits
digits _vec <- substring (x_char , 1:n, 1:n)

# Convert to numeric vector
as. numeric (digits _vec)
}

# --- Counting matrix N(x, n) ---
N <- function (x, n) {
d <- digits (x, n)
M <- matrix (0, nrow = 10, ncol = 10)

for (k in 1:( length (d) -1)) {
i <- d[k]
j <- d[k+1]
M[i+1, j+1] <- M[i+1, j+1] + 1
}

rownames (M) <- 0:9
```

```
colnames (M) <- 0:9
M
}

# --- Transition matrix P(x, n) ---
P <- function (x, n) {
M <- N(x, n)
R <- rowSums (M)
T <- matrix (0, nrow = 10, ncol = 10)

for (i in 1:10) {
if (R[i] > 0) {
T[i, ] <- M[i, ] / R[i]
}
}

rownames (T) <- 0:9
colnames (T) <- 0:9
T
}

# --- Stationary distribution pi_stat(x, n) ---
pi_stat <- function (x, n) {
T <- P(x, n)
states <- as. character (0:9)
mc <- new(" markovchain ", states = states , transitionMatrix = T)
steadyStates (mc)
}

# ########### Example usage ############

# Extract 100 ,000 digits of e and compute stationary distribution
e_val <- exp(mpfr (1, prec = 400000) )
distr_e <- pi_stat(e_val , 100000)
round(distr_e, 3)

# Extract 50 ,000 digits of sqrt (2)
sqrt2_val <- sqrt(mpfr (2, prec = 200000) )
distr_sqrt2 <- pi_stat(sqrt2_val , 50000)
round(distr_sqrt2 , 3)
```

For the irrational numbers e and $\sqrt{2}$, the stationary distributions obtained are:

$$\pi_e = \left(0.099, 0.103, 0.099, 0.100, 0.100, 0.100, 0.102, 0.099, 0.100, 0.099\right),$$

$$\pi_{\sqrt{2}} = \left(0.099, 0.101, 0.100, 0.100, 0.103, 0.099, 0.099, 0.100, 0.099, 0.100\right)$$

which clearly confirms the uniformity of e and $\sqrt{2}$, in the same way as discussed for π.

3.5. Uniformity of π in other bases

In the same line as [6], we use R to construct a set of functions that extract the first n digits of π in an arbitrary base b. These tools enable us to replicate all computations previously performed in base 10, but now for any chosen base. The procedure consists in generating the digits with sufficiently high precision, building the empirical transition matrix associated with successive digits, and then computing the stationary distribution of the resulting Markov chain.

```
library (Rmpfr)

digits _base <- function (n, base = 8) {
# Required bits: log2(base) per digit
prec_bits <- ceiling (n * log2(base)) + 20

# High precision pi
```

```
pi_mpfr <- Const("pi", prec = prec_bits)
# Convert to base ’base ’ string
s <- formatMpfr (pi_mpfr , base = base , digits = n + 5) # extra precision
# Remove integer part "3."
s <- sub("ˆ[0 -9a-zA -Z]+\\.", "", s)
# Keep exactly n digits
s <- substring (s, 1, n)
# Convert characters to numbers in 0:( base -1)
as. numeric ( strsplit (s, "") [[1]])
}
# Test: First 100000 base -8 digits of pi
pi8 <- digits _base (100000 , base = 8)
# Show last 10 digits
pi8 [99991:100000]
N_from_digits <- function (pib , base) {
N <- matrix (0, nrow = base , ncol = base)
for (k in 1:( length (pib) - 1)) {
i <- pib[k] + 1
j <- pib[k + 1] + 1
N[i, j] <- N[i, j] + 1
}
rownames (N) <- 0:( base - 1)
colnames (N) <- 0:( base - 1)
N
}
P_from_digits <- function (pib , base) {
N <- N_from_digits (pib , base)
rs <- rowSums (N)
P <- matrix (0, nrow = base , ncol = base)
for (i in 1: base) {
if (rs[i] > 0) {
P[i, ] <- N[i, ] / rs[i]
}
}
rownames (P) <- 0:( base - 1)
colnames (P) <- 0:( base - 1)
P
}
library ( markovchain )
pi_stat_from_digits <- function (pib , base) {
P <- P_from_digits (pib , base)
mc <- new(" markovchain ",
states = as. character (0:( base - 1)),
transitionMatrix = P)
```

```

as. numeric ( steadyStates (mc))
}
```

For example, in base 4, we compute:
pi4 ← digits base(100000, base = 4),
N4 ← N from digits(pi4, 4),
P4 ← P from digits(pi4, 4),
pi stat4 ← pi stat from digits(pi4, 4).

After rounding, the stationary distribution is

$$\pi_{\pi}^{(4)} = (0.2498, 0.2497, 0.2496, 0.2509)$$

which is fully consistent with the expected uniform distribution in base 4.

3.6. Franklin's Framework of Logical Probability and its Connection to Our Study

Our investigation of digit-distribution uniformity in mathematical constants intersects with James Franklin's frame-work of logical probability [7] in several important ways, though with necessary qualifications. Franklin argues that mathematical conjectures can accumulate substantial non-deductive support from empirical and probabilistic considerations; this view legitimizes the use of large-scale computational evidence as evidence-bearing material in mathematics.Our approach treating digit sequences statistically accords with that spirit, but it goes further by embedding empirical counts in a structured probabilistic model. In particular, Franklin discusses inductive reasoning from computed examples (for instance, inferring that later segments of a sequence will resemble earlier ones) and the ways in which such evidence increases the plausibility of conjectures. The Markov-chain methodology we employ here refines that idea: rather than relying only on marginal digit frequencies, we analyse transition structures and compute stationary distributions, thereby supplying a model-based, dynamical form of support that captures both local dependencies and global equilibrium. Franklin also emphasises the limits of non-deductive support: probabilistic assignments depend on modeling choices and reference classes, and one must distinguish between model-generated quantities and literal measures of logical probability. In that vein, we present stationary distributions as model properties robust diagnostics that strengthen empirical claims while acknowledging they are not direct proofs or definitive logical-probability values for normality. Thus, Franklin's central insight that probabilistic methods can legitimately augment mathematical belief supports our methodology, provided we remain explicit about the distinction between model evidence and deductive proof [7].

4. Interpretation

Leibniz famously claimed that our world is the best of all possible worlds [8]. Yet if this is indeed the case, how are we to reconcile his assertion with the widespread oppression, injustice, and inequality that we observe in the world? If we allow ourselves to treat π as a kind of silent protagonist, its digits reveal a story of patience and eventual fairness. Consider the first hundred digits: computing the stationary distribution over this short interval yields

$$\begin{bmatrix}0.08 & 0.069 & 0.122 & 0.114 & 0.101 & 0.080 & 0.088 & 0.082 & 0.123 & 0.142\end{bmatrix}.$$

Here, digit 1 appears unusually seldom, almost as if π were neglecting or oppressing it. On such a brief stage, imbalance dominates; some digits shine, others languish. Yet, this local inequity is deceptive. Extend the gaze to a longer span the first million, or billion digits and the stationary distribution smooths into near-perfect uniformity. Each digit, without exception, eventually claims its due share, and π exhibits the impartiality of a just arbiter.

In this mathematical drama, π enacts a kind of cosmic patience: early injustices are not evidence of permanent disorder. Transient inequalities give way to enduring equilibrium, reminding us that apparent chaos can mask an underlying harmony. From the viewpoint of Leibniz's philosophy, our universe like π may temporarily privilege or suppress certain outcomes, yet the grand design ensures that, in the long run, justice is done and every state receives its rightful share. Viewed through the lens of Markov chains and stationary distributions, π becomes a character who enforces fairness not instantaneously, but through a subtle, cumulative process. Local imbalances are real and instructive, yet they do not contradict the ultimate uniformity. In this way, π offers a tangible metaphor for Leibniz's optimism: even in a world that seems unfair at first glance, a deeper equilibrium prevails over time. A second, complementary interpretation deepens this insight. Suppose we regard the digit 1 as a "preferred state" of π. Although its occurrence may be infrequent in early segments, over the long term it attains its rightful proportion. Indeed, π is nothing but the collection of all its decimal digits; each state preferred or unpreferred is essential to its identity.

By analogy, each of us is the sum of the sequence of our days, each day representing a distinct state, whether joyous or challenging. Every experience, every seemingly unpreferred moment, contributes indispensably to the richness and coherence of our lives. This perspective encourages acceptance of life as it is, fostering the understanding that fairness and peace, whether personal or societal, emerge gradually over time rather than instantaneously.

5. Conclusion

In this essay, we have explored the digits of π through multiple lenses: historical, mathematical, probabilistic, and philosophical. Beginning with a cultural history of π's computation, we traced the evolution of human ingenuity from ancient polygonal approximations to modern algorithmic feats, highlighting the enduring fascination with this transcendental number. We then examined the digits as a Markov chain, emphasizing ergodicity and the stationary distribution. This perspective provides a deeper, model-based measure of uniformity than traditional frequency counts. By treating π's digits dynamically, we revealed how local imbalances even the temporary neglect of certain digits eventually give way to global equilibrium, illustrating the subtle harmony hidden within deterministic sequences. It is important to acknowledge the limitations and challenges of applying a Markov chain model to the digits of a deterministic sequence such as π. While the model provides a convenient probabilistic framework to study the long-term behavior and approximate uniformity of digits, the underlying sequence is fully deterministic, which means that traditional stochastic assumptions of independence and randomness do not strictly hold. As a result, conclusions drawn from the stationary distribution should be interpreted as strong empirical and statistical evidence rather than a formal proof of randomness or uniformity. Nevertheless, the Markov chain approach is flexible and can be readily applied to other irrational numbers or deterministic sequences, such as e or $\sqrt{2}$, by constructing analogous transition matrices based on observed digit sequences. Our investigation of the ergodic properties of digit sequences in mathematical constants aligns with the dynamical systems approach exemplified by Ghazel et al. [11], who conducted a comprehensive stability and periodicity analysis of rational difference equations. Both studies share a common framework of analyzing long-term behavior in deterministic systems. Our discussion also connected these findings to Franklin's framework of logical probability, demonstrating that structured probabilistic reasoning can serve as a rigorous form of non-deductive support for mathematical conjectures. The stationary distributions computed here exemplify how empirical patterns in deterministic objects can be quantitatively analyzed, providing robust, model-based evidence for the conjectured uniformity of π. Finally, we reflected on the philosophical implications of these results. By personifying π and examining its long-term fairness, we drew an analogy to Leibniz's claim that our world is the best of all possible worlds. Temporary inequities whether in digits or in human affairs do not negate the emergence of global balance. In this sense, the mathematics of π offers a poetic insight into the coexistence of determinism, randomness, and eventual equilibrium, bridging the conceptual gap between law and chance. Taken together, the historical, probabilistic, and philosophical threads of this paper illuminate not only the structure of π's digits but also the broader significance of probabilistic reasoning in mathematics. The interplay between deterministic processes and emergent uniformity invites reflection on the deeper harmony underlying both numbers and the world, suggesting that order and fairness can manifest even in seemingly chaotic systems.

Declarations

Funding: This research received no external funding.

Acknowledgments

The corresponding author would like to acknowledge Prof. Anis Rezgui (Département de Mathématiques, FST) for fruitful discussions.

References

[1] David H. Bailey and Richard E. Crandall, "Random generators and normal numbers", Experimental Mathematics, Vol.11, No.4, pp. 527-546, 2012.
[2] Emile Borel, "Sur les chiffres décimaux de $\sqrt{2}$ et divers problèmes de probabilités en chaîne", Comptes Rendus de l'Académie des Sciences, Vol.230, pp. 591-593, 1950.
[3] Yann Bugeaud, "Distribution Modulo One and Diophantine Approximation", Cambridge University Press, 2012.
[4] Jonathan M. Borwein and David H. Bailey, "Mathematics by Experiment: Plausible Reasoning in the 21st Century", A K Peters, 2004.
[5] Yasumasa Kanada, "Numerical computations of π and applications", Journal of Computational and Applied Mathematics, Vol.159, No.1, pp. 1-8, 2003.
[6] Fethi Kadhi, "Bridging Category Theory and Functional Programming for Enhanced Learning", International Journal of Mathematical Sciences and Computing (IJMSC), Vol.11, No.3, pp. 1-18, 2025. DOI: 10.5815/ijmsc.2025.03.01. James
[7] Franklin, "Logical Probability and the Strength of Mathematical Conjectures", The Mathematical Intelligencer, Vol.38, No.3, pp. 14-19, 2016.

[8] Gottfried Wilhelm Leibniz, "Essais de Th´eodic´ee sur la bont´e de Dieu, la libert´e de l'homme et l'origine du mal", Isaac Troyel, Amsterdam, 1710.
[9] James R. Norris, "Markov Chains", Cambridge University Press, 1998.
[10] J. Liu and R. Wilson, "The Beauty of Mathematics", The Mathematical Intelligencer, 2025. DOI: 10.1007/s00283025-10429-7.
[11] Ghazel M, Alraqad T, and Kadhi F (2020). Dynamical study of the class of difference equation $x_{n+1} = \frac{x_{n-2q+1}}{A + Bx_{n-2q+1}x_{n-q+1}}$. International Journal of Advanced and Applied Sciences, 7(5): 66–78.

Authors' Profiles

Fethi Kadhi is an Assistant Professor at the National School of Computer Science (ENSI), University of Manouba, Tunisia. In addition to his teaching duties, he holds several leadership roles within the university's research groups and laboratories. Notably, he led the Department of Image and Mathematical Modeling at ENSI and coordinated with other departments to establish two Master's degree programs. Dr Kadhi earned his B.S. in Mathematics and his M.S. in Optimization Theory from the Faculty of Sciences of Tunis, where he later completed his Ph.D. in Applied Mathematics. His research interests are interdisciplinary, encompassing Optimization, Dynamical Systems, Computer Science, and Category Theory.

Moncef Ghazel is a Tunisian mathematician and researcher at the Faculty of Sciences of Tunis, University of Tunis El Manar. He earned his Ph.D. in Mathematics from the University of Wisconsin–Madison in 1995, focusing on algebraic topology. His research interests include algebraic topology, category theory, and topological structures. Dr. Ghazel has contributed to the study of loop groups, fibrewise topology, and Reedy diagrams in model categories. His recent work includes the articles \Kan extendable subcategories and fibrewise topology (2024), Thin loop groups (2020), and Reedy Diagrams in V-Model Categories (2019), reflecting his focus on combining categorical methods with topological and homotopy-theoretic structures.

Malek Ghazel is a mathematician whose research focuses on nonlinear discrete dynamical systems, with a particular emphasis on rational and higher-order difference equations. He has contributed to the analysis of stability, periodicity, and asymptotic behavior of such systems, and has collaborated with several researchers on analytical studies in this area. His academic affiliations include the University of Ha'il and Tunis El Manar University.

How to cite this paper: Fethi Kadhi, Moncef Ghazel, Malek Ghazel, "Ergodicity and the Emergence of Long-Term Balance in the Dynamical States of π", International Journal of Mathematical Sciences and Computing(IJMSC), Vol.12, No.1, pp. 42-54, 2026. DOI: 10.5815/ijmsc.2026.01.04