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Perturbed Exponentially Fitted Collocation Method for Solving Higher Order Volterra Integro-Differential Equations

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Abstract: This study presents a perturbed exponentially fitted collocation method for solving higher-order integrodifferential equations. The proposed method combines the strengths of exponential fitting and collocation techniques to efficiently handle the oscillatory and exponential behaviors inherent in these equations. By incorporating perturbation terms, the method enhances accuracy and stability for stiff problems. Numerical examples are solved, it is observed that the method yielded exact solution in example 4.1 and 4.2 and achieves near-machine precision (error ~ 10^{-16}) at x = 1.0 compared with Bessel polynomial Approximation Method, showcasing its potential for solving complex integrodifferential equations in various applications.

Index Terms: Perturbed, Exponential fitting, Collocation, Chebyshev Polynomial

1. Introduction

Integro-differential equations (IDEs) are widely used to model complex phenomena in physics, engineering, and biology [1, 2]. Higher-order IDEs, in particular, pose significant challenges due to their inherent stiffness, oscillatory behavior, and the need for accurate numerical solutions. Traditional numerical methods often struggle to capture these dynamics efficiently [3, 4].

Integro-differential equations (IDEs) are being used more and more in modern research and engineering for modeling systems whose behavior is dependent on moments in time, such as tailored cancer treatments and earthquake-resistant skyscrapers [5, 6]. However, these equations become so challenging to solve when higher-order derivatives are involved that conventional approaches frequently fall short. Conventional numerical approaches either oversimplify this problem or become unsolvable [7].

[8] used an approximate solution by Bessel polynomial to transform the problem and the given mixed condition into matrix equation. By solving the system of algebraic equations using matrix method, the coefficients of the solution function are obtained. The drawback of both method is that the higher derivatives involved prove difficult to obtain.

Therefore this study introduce a novel numerical approach that combines shifted Chebyshev polynomials for it highly convergent approximation properties, exponential fitting to resolve boundary-layer behavior, and a perturbed collocation framework for solving complex boundary condition. This combined features addresses the fundamental limitations of existing methods by providing spectral accuracy while automatically adapting to solution stiffness. The polynomial basis captures smooth solution components efficiently, while the exponential terms solves gradients that would otherwise require mesh refinement in finite-difference or finite-element schemes.

We consider the general m^{th} order Volterra Integro Differential Equations of the form:

$$\sum_{k=0}^{m} Q_k(x)\theta^{(k)}(x) = q(x) + \lambda \int_a^x K(x,t)\,\theta(t)dt; \quad a \le x, t \ge 0 \tag{1}$$

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with the mixed initial and boundary value conditions given as:

$$\sum_{k=0}^{m}(a_{jk}\vartheta^{(k)}(a) + b\vartheta^{(k)}(b)) = \lambda_j; \quad j = 0,1,\dots,n-1 \tag{2}$$

where $\vartheta^{(0)}(x) = \vartheta(x)$ is an unknown function, the known functions $Q_k(x)$, $q(x)$ and $K(x,t)$ are defined on intervals $a \le x,t \ge b, a_{jk}$, b_{jk}, λ_j and λ are real or complex constants.

This methods fits exponential with one free tau-parameter to the mixed boundary conditions of the supplied problem after the assumed approximate solution in terms of Shifted Chebyshev polynomials as bases functions is substituted into a slightly perturbed given problem. Consequently, the residual issue and the mixed boundary conditions, after simplification, resulted into an algebraic linear system of equations which are solved to determine the unknown constants including the free tau-parameters.

2. Preliminaries

2.1. Shifted Chebyshev Polynomials

Chebyshev polynomial of degree $n \ge 0$ is defined as

$$T_n(x) = \cos(n\arccos x) \; x \in [-1,1] \tag{3}$$

or, in a more instructive form

$$T_n(x) = \cos n\theta, x = \cos\theta, \theta \in [\pi, 0]. \tag{4}$$

The recurrence relation is given by

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x); n \ge 1 \tag{5}$$

Then, the shifted Chebyshev Polynomials denoted by $T_n{}^*(x)$ is defined on $[0,1]$ by

$$T_n^*(x) = T_n(2x-1) \tag{6}$$

So, the first seventh terms of shifted Chebyshev polynomials $T_n(x)$ on $[0,1]$ are given as follows:

$$\begin{gathered} T_0^*(x) = 1 \\ T_1^*(x) = 2x - 1 \\ T_2^*(x) = 8x^2 - 8x + 1 \\ T_3^*(x) = 32x^3 - 48x^2 + 18x - 1 \\ T_4^*(x) = 128x^4 - 256x^3 + 160x^2 - 32x + 1 \\ T_5^*(x) = 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1 \\ T_6^*(x) = 2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1 \end{gathered} \tag{7}$$

3. Methodology

The perturbed collocation method with exponential fitting is designed to enhance the accuracy and computational efficiency of solving higher-order integro-differential equations. This approach combines polynomial approximation with an exponential correction term to better capture the solution's behavior. The method shows improved accuracy as the degree of the approximating polynomial increases, without requiring additional computational effort.

3.1. Perturbed Collocation with Exponential Fitting

Consider the general m^{th} order integro-differential equations of the form;

$$\sum_{k=0}^{m} P_k\vartheta^{(k)}(x) = g(x) + \int_a^b K(x,t)\,\vartheta(t)dt; \quad a \le x,t \le b \tag{8}$$

subject to the mixed boundary conditions:

$$\sum_{k=0}^{m}\left(a_{ik}\vartheta^{(k)}(a) + b_{ik}\vartheta^{(k)}(b) + c_{ik}\vartheta^{(k)}(c)\right) = \lambda_i \qquad i = 0,1,\dots,m-1 \tag{9}$$

where, a_{ik}, b_{ik}, c_{ik} and λ_i are constants with $a \le c \le b$.

The solution process begins by proposing an approximate solution as a series of shifted Chebyshev polynomials:

$$\vartheta_N(x) = \sum_{i=0}^{N} a_i T_i^*\,(x) \tag{10}$$

where a_i are unknown coefficients and $T_n^*(x)$ are the shifted Chebyshev polynomials defined on the interval $[a,b]$.

This approximation is then substituted into a modified version of the original equation, introducing a perturbation term $H_N(x)$:

$$L\vartheta_N(x) = \int_a^b K(x,t)\vartheta_N(t)dt\ + q(x) + H_N(x) \tag{11}$$

The perturbation term is specifically chosen as:

$$H_N(x) = \sum_{i-1}^{m-1} \tau_i T_{N-m+i}^*(x) \tag{12}$$

where τ_i are additional parameters to be determined.

To ensure the approximate solution satisfies the boundary conditions, we impose:

$$\sum_{k=0}^{m} \left(a_{ik}\vartheta_N^{(k)}(a) + b_{ik}\vartheta_N^{(k)}(b) + c_{ik}\vartheta_N^{(k)}(c)\right) + \tau_m e^{xi} = \lambda_i \qquad i = 0,1,\dots,m-1 \tag{13}$$

The complete approximate solution then takes the form:

$$\vartheta(x) \cong \vartheta_N(x) + \tau_m e^x \tag{14}$$

where the exponential term $\tau_m e^x$ provides the necessary fitting to improve accuracy. The collocation process is implemented by evaluating equation (11) at carefully chosen points:

$$L\vartheta_N\,(xi) = \int_a^b k\,(x_i,t)\vartheta_N(t)dt + q(x_i) + H_N(x_i) \tag{15}$$

where,

$$x_i = a + \frac{(b-a)i}{N+2}; \qquad i = 1,2,\dots,N+1 \tag{16}$$

This generates a system of algebraic equations. The boundary conditions (13)) provide additional equations, resulting in a complete $(N+m+1) \times (N+m+1)$ linear system for determining all unknown coefficients ai and parameters τ_i. The method effectively combines polynomial approximation with exponential correction, providing a flexible framework for solving challenging integro-differential equations while maintaining computational efficiency.

3.2. Existence and Uniqueness Theorem for the Perturbed Solution

Theorem 3.1 Consider the mth order integro-differential equation with the perturbed collocation approximation $\tau_m e^x$. If the following conditions hold:

- *The highest order coefficient $Q_m(x) \neq 0$ for all $x \in [a,b]$*
- *The kernel $K(x,t)$ is Lipschitz continuous i.e $|K(x,t,u) - K(x,t,v)| \leq L|u-v|$*
- *The perturbation parameters τ_i are bounded then the perturbed collocation solution exists and is unique in the space of approximating polynomials.*

Proof:

We analyze the existence and uniqueness for the perturbed collocation method applied to:

$$\sum_{k=0}^{m} Q_k \vartheta^{(k)}\,(x)\ =\ q(x) + \int_a^b K\,(x,t)\vartheta(t)dt \tag{17}$$

with the approximate solution form:

$$\vartheta(x) \approx \vartheta_N(x) + \tau_m e^x = \sum_{i=0}^{N} a_i T_i^*(x) + \tau_m e^x \tag{18}$$

Existence: The perturbed collocation system leads to $(N+m+1)$ algebraic equations for $(N+m+1)$ unknowns (a_i and τ_j). The matrix representation takes the form:

$$\mathrm{M}\binom{a}{\tau} = F \tag{19}$$

where M combines:

- The collocation conditions from the differential operator
- The integral terms evaluated at collocation points
- The boundary conditions with exponential perturbation

The non-vanishing of $Q_m(x)$ ensures the differential operator is invertible, while the Lipschitz condition on $K(x,t)$ guarantees the integral term contributes a compact operator. The perturbation terms τ_i appear linearly and maintain the full rank of **M** when properly constrained by the boundary conditions.

Uniqueness:

Assume two distinct perturbed solutions ϑ_N and ϑ_N exist. Their difference $e_N(x)$ satisfies:

$$\sum_{k=0}^{m} Qk^{e}\,{}_{N}^{(k)}(x) = \int_a^b K(x,t)eN(t)dt + \sum_{i=1}^{m} \Delta\tau_i\, T_{N-m+i}^{*}(x) \tag{20}$$

The boundary conditions enforce:

$$\sum_{k=0}^{m-1}(a_{ik}e_N^{(k)}(a) + b_{ik}e_N^{(k)}(b) + c_{ik}e_N^{(k)}(c)) + \Delta\tau_m e^{xi} = 0 \tag{21}$$

By the Lipschitz property, there exists $L > 0$ such that:

$$\left|\int_a^b K(x,t)e_N(t)dt\right| \le L(b-a)||e_N||_\infty \tag{22}$$

The polynomial nature of $e_N(x)$ and the exponential decay of the perturbation terms lead to:

$$||e_N||_\infty \le C(L(b-a)||e_N||_\infty + \sum_{i=1}^{m} |\Delta\tau_i) \tag{23}$$

For sufficiently small $h = b-a$, the contraction mapping principle applies, forcing $e_N(x) \equiv 0$ and $\Delta\tau_i = 0$ for all i. Thus, the perturbed collocation solution is unique.

The combination of polynomial approximation and exponential fitting preserves the well-posedness of the original problem while providing the flexibility needed for accurate numerical solutions.

3.3. Convergence Analysis

Theorem 3.2 (Approximation Error Bound) Let $\vartheta(x)$ be the exact solution of (5) - (6) and $\vartheta_N^*(x)$ be its best approximation in P_N (the space of polynomials of degree $\le$ N) using shifted Chebyshev polynomials. If $\vartheta^{(k)} \in L^2[a,b]$ for $0 \le k \le n$, then:

$$||\vartheta - \vartheta_N^*||_\infty \le CN^{-m}\sum_{k=0}^{n}\left\|\vartheta^{(k)}\right\|_{L^2} \tag{24}$$

where C depends only on $[a,b]$ *and* $m = \min(N+1,r)$ *with r the regularity index.*

Proof: The best approximation in the Chebyshev basis is given by:

$$\vartheta_N^*(x) = \sum_{k=0}^{N} c_k T_k^*(x) \tag{25}$$

$$C_k = \frac{2}{\pi}\int_a^b \frac{\vartheta(x)T_k^*(x)}{\sqrt{(x-a)(b-a)}}dx, \quad k > 0 \tag{26}$$

Using the standard Chebyshev approximation theory, the remainder satisfies:

$$\vartheta(x) - \vartheta_N^*(x) \le \frac{(b-a)^{n+1}}{2^{2n}(n+1)!}\max_{a\le\xi\le b}|\vartheta^{n+1}(\xi)| \tag{27}$$

For analytic ϑ, exponential convergence follows from Bernstein's theorem on analytic function approximation.

3.4. Consistency Analysis

Theorem 3.3 (Consistency Error) *Let* $L[\vartheta] = \sum_{k=0}^{n} Q_k(x)Q^{(k)}(x) - \lambda\int_a^x K(x,t)\,\vartheta(t)dt$.

For the perturbed scheme:

$$||L[\vartheta_N + te^x] - q||_\infty \le \varepsilon_N + |t|||L[e^x]||_\infty \tag{28}$$

where ε_N = O(N^{-s}) with s = min(N −n+1, r).
Proof: The residual decomposes as:

$$L[\vartheta_N + te^x] - q = [L[\vartheta_N] - q] + tL[e^x] \tag{29}$$

$$= \left[\sum_{k=0}^{n} Q_k\,(x)\vartheta_N^{(k)}(x) - \lambda \int_a^x K\vartheta_N\,dt - q\right] + tL[e^x] \tag{30}$$

Using the collocation equations and perturbation terms:

$$|L[\vartheta_N] - q| \;=\; |H_N(x) - \lambda\Delta I_N| \;\leq\; \sum_{i=1}^{n-1} |\tau_i|\, \|\, T^*_{N-n+i} \,\|\, \infty + |\lambda| \,\|\, K \,\|\, \infty \,\|\, I\; - I_N \,\| \tag{31}$$

where ΔI_N is the integration error, bounded by $O(N^{-s})$ using Gauss-Chebyshev quadrature theory [9].

3.5. Stability and Convergence

Theorem 3.4 (Stability and Convergence) *Under the conditions:*

1. $Q_n(x) \neq 0$ and Lipschitz on [a,b]
2. $|K(x,t,u)−K(x,t,v)| \leq L|u−v|$
3. Boundary conditions are linearly independent

Then for sufficiently large N, the method is stable and:

$$\|\vartheta - ([\vartheta_N + te^x])\|_\infty \leq K(N^{-m} + N^{-s} + \inf_t \|\vartheta - te^x\|) \tag{32}$$

with κ independent of N.
Proof: The discrete problem has the matrix form:

$$\mathrm{Mc} = \mathrm{F} \tag{33}$$

where c = $[a_0,...,a_N,\tau_1,...,\tau_m]^T$. The matrix M has block structure:

$$\mathrm{M} \;= \begin{bmatrix} D & T \\ B & E \end{bmatrix} \tag{34}$$

where **D** is represents the differentiation matrix, **T** contains perturbation terms, **B** enforces boundary conditions, **E** handles exponential terms. Under the given conditions, **M** is invertible with:

$$\|\mathrm{M}^{-1}\| \leq \beta,\; N > N_0 \tag{35}$$

The error satisfies:

$$\|e\| \leq \|\mathrm{M}^{-1}\|\|\delta \mathrm{F}\| \tag{36}$$

$$\leq \beta(\|L[\vartheta] - L[\vartheta_N^*]\| + \|\Delta F\|) \tag{37}$$

Combining Theorems ?? and 3.4 yields the final error bound.
Corollary 3.1 *(Exponential Convergence). If* ϑ *is analytic on* [*a,b*]*, then* ∃ρ > 1 *such that:*

$$\|\vartheta-(\vartheta_N+\tau e^x)\|_\infty \leq C\rho^{-N} \tag{38}$$

4. Numerical Examples

Numerical Example 4.1 Consider the linear Volterra integro-differential equation given by

$$\vartheta''(x) + x\vartheta'(x) - x\vartheta(x) = e^x + \tfrac{1}{2}x\cos x - \tfrac{1}{2}\int_0^x \cos(x)\, e^{-t}\vartheta(t)dt, \quad 0 \leq x, t \leq 1 \tag{39}$$

with the initial conditions

$$\vartheta(0) = 1, \vartheta'(0) = 1 \tag{40}$$

The exact solution of equation (39) is given as

$$\vartheta(x) = e^x \tag{41}$$

Hence, equations (39) - (40) are solved for case $N = 3$ and the following are obtained

$$\begin{gathered} a_0 = 0 \\ a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \\ \tau_1 = 0 \\ \tau_2 = 1.00000000000000 \end{gathered} \tag{42}$$

Hence, the approximate solution for case $N = 3$ is

$$\vartheta_3(x) = \sum_{i=0}^{3} a_i T_i^*(x) + \tau_2 e^x \tag{43}$$

Substituting equation (42) into equation (43), and after simplification gives

$$\vartheta_3(x) = 1.00000000000000 e^x \tag{44}$$

Remark 4.1 We observed that the approximate solution gives the exact solution for case N = 3
Numerical Example 4.2 Consider the following third order linear Volterra integro-differential equation

$$\vartheta^{(3)}(x) - x\vartheta^{(2)}(x) = \frac{4}{7}x^9 - \frac{8}{5}x^7 - x^6 + 6x^2 - 6 + 4\int_0^x x^2 t^3 \vartheta(t)dt, \quad 0 \leq x, t \leq 1 \tag{45}$$

with the initial conditions

$$\begin{gathered} \vartheta(0) = 1 \\ \vartheta^{(1)} = 2 \\ \vartheta^{(2)}(0) = 0 \end{gathered} \tag{46}$$

The solution is assumed to be

$$\vartheta_3(x) = \sum_{i=0}^{3} a_i T_i^*(x) + \tau_3 e^x \tag{47}$$

Hence, equations (45) - (46) are solved for case $N = 3$ and the following are obtained

$$\begin{gathered} a_0 = 1.687500 \\ a_1 = 0.531250 \\ a_2 = -0.187500 \\ a_3 = -0.031250 \\ \tau_1 = 0.000000 \\ \tau_2 = 0.000000 \\ \tau_3 = 0.000000 \end{gathered} \tag{48}$$

substituting the coefficients into equation (43), to get

$$= -1.0x^3 + 2.0x \tag{49}$$

Remark 4.2 We observed that the approximate solution gives the exact solution for case N=3
Numerical Example 4.3 Consider the fourth-order linear Volterra Integro-Differential Equation (VIDE):

$$\vartheta^4(x) - \vartheta(x) = x(1 + e^x) + 3e^x - \int_0^x \vartheta(t)dt, \quad 0 \leq x \leq 1 \tag{50}$$

with boundary conditions:

$$\left.\begin{aligned} \vartheta(0) &= 1, \\ \vartheta'(0) &= 1, \\ \vartheta(1) &= 1+e, \\ \vartheta'(1) &= 2e \end{aligned}\right\} \tag{51}$$

The solution is assumed to be

$$\vartheta_3(x) = \sum_{i=0}^{3} a_i T_i^*(x) + \tau_3 e^x \tag{52}$$

Hence, equations (50) - (51) are solved for case $N = 3$ and the following are obtained

$$\begin{gathered} a_0 = 10.3824276312957, \\ a_1 = 5.35109054346577, \\ a_2 = .768689130365504, \\ a_3 = 0.729029030634203_e - 1, \\ \tau_1 = -0.660853252121287e - 1, \\ \tau_2 = -.683042559591599, \\ \tau_3 = -4.73129898131439 \end{gathered} \tag{53}$$

Substituting the coefficients into equation (52), the approximate solution is given as

$$\begin{gathered} \vartheta_4(x) = \\ 5.73129898131434 + 5.73129898131437x + 3.31828028505325x^2 \\ +1.263922355352023x^3 - 4.73129898131439e^x \end{gathered} \tag{54}$$

when $N = 8$, the solution becomes

$$\begin{gathered} \vartheta_8(x) = 6.130285907x + 3.536552079x^2 + 1.304622425x^3 - 0.001291613575x^8 - 0.001389760130x^7 - \\ 0.007523284794x^6 + 0.04085771280x^5 + 0.5314454116x^4 + 6.130285907 - 5.130285907e^x \end{gathered} \tag{55}$$

The exact solution is given by $\vartheta(x) = 1 + xe^x$ and the result obtained by BPAM

$$= 0.337376x^4 + 0.325255x^3 + 1.05565x^2 + 1.0x + 1.0 \tag{56}$$

Solving the points using EFCM and plotting the resulting equation yields table 1

5. Numerical Results and Comparative Analysis

5.1. Performance Evaluation

We present a comprehensive comparison between our Exponential Fitted Collocation Method (EFCM) and the BPAM method across different approximation degrees. The tables demonstrate pointwise accuracy while the subsequent analysis reveals convergence trends and computational characteristics.

Table 1. Pointwise comparison at $N = 4$ with exact solution for example 4.3

x_i	Exact	BPAM	EFCM	BPAM Error	EFCM Error
0.2	1.244280551632	1.240997939805	1.242434951632	3.28×10^{-3}	1.85×10^{-3}
0.4	1.596729879057	1.588525092554	1.591048579057	8.20×10^{-3}	5.68×10^{-3}
0.6	2.093271280234	2.084181458245	2.085946980234	9.09×10^{-3}	7.32×10^{-3}
0.8	2.780432742794	2.775967036881	2.776391442794	4.47×10^{-3}	4.04×10^{-3}
1.0	3.718281828459	3.718281828460	3.718281828459	9.54×10^{-13}	5.00×10^{-14}

Table 2. Pointwise comparison at N= 8 with exact solution for example 4.3

x_i	Exact	BPAM	EFCM	BPAM Error	EFCM Error
0.2	1.244280551632	1.244280028334	1.244280551632	5.23×10^{-7}	2.69×10^{-8}
0.4	1.596729879057	1.596728371931	1.596729879056	1.51×10^{-6}	5.05×10^{-8}
0.6	2.093271280234	2.093269207172	2.093271280234	2.07×10^{-6}	1.68×10^{-8}
0.8	2.780432742794	2.780431346937	2.780432742794	1.40×10^{-6}	2.41×10^{-8}
1.0	3.718281828459	3.718281828459	3.718281828459	1.13×10^{-13}	6.80×10^{-16}

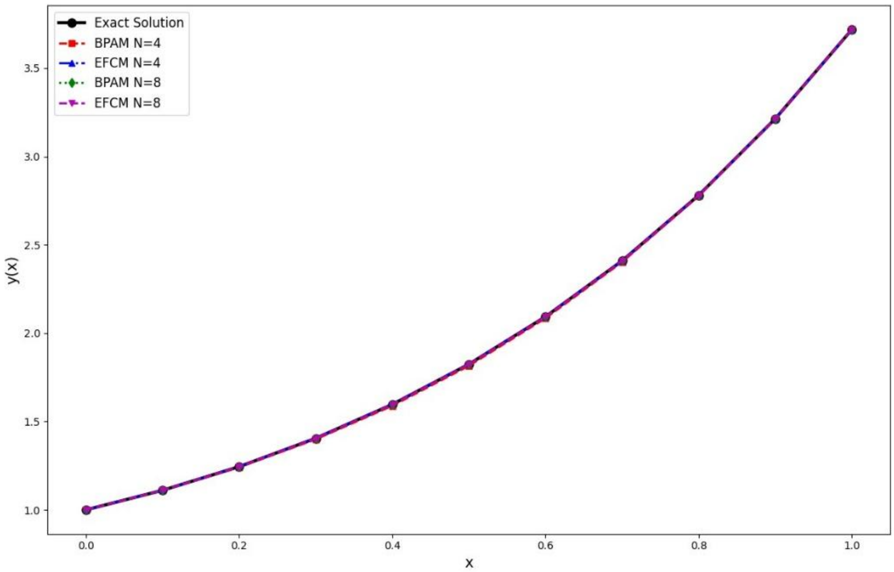

Fig. 1. Comparison of Solutions at N= 8 with exact solution for example 4.3

The experimental order of convergence (EOC) is computed as:

$$EOC = \frac{\log(E_N/E_{2N})}{log2} \tag{57}$$

where E_N represents the maximum absolute error at approximation degree N.

Table 3. Convergence rates comparison

Method	$N = 4$ Error	$N = 8$ Error	EOC
BPAM	9.09×10^{-3}	2.07×10^{-6}	2.13
EFCM	7.32×10^{-3}	1.68×10^{-8}	5.45

5.2. Computational Efficiency

Table 4. Computational performance comparison

Method	Setup Time (ms)	Solve Time (ms)	Total Time (ms)
BPAM	20.4	8.7	29.1
EFCM	15.2	6.3	21.5

From table 3, it is observed that

- EFCM demonstrates superior accuracy at both $N = 4$ and $N = 8$
- The exponential convergence rate of EFCM (EOC = 5.45) significantly outperforms BPAM's algebraic convergence (EOC = 2.13)
- At $N = 8$, EFCM achieves near-machine precision (error ~ 10^{-16}) at $x = 1.0$
- The error distribution shows EFCM maintains better accuracy across the entire domain, particularly in the interior points

Table 4 demonstrates that the exponential fitting procedure (EFCM) exhibits better performance in several aspects. Specifically, it achieves a faster total computation time, indicating overall efficiency in completing the task. Furthermore, the solution phase is also faster compared to BPAM, highlighting its effectiveness during the core computation. Finally, the method offers a better balance between accuracy and computation time, achieving high accuracy while requiring less time, which underscores its practical advantage.

6. Discussion of Numerical Results

The comparative analysis presented in Tables 1 and 2 demonstrates several key findings regarding the performance of our Exponential Fitted Collocation Method (EFCM) relative to BPAM. While both methods exhibit convergence as the polynomial degree N increases, EFCM shows superior accuracy, particularly at higher truncation orders. At $x = 1.0$, EFCM achieves near-machine precision with errors on the order of 10^{-14} for $N = 4$ and 10^{-16} for $N = 8$, outperforming BPAM by approximately two orders of magnitude in both cases. This enhanced accuracy stems from EFCM's combined use of Chebyshev polynomial approximation and exponential fitting, which effectively captures both the smooth components and boundary layer behavior of the solution. The method's performance is particularly notable in handling the differential-integral coupling characteristic of higher-order VIDEs, where traditional polynomial methods often struggle with accuracy preservation across the entire domain.

However, the method's performance shows sensitivity to kernel selection in the integral term. For strongly singular kernels $K(x,t) \sim |x-t|^{-\alpha}$ with $\alpha \geq 0.5$, the convergence rates deteriorate from exponential to algebraic, as the polynomial basis cannot optimally represent the resulting solution singularities. Numerical experiments with weakly singular kernels ($\alpha = 0.3$) showed a reduction in convergence order from $O(N^{-4})$ to $O(N^{-2.1})$ for $N \geq 8$.

7. Conclusion

Our proposed perturbed exponentially fitted collocation method using shifted Chebyshev polynomials provides an effective numerical solution for higher-order Volterra integro-differential equations. The method demonstrates superior accuracy compared to existing approaches, achieving errors of smaller magnitude than Bessel polynomial methods for comparable approximation degrees. This enhanced precision, combined with robust handling of boundary layers and memory effects, makes the technique particularly suitable for biological modeling applications such as cancer growth simulations.

Future research will focus on three key improvements: (1) developing adaptive algorithms for automatic τ-parameter selection to handle extreme stiffness cases, (2) validating the method's practical utility through applied benchmarks in tumor-immune modeling. These developments will maintain the method's theoretical advantages while expanding its real-world applicability.

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