

Fixed Point Results in the Fuzzy Multiplicative Metric Spaces and Application

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Abstract: This paper presents the development of fuzzy multiplicative metric spaces, an extended framework that combines the principles of multiplicative metric spaces with fuzzy logic to better address uncertainty and imprecision inherent in many real-world problems. By replacing additive distance measures with multiplicative ones, this approach proves particularly effective in contexts where relative variations or proportional relationships are more meaningful than absolute differences. Within this generalized setting, we establish a series of fixed-point theorems of Banach, Kannan, and Chatterjea types, along with related corollaries, each supported by concrete examples. The theoretical results are further validated through their application to the solution of nonlinear integral equations, demonstrating the versatility and applicability of the proposed framework across mathematical analysis and applied disciplines.

Index Terms: Fixed Point, Multiplicative Metric Spaces, Fuzzy Multiplicative Metric Spaces, Integral Equation

1. Introduction

Fixed point theory is a central pillar of modern mathematical analysis, deeply connected with numerous disciplines such as optimization, nonlinear analysis, control theory, game theory, topology, differential and integral equations, partial differential equations, and variational inequalities. In 1922, Banach [1] initiated a powerful approach to fixed point theory through what is now known as the “Banach Fixed Point Theorem,” which guarantees the existence and uniqueness of fixed points for contraction mappings in complete metric spaces. Since then, fixed point theory has been significantly developed and generalized in various directions (see [2, 3, 4, 5] and references therein).

The concept of multiplicative calculus was first introduced by Grossman and Katz in 1972 [6], laying the foundation for multiplicative metric spaces. Later, Bashirov et al. [7] formalized the structure of these spaces to model systems governed by multiplicative relationships rather than additive ones. This framework is particularly useful in disciplines such as economics, population dynamics, and growth models, where proportional change and geometric scaling are more appropriate than additive differences. Ozavsar and Cevikel [8] extended this setting by obtaining fixed point results for various contraction mappings, while others, including Hxiaoju et al. [9], Rasham et al. [10], and Hussain et al. [11], contributed to the growing body of literature by exploring generalized contractions, commutativity conditions, and coupled fixed points. Radenovic and Samet [12] further enriched the theory with broader classes of contractions in the multiplicative setting.

Parallel to this, Zadeh's introduction of fuzzy set theory led to the formulation of fuzzy metric spaces, which model uncertainty and vagueness in distance. Fuzzy metrics have found extensive use in real-world systems where imprecision is inherent. Several researchers have investigated fixed point theory in the fuzzy context (see [13, 14, 15, 16, 17] and references therein), further enhancing the analytical toolkit available for dealing with non-crisp systems.

The concept of fuzzy multiplicative metric spaces was introduced by Farhan et al. [18] as a natural extension of classical fuzzy metric spaces, aimed at modeling systems where both uncertainty and proportional (multiplicative) relationships coexist. Unlike traditional fuzzy metrics, which are built on additive distance structures (as in George and Veeramani [14]), the fuzzy multiplicative framework replaces the additive triangle inequality with a multiplicative one, making it suitable for systems governed by exponential growth, relative change, or geometric progression common in fields like population dynamics, epidemiology, and economics. In classical fuzzy metric spaces, the closeness between two points $p, q \in X$ is measured with respect to a positive real parameter $s > 0$, representing the time or distance scale. In fuzzy multiplicative metric spaces, this parameter is taken from the interval $(1, \infty)$, and the composition of closeness values is governed by multiplicative scaling rather than addition. This change not only generalizes the underlying geometry but also captures behavior in systems where changes occur proportionally rather than incrementally.

Recent developments in multiplicative metric spaces have advanced the theory of fixed points under multiplicative settings. In particular, Özavsar [8] established foundational results for fixed points of multiplicative contraction mappings, providing insights into the topological behavior of such mappings. Radenovic and Bassem [12] further enriched this framework by introducing new classes of contractions and expanding the applicability of fixed-point theory in multiplicative metric spaces. However, these works are primarily confined to crisp (non-fuzzy) environments and do not fully address scenarios where both multiplicative scaling and uncertainty coexist. To bridge this gap, the present study builds on these contributions by extending Banach, Kannan, and Chatterjea-type fixed point theorems to the setting of fuzzy multiplicative metric spaces. This unified framework integrates the flexibility of fuzzy logic with the proportional structure of multiplicative distances, enabling a more comprehensive treatment of fixed-point problems in uncertain and non-additive environments. The unified approach presented here not only enhances the existing theory but also opens new pathways for further research in fixed point theory under hybrid metric structures. To demonstrate its utility, we conclude the paper with an application to nonlinear integral equations, illustrating the practicality of the proposed framework in solving complex nonlinear problems.

2. Preliminary

This section provides some useful definitions, terms and related concepts necessary for further use.

Definition 2.1. [7] Consider the set X to be nonempty. A self-mapping $M_m : X^2 \rightarrow (1, \infty)$ is known as multiplicative metric, if for every p, q, r in X , we have

M1) Identity: $M_m(p, q) = 1$ if and only if $p = q$,

M2) Symmetry: $M_m(p, q) = M_m(q, p)$,

M3) Multiplicative triangle inequality: $M_m(p, q) \cdot M_m(q, r) \geq M_m(p, r)$.

The pair (X, M_m) is known as multiplicative metric space.

Definition 2.2. [12] Let (X, M_m) be a multiplicative metric space, and $\{p_n\}$ be a sequence in X .

1. The sequence $\{p_n\}$ is said to be multiplicative convergent to a point $p \in X$, if for every $\sigma > 1$, there exists a natural number n_0 such that $M_m(p_n, p) < \sigma$ for any $n \geq n_0$, that is,

$$\lim_{n \rightarrow \infty} M_m(p_n, p) = 1. \quad (1)$$

2. The sequence $\{p_n\}$ is said to be multiplicative Cauchy sequence, if for every $\sigma > 1$, there exists a natural number n_0 such that $M_m(p_n, p_k) < \sigma$ for any $n, k \geq n_0$, that is,

$$\lim_{n \rightarrow \infty} M_m(p_n, p_k) = 1. \quad (2)$$

3. If every multiplicative Cauchy sequence $\{p_n \subseteq X\}$ is multiplicative convergent to a certain p in X , then (X, M_m) is named as complete multiplicative metric space.

Remark 2.3. If $\{p_n\}$ is a multiplicative convergent sequence, then there exists a unique p in X such that equation (1) holds.

Ozavsar and Cevikel [8] established the multiplicative versions of Banach, Kannan and Chatterjea's fixed-point Theorems as follows.

Theorem 2.4. “[8] Suppose that $T : X \rightarrow X$ is a self-mapping and (X, M_m) is a complete multiplicative metric space. Then a function T has a unique fixed point if there exist $\sigma \in (0, 1)$ such that

$$M_m(Tp, Tq) \leq [M_m(p, q)]^\sigma, \quad (3)$$

for every $p, q \in X$.”

Theorem 2.5. “[8] Suppose that $T : X \rightarrow X$ is a self-mapping and (X, M_m) is a complete multiplicative metric space. Then a function T has a unique fixed point if there exist $\sigma \in (0, 1/2)$ such that

$$M_m(Tp, Tq) \leq [M_m(Tp, p)M_m(Tq, q)]^\sigma, \quad (4)$$

for every $p, q \in X$.”

Theorem 2.6. [8] Suppose that a self-mapping $T : X \rightarrow X$ is given and (X, M_m) is a complete multiplicative metric space. Then a function T has a unique fixed point if there exist $\sigma \in (0, 1/2)$ such that

$$M_m(Tp, Tq) \leq \sigma [M_m(Tp, q)M_m(Tq, p)]^\sigma, \quad (5)$$

for every $p, q \in X$.

Radenovic and Samet [12] discussed the relationship of metric and multiplicative metric spaces as follows. Consider M_m and m be a multiplicative metric and metric on X respectively. For every $p, q \in X$, denoting m_M by the function $m_M : X \times X \rightarrow [0, \infty)$ defined as

$$m_M(p, q) = \ln(M(p, q)),$$

and M_m the function $M_m : X \times X \rightarrow [1, \infty)$ defined as

$$M_m(p, q) = e^{m(p, q)}.$$

The following observations are obtained:

- (i) If M is a multiplicative metric on X , then m_M is a metric on X ,
- (ii) If (X, M_m) is a complete multiplicative metric space, then (X, m_M) is a complete metric space,
- (iii) If m is a metric on X , then M_m is a multiplicative metric on X ,
- (iv) If (X, m) is a complete metric space, then (X, M_m) is a complete multiplicative metric space.

Theorem 2.7. [12] Assume that a self-mapping $T : X \rightarrow X$ is given and (X, M_m) is a complete multiplicative metric space. Suppose that the next situations are met:

- (i) There exist $\sigma \in (0, 1)$ and $\phi \in \Phi$ such that

$$\phi(M_m(Tp, Tq)) \leq \sigma \phi(M_m(p, q)) \quad (6)$$

for any $p, q \in X$ with $M_m(p, q) > 1$,

- (ii) For any $p, q \in X$, if

$$\lim_{n \rightarrow \infty} M_m(T^n p, q) = 1,$$

then there exists a subsequence $\{T^{n_k} p\}$ of $\{T^n p\}$ such that

$$\lim_{k \rightarrow \infty} M_m(T^{n_k+1} p, Tq) = 1.$$

Then, T has a unique fixed point.

Theorem 2.8. “[12] Assume that a self-mapping $T : X \rightarrow X$ is given and (X, M_m) is a complete multiplicative metric space. Suppose that the next situations are met:

- (i) There exist $\sigma \in (0, 1/2)$ and $\phi \in \Phi$ such that

$$\phi(M_m(Tp, Tq)) \leq \sigma [\phi(M_m(p, Tp)) + \phi(M_m(q, Tq))] \quad (7)$$

for all $p, q \in X$ with $M_m(p, q) > 1$,

- (ii) For all $p, q \in X$, if

$$\lim_{n \rightarrow \infty} M_m(T^n p, q) = 1,$$

then there exists a subsequence $\{T^{n_k+1}p\}$ of $\{T^n p\}$ such that

$$\lim_{k \rightarrow \infty} M_m(T^{n_k+1}p, Tq) = 1.$$

Then, T has a unique fixed point."

Definition 2.9. [19] A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a *continuous t-norm* if it satisfies the following properties for all $p, q, r, s \in [0, 1]$:

- (i) $p * q = q * p$,
- (ii) $(p * q) * r = p * (q * r)$,
- (iii) $*$ is continuous,
- (iv) $p * 1 = p$,
- (v) If $p \leq q$ and $r \leq s$, then $p * r \leq q * s$.

George and Veeramani [14] refined the "fuzzy metric spaces" idea presented in the work of Michalek and Kramosil [16] using the triangular norm as follow.

Definition 2.10. [14] A triplet $(X, M_F, *)$ of any arbitrary set $X \neq \emptyset$, a continuous triangular norm $*$ with a fuzzy set M_F on $X \times X \times (0, \infty)$ is called fuzzy metric spaces satisfying the next conditions for every p, q, r in X and all $s, t > 0$:

- (GV1) $M_F(p, q, s) > 0$,
- (GV2) $M_F(p, q, s) = 1$ if and only if $p = q$,
- (GV3) $M_F(p, q, s) = M_F(q, p, s)$,
- (GV4) $M_F(p, r, s) * M_F(p, q, t) \leq M_F(q, r, s + t)$,
- (GV5) $M_F(p, q, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

A notation $M_F(p, q, s)$ stand for the grade of closeness between p and q with regard to s . Farhan et al., [18] introduced the concept of fuzzy multiplicative metric spaces, thereby incorporating the principles of fuzzy logic into the multiplicative structure of distance measurement.

Definition 2.11. [18] A triplet $(X, M_F, *)$ of any arbitrary set $X \neq \emptyset$, a continuous triangular norm $*$ with a fuzzy set M_F on $X \times X \times (0, \infty)$ is called fuzzy multiplicative metric spaces satisfying the next conditions for every p, q, r in X and all $s, t > 1$:

- (M_F 1) $M_F(p, q, s) > 0$,
- (M_F 2) $M_F(p, q, s) = 1$ if and only if $p = q$,
- (M_F 3) $M_F(p, q, s) = M_F(q, p, s)$,
- (M_F 4) $M_F(p, r, s) * M_F(p, q, t) \leq M_F(q, r, s \cdot t)$,
- (M_F 5) $M_F(p, q, \cdot) : (1, \infty) \rightarrow [0, 1]$ is continuous.

Example 2.11.1. Consider $X = \mathbb{R}^+$ and $M_F(p, q, s) = \frac{s}{|p - q| + s}$ for all $p, q, r \in X$ and $s > 1$. Suppose that a continuous t-norm defined by $p * q = p \cdot q$. Then, $(X, M_F, *)$ is fuzzy multiplicative metric space.

Definition 2.12. [18] Suppose that $(X, M_F, *)$ is a fuzzy multiplicative metric space and $\{p_n\}$ is a sequence in X . Then,

- (i) The sequence $\{p_n\}$ is said to be fuzzy multiplicative convergent to a point p in X , if for all $s > 1$ such that

$$\lim_{n \rightarrow \infty} M_F(p, q, s) = 1. \quad (8)$$

- (ii) The sequence $\{p_n\}$ is said to be fuzzy multiplicative Cauchy sequence, if for all $s > 1$ and $\sigma \in (0, 1)$, there exists a natural number n_0 so that $M_F(p_n, p_m, s) > 1 - \sigma$ for any $n, m \geq n_0$. Or, equivalently, we have

$$\lim_{n \rightarrow \infty} M_F(p_n, p_m, s) = 1. \quad (9)$$

- (iii) If every fuzzy multiplicative Cauchy sequence $\{p_n\} \subseteq X$ is fuzzy multiplicative convergent to a certain p in X , then $(X, M_F, *)$ is called a complete fuzzy multiplicative metric space.

3. Main Results

Building upon the framework of fuzzy multiplicative metric spaces, generalized fixed-point theorems of Banach, Kannan, and Chatterjea types are developed and established, thereby extending classical results into a more flexible and uncertainty aware setting.

Consider Φ represents the collection of all functions $\phi : [1, \infty) \rightarrow (0, \infty)$ such that ϕ is monotone non-decreasing and continuous, with $\phi(s) > 0, s > 1$.

Theorem 3.1. Assume that a self-mapping $T : X \rightarrow X$ is given and $(X, M_F, *)$ is a complete fuzzy multiplicative metric space. Suppose that there exist $\sigma \in (0, 1)$ and $\phi \in \Phi$ such that

$$\phi(M_F(Tp, Tq, s^\sigma)) \geq \phi(M_F(p, q, s)) \quad (10)$$

for any p, q in X and all $s > 1$ with $\lim_{s \rightarrow \infty} M_F(p, q, s) = 1$. Then, T has a fixed point which is unique.

Proof: Let $p_0 \in X$ be arbitrary and define the sequence $\{p_n\}$ by $p_n = T^n p_0$ for all $n \in \mathbb{N}$. We aim to prove that $\{p_n\}$ converges to a unique fixed point of T . From the assumption (10), for all $p, q \in X$ and $s > 1$, we have

$$\phi(M_F(Tp, Tq, s^\sigma)) \geq \phi(M_F(p, q, s)).$$

Apply this with $p = p_{n-1}$ and $q = p_n$, and recall that $p_n = Tp_{n-1}$, so

$$\phi(M_F(p_n, p_{n+1}, s^\sigma)) \geq \phi(M_F(p_{n-1}, p_n, s)). \quad (11)$$

This inequality is the basis for recursion. To proceed, replace s with $s^{1/\sigma}$ in the inequality

$$\phi(M_F(p_{n+1}, p_{n+2}, s^\sigma)) \geq \phi(M_F(p_n, p_{n+1}, s)).$$

Thus, applying (11) recursively yields

$$\phi(M_F(p_n, p_{n+1}, s)) \leq \phi(M_F(p_{n+1}, p_{n+2}, s^\sigma)) \leq \dots \leq \phi(M_F(p_{n+m}, p_{n+m+1}, s^{\sigma^m})).$$

Since ϕ is non-decreasing and $M_F(p_{n+m}, p_{n+m+1}, s^{\sigma^m}) \leq 1$, the sequence $\phi(M_F(p_n, p_{n+1}, s))$ is bounded above by $\phi(1)$, hence convergent. So, we can write

$$\lim_{n \rightarrow \infty} \phi(M_F(p_n, p_{n+1}, s)) = L, \text{ for some } L \leq \phi(1).$$

Because ϕ is continuous and $\phi(t) > 0$ for $t > 0$, we have

$$\lim_{n \rightarrow \infty} M_F(p_n, p_{n+1}, s) = 1, \text{ for all } s > 1.$$

Now we prove that $\{p_n\}$ is a Cauchy sequence. From the fuzzy multiplicative triangle inequality (M_F 4), for $m > n$, we have that

$$M_F(p_n, p_{n+m}, s) \geq M_F(p_n, p_{n+1}, s^{1/m}) * M_F(p_{n+1}, p_{n+2}, s^{1/m}) * \dots * M_F(p_{n+m-1}, p_{n+m}, s^{1/m}). \quad (12)$$

Now apply ϕ on both sides and use its monotonicity

$$\begin{aligned} \phi(M_F(p_n, p_{n+m}, s)) &\geq \phi(M_F(p_n, p_{n+1}, s^{1/m})) * \phi(M_F(p_{n+1}, p_{n+2}, s^{1/m})) \\ &\quad * \dots * \phi(M_F(p_{n+m-1}, p_{n+m}, s^{1/m})). \end{aligned} \quad (13)$$

From recursive application of (11) as done earlier, we have

$$\phi(M_F(p_k, p_{n+1}, s^{1/m})) \geq \phi(M_F(p_0, p_1, s^{1/m\sigma^k})), \quad (14)$$

for each $k = n, \dots, n+m-1$. Therefore, (13) becomes

$$\begin{aligned} \phi(M_F(p_n, p_{n+m}, s)) &\geq \phi(M_F(p_0, p_1, s^{1/m\sigma^n})) * \phi(M_F(p_0, p_1, s^{1/m\sigma^{n+1}})) \\ &\quad * \dots * \phi(M_F(p_0, p_1, s^{1/m\sigma^{n+m-1}})). \end{aligned} \quad (15)$$

As $n, m \rightarrow \infty$, the exponents $\frac{1}{m\sigma^{n+k}} \rightarrow \infty$, and since $\lim_{s \rightarrow \infty} \phi(M_F(p_0, p_1, s)) = 1$, we get

$$\lim_{n, m \rightarrow \infty} \phi(M_F(p_n, p_{n+m}, s)) = 1 * 1 * \dots * 1 = 1. \quad (16)$$

Hence, $\{p_n\}$ is a Cauchy sequence. Since $(X, M_F, *)$ is complete, there exists $p \in X$ such that $p_n \rightarrow p$ as $n \rightarrow \infty$. Then we demonstrate that T has a fixed point p in X , i.e., $T(p) = p$. By the continuity of M_F and using (M_F4) , for any $s > 1$

$$M_F(Tp, p, s) \geq M_F(Tp, p_n, s^{1/2}) * M_F(Tp_n, p, s^{1/2}). \quad (17)$$

Taking ϕ on both sides and passing to the limit as $n \rightarrow \infty$, using (10) and continuity of ϕ , we obtain

$$\phi(M_F(Tp, p, s)) \geq \phi(1) = 1,$$

which implies $M_F(Tp, p, s) = 1$ for all $s > 1$, and thus $Tp = p$. To prove uniqueness, suppose $q \in X$ is another fixed point of T , with $q \neq p$. Then, using (10) recursively,

$$\begin{aligned} \phi(M_F(p, q, s)) &\leq \phi(M_F(Tp, Tq, s^\sigma)) = \phi(M_F(p, q, s^\sigma)) \\ &\leq \phi(M_F(p, q, s^{\sigma^2})) \leq \dots \leq \phi(M_F(p, q, s^{\sigma^n})). \end{aligned} \quad (18)$$

Taking the limits as $n \rightarrow \infty$ (noting $\sigma^n \rightarrow 0$, hence $s^{\sigma^n} \rightarrow 1$, and $\lim_{s \rightarrow 1^+} M_F(p, q, s) < 1$ if $p \neq q$), we get

$$\phi(M_F(p, q, s)) \leq \phi(M_F(p, q, 1)) = \phi(1) \text{ for all } s > 1.$$

Since ϕ is increasing and $\phi(M_F(Tp, p, s)) \rightarrow \phi(1)$, this implies $M_F(Tp, p, s) \rightarrow 1$, and by (M_F2) , it follows that $p = q$ a contradiction with $p \neq q$. Hence, the fixed point is unique.

Remark 3.2. Theorem 3.1 extends the classical Banach Fixed Point Theorem (see Theorem 3.2 of [8]) from the setting of multiplicative metric spaces to the more general framework of fuzzy multiplicative metric spaces. This extension captures uncertainty and imprecision through the use of fuzzy membership functions in the metric structure.

Example 3.3. Let $X = [0, 1]$, and define the mapping $T : X \rightarrow X$ by $T(p) = \sqrt{p}$. Define a fuzzy multiplicative metric $M_F : X \times X \times (1, \infty) \rightarrow [0, 1]$ as

$$M_F(p, q, s) = \frac{s}{s + |p - q|}, \text{ for all } p, q \in X, s > 1.$$

Then, M_F satisfies the properties of a fuzzy multiplicative metric, that is

- (i) $M_F(p, q, s) > 0$ for all $s > 1$,
- (ii) $M_F(p, q, s) = 1$ if and only if $p = q$,
- (iii) $M_F(p, q, s) = M_F(q, p, s)$,
- (iv) The triangle inequality holds in multiplicative form

$$M_F(p, r, s) * M_F(r, q, t) \leq M_F(p, q, s \cdot t),$$

which follows since $\frac{s}{s + |p - r|} \cdot \frac{t}{t + |r - q|} \leq \frac{st}{st + |p - q|}$.

- (v) $M_F(p, q, \cdot)$ is continuous on $(1, \infty)$.

Let us choose $\phi(s) = \sqrt{s} \in \Phi$, which is continuous, non-decreasing, and satisfies $\phi(s) > 0$ for all $s > 1$. Take $\sigma = 1/2 \in (0, 1)$. We will verify the contractive condition of Theorem 3.1 as follows

$$\phi(M_F(Tp, Tq, s^\sigma)) \geq \phi(M_F(p, q, s)).$$

Consider $p = 0.4, q = 0.7$, and $s = 2$. Then we calculate

$$\begin{aligned} T(p) &= \sqrt{0.4} \approx 0.6325, T(q) = 0.7 \approx 0.8367, \\ |Tp - Tq| &\approx |0.6325 - 0.8367| = 0.2042, |p - q| = 0.3, \\ M_F(Tp, Tq, s^\sigma) &= \frac{2^{1/2}}{2^{1/2} + 0.2042} \approx \frac{1.4142}{1.6184} \approx 0.8744, \end{aligned}$$

$$\begin{aligned} M_F(p, q, s) &= \frac{2}{2 + 0.3} = \frac{2}{2.3} \approx 0.8696, \\ \phi(M_F(Tp, Tq, s^\sigma)) &= \sqrt{0.8744} \approx 0.9351, \\ \phi(M_F(p, q, s)) &= \sqrt{0.8696} \approx 0.9323. \end{aligned}$$

Hence, we obtain

$$\phi(M_F(Tp, Tq, s^\sigma)) \approx 0.9351 \geq 0.9323 = \phi(M_F(p, q, s)),$$

which verifies the contractive condition. Therefore, the mapping $T(p) = \sqrt{p}$ satisfies all the assumptions of Theorem 3.1, and hence, has a unique fixed point in X . In fact, solving $T(p) = p$ gives the unique fixed-point $p = 1$.

Corollary 3.2: Let $(X, M_F, *)$ be a complete fuzzy multiplicative metric space, and let $T : X \rightarrow X$ be a self-mapping. Suppose there exist $\sigma \in (0, 1)$ and $\alpha \in (0, 1)$ such that for all $p, q \in X$ and $s > 1$ with $\lim_{s \rightarrow \infty} M_F(p, q, s) = 1$, the following inequality holds:

$$M_F(Tp, Tq, s^\sigma)^\alpha \geq M_F(p, q, s) \quad (19)$$

Then, T has a unique fixed point in X .

Proof: Define $\phi(s) = s^\alpha$ for $s \in [1, \infty)$, where $\alpha \in (0, 1)$. Clearly, ϕ is continuous, monotone non-decreasing, and $\phi(s) > 0$ for all $s > 1$, so $\phi \in \Phi$. Then, inequality (19) can be rewritten as

$$\phi(M_F(Tp, Tq, s^\sigma)) \geq M_F(p, q, s) = \phi(M_F(p, q, s^{1/\alpha}))$$

but since ϕ is increasing and $\alpha < 1$, this inequality is stronger than the inequality (10). Thus, all conditions of Theorem 3.1 are satisfied, and hence T has a unique fixed point.

Example 3.2. Let $X = [0.5, 2] \subset \mathbb{R}$, and define a fuzzy multiplicative metric $M_F : X \times X \times (1, \infty) \rightarrow [0, 1]$ by

$$M_F(p, q, s) = \frac{1}{1 + |p - q|^s},$$

which satisfies the axioms of a fuzzy multiplicative metric. Define the mapping $T : X \rightarrow X$ by $T(p) = \frac{1+p}{3}$, and choose $\sigma = 0.7$ and $\alpha = 0.8$. Now, we verify the contractive condition of Corollary 3.2, i.e.,

$$M_F(Tp, Tq, s^\sigma)^\alpha \geq M_F(p, q, s),$$

for nontrivial choices $p = 1.5, q = 0.8$, and $s = 2.5$, we compute all the values as

$$\begin{aligned} |p - q| &= |1.5 - 0.8| = 0.7, s = 2.5, |p - q|^s \approx 0.7^{2.5} \approx 0.4085, \\ M_F(p, q, 2.5) &= \frac{1}{1 + 0.4085} \approx 0.7101, T(1.5) = \frac{1 + 1.5}{3} = \frac{2.5}{3} \approx 0.8333, \\ T(0.8) &= \frac{1 + 0.8}{3} = \frac{1.8}{3} = 0.6, |0.8333 - 0.6| = 0.2333, s^\sigma = 2.5^{0.7} \approx 1.8661, \\ |Tp - Tq|^{s^\sigma} &\approx 0.2333^{1.8661} \approx 0.0627, M_F(Tp, Tq, s^\sigma) = \frac{1}{1 + 0.0627} \approx 0.9410. \end{aligned}$$

By applying exponent α , we get

$$M_F(Tp, Tq, s^\sigma)^\alpha = 0.9410^{0.8} \approx 0.7619.$$

Hence, since

$$M_F(Tp, Tq, s^\sigma)^\alpha \approx 0.7619 \geq 0.7101 \approx M_F(p, q, s),$$

the contractive condition is satisfied for nontrivial values of $p = 1.5, q = 0.8, s = 2.5, \sigma = 0.7$, and $\alpha = 0.8$. Therefore, the mapping $T(p) = \frac{1+p}{3}$ verifies the hypothesis of Corollary 3.2, establishing the existence and uniqueness of a fixed point.

Theorem 3.3. Assume that a self-mapping $T : X \rightarrow X$ is given and $(X, M_F, *)$ is a complete fuzzy multiplicative metric space. Suppose that there exist $\sigma \in (0, 1/2)$ and $\phi \in \Phi$ such that

$$\phi(M_F(Tp, Tq, s)) \geq [\phi(M_F(p, Tp, s)) \cdot \phi(M_F(q, Tq, s))]^\sigma \quad (20)$$

for each p, q in X with

$$\lim_{n \rightarrow \infty} M_F(T^n p, q, s) = 1.$$

Then, T has a unique fixed point.

Proof: Let $p_0 \in X$ be arbitrary, and define a sequence $\{p_n\}$ by $p_n = T^n p_0$ for $n \in \mathbb{N}$. We will show that $\{p_n\}$ converges to a unique fixed point of T in X . Applying the contractive condition (20) with $p = p_n$ and $q = p_{n-1}$, we get

$$\phi(M_F(p_{n+1}, p_n, s)) = \phi(M_F(Tp_n, Tp_{n-1}, s)) \geq [\phi(M_F(p_n, p_{n+1}, s)) \cdot \phi(M_F(p_{n-1}, p_n, s))]^\sigma. \quad (21)$$

Assuming $\phi(M_F(p_n, p_{n+1}, s)) > 0$ (as $\phi(s) > 0$ for all $s > 1$), we divide both sides of (21) by $[\phi(M_F(p_n, p_{n+1}, s))]^\sigma$ to isolate $\phi(M_F(p_n, p_{n+1}, s))$, yielding

$$[\phi(M_F(p_n, p_{n+1}, s))]^{1-\sigma} \geq [\phi(M_F(p_{n-1}, p_n, s))]^\sigma.$$

Set $t = \frac{\sigma}{1-\sigma} \in (0, 1)$ since $\sigma \in (0, 1/2)$. Then, we obtain

$$\phi(M_F(p_n, p_{n+1}, s)) \geq [\phi(M_F(p_{n-1}, p_n, s))]^t. \quad (22)$$

This inequality shows that the sequence $\{\phi(M_F(p_n, p_{n+1}, s))\}$ is non-decreasing. Since $M_F(p, q, s) \in [0, 1]$ and ϕ is continuous and non-decreasing with $\phi(s) > 0$ for all $s > 1$, the sequence is bounded above by 1. Hence, by the monotone convergence of ϕ , we have

$$\lim_{n \rightarrow \infty} \phi(M_F(p_n, p_{n+1}, s)) = 1 \Rightarrow \lim_{n \rightarrow \infty} M_F(p_n, p_{n+1}, s) = 1.$$

Now, we show that $\{p_n\}$ is Cauchy sequence. For any $n > m$, repeatedly applying (22) gives

$$\begin{aligned} \phi(M_F(p_n, p_m, s)) &\geq \prod_{k=m}^{n-1} [\phi(M_F(p_{k+1}, p_k, s))]^t \\ &= \prod_{k=m}^{n-1} (\phi(M_F(p_1, p_0, s)))^t \\ &= (\phi(M_F(p_1, p_0, s)))^{\sum_{k=m}^{n-1} t^k}. \end{aligned} \quad (23)$$

Since $t \in (0, 1)$, the geometric sum converges

$$\sum_{k=m}^{n-1} t^k = \frac{t^m(1-t^{n-m})}{1-t} \leq \frac{t^m}{1-t}.$$

Thus, we have

$$\phi(M_F(p_n, p_m, s)) \geq (\phi(M_F(p_1, p_0, s)))^{t^m/(1-t)}.$$

As $m \rightarrow \infty$, this lower bound tends to 1. Hence,

$$\lim_{n, m \rightarrow \infty} \phi(M_F(p_n, p_m, s)) = 1 \Rightarrow \lim_{n, m \rightarrow \infty} M_F(p_n, p_m, s) = 1,$$

which implies that $\{p_n\}$ is a Cauchy sequence. Since $(X, M_F, *)$ is complete fuzzy multiplicative metric space, there exists $p \in X$ such that $p_n \rightarrow p$. Next, we show that T has a fixed point. Applying the contractive condition (20) with $p = p, q = p_n$, and using (M_F4), we have

$$\begin{aligned} \phi(M_F(Tp, p, s)) &\geq [\phi(M_F(Tp, p_n, s^{1/2})) \cdot \phi(M_F(p_n, p, s^{1/2}))]^\sigma \\ &\geq [\phi(M_F(p, p_n, s^{1/2})) \cdot \phi(M_F(p_n, p, s^{1/2}))]^\sigma \rightarrow 1 \text{ as } n \rightarrow \infty. \end{aligned} \quad (24)$$

Therefore, $\phi(M_F(Tp, p, s)) = 1 \Rightarrow M_F(Tp, p, s) = 1$ for all $s > 1$, so $Tp = p$. Hence, p is a fixed point. To prove uniqueness, let q be another fixed point of T . Then, using (20) and (M_F4) again, we obtain

$$\begin{aligned} \phi(M_F(p, q, s)) &= \phi(M_F(Tp, Tq, s)) \\ &\geq [\phi(M_F(p, Tp, s^{1/2})) \cdot \phi(M_F(q, Tq, s^{1/2}))]^\sigma \\ &= [\phi(M_F(p, p, s^{1/2})) \cdot \phi(M_F(q, q, s^{1/2}))]^\sigma = 1. \end{aligned}$$

So $M_F(p, q, s) = 1$ for all $s > 1$ implies $p = q$ by (M_F2) . Therefore, the fixed point is unique.

Remark 3.2. Theorem 3.3 generalizes the classical Kannan fixed point theorem (Theorem 3.5 of [8]) from multiplicative metric spaces to fuzzy multiplicative metric spaces. This transition incorporates the fuzzy logic framework to handle uncertainties inherent in many real-world systems. The additional condition $\lim_{n \rightarrow \infty} M_F(T^n p, q, s) = 1$ further strengthens the uniqueness result by ensuring convergence behavior the orbit of T .

Definition 3.1. [20] Let $(X, M_F, *)$ be a fuzzy multiplicative metric space, and let $T : X \rightarrow X$ be a self-mapping. Then T is Orbitally continuous at a point $p \in X$ if for every sequence $\{p_n\}$ in the orbit $O_{T(p)} : \{T^n p : n \in \mathbb{N}\}$ such that $p_n \rightarrow p$ (with respect to M_F), it follows that $Tp_n \rightarrow p$.

In other words, if $\lim_{n \rightarrow \infty} M_F(p_n, p, s) = 1$ for all $s > 1$, then $\lim_{n \rightarrow \infty} M_F(Tp_n, Tp, s) = 1$.

Corollary 3.4. Let $(X, M_F, *)$ be a complete fuzzy multiplicative metric space and let $T : X \rightarrow X$ be a self-mapping. Suppose there exist $\sigma \in (0, 1/2)$ and $\phi \in \Phi$ such that

$$\phi(M_F(Tp, Tq, s)) \geq [\phi(M_F(p, Tp, s)) \cdot \phi(M_F(q, Tq, s))]^\sigma \quad (25)$$

for all $p, q \in X$ with $(p, q) \in O_T$, where $O_T : \{(T^n p_0, T^m p_0) : n, m \in \mathbb{N}\}$ denotes the orbit of p_0 under T . Assume further that:

- T is Orbitally continuous at some $p_0 \in X$, i.e., if $T^n p_0 \rightarrow p$, then $T(T^n p_0) \rightarrow Tp$,
- and $\lim_{n \rightarrow \infty} M_F(T^n p_0, q, s) = 1$ for some $q \in X$.

Then, T has a unique fixed point.

Proof: Define the Picard sequence $\{p_n\}$ by $p_n = T^n p_0$. Since T is orbitally continuous at p_0 and the inequality (25) holds along the orbit O_T , we can apply the recursive technique used in Theorem 3.3, restricted to the orbit of p_0 . The inequality and continuity together ensure that $\{p_n\}$ is a Cauchy sequence in $(X, M_F, *)$, and since the space is complete, it converges to a point $p \in X$. Using orbital continuity, we have

$$Tp = \lim_{n \rightarrow \infty} Tp_n = \lim_{n \rightarrow \infty} p_{n+1} = p,$$

so p is fixed point. Uniqueness follows by repeating the uniqueness argument from Theorem 3.3, since the inequality (25) is valid for any pair in the orbit, and hence in the limit. Thus, T has a unique fixed point in X .

Example 3.3. Let $X = [0, \infty)$, and define a fuzzy multiplicative metric $M_F : X \times X \times (1, \infty) \rightarrow [0, 1]$ by

$$M_F(p, q, s) = \frac{1}{1 + |p - q|}, \text{ for all } p, q \in X, s > 1.$$

We can verify that $(X, M_F, *)$ is a fuzzy multiplicative metric under the standard t-norm $a * b = a \cdot b$. Define the self-mapping $T : X \rightarrow X$ by $T(p) = \frac{p+1}{3}$. Choose $\phi(s) = \sqrt{s}$, which belongs to the class Φ (monotone, continuous, and $\phi(s) > 0$ for $s > 1$). Let $\sigma = 1/3 \in (0, 1/2)$. Now, we verify the contractive condition of Theorem 3.3, i.e.,

$$\phi(M_F(Tp, Tq, s)) \geq [\phi(M_F(p, Tp, s)) \cdot \phi(M_F(q, Tq, s))]^\sigma.$$

Choose $p = 2, q = 3$, and $s = 5$. Then, we compute all the values as

$$\begin{aligned} T(2) &= 3/3 = 1, T(3) = 4/3, \\ M_F(Tp, Tq, 5) &= \frac{5}{5 + |1 - 4/3|} = \frac{5}{5 + 1/3} = \frac{15}{16}, \\ M_F(p, Tp, 5) &= \frac{5}{5 + |2 - 1|} = \frac{5}{6}, \\ M_F(q, Tq, 5) &= \frac{5}{5 + |3 - 4/3|} = \frac{5}{5 + 5/3} = \frac{15}{20}. \end{aligned}$$

Now by applying the function ϕ , we evaluate

$$\begin{aligned} \phi(M_F(Tp, Tq, 5)) &= \sqrt{\frac{15}{16}} \approx 0.9682 \\ \phi(M_F(p, Tp, 5)) &= \sqrt{\frac{5}{6}} \approx 0.9129, \\ \phi(M_F(q, Tq, 5)) &= \sqrt{\frac{3}{4}} \approx 0.8660, \\ [\phi(M_F(p, Tp, 5)) \cdot \phi(M_F(q, Tq, 5))]^\sigma &= (0.9129 \times 0.8660)^{1/3} \approx (0.7910)^{1/3} \approx 0.9282. \end{aligned}$$

Hence, since

$$\phi(M_F(Tp, Tq, 5)) \approx 0.9682 \geq 0.9282,$$

the contractive condition is satisfied. Moreover, we can check that

$$\lim_{n \rightarrow \infty} M_F(T^n p, q, s) = 1,$$

since $T^n p \rightarrow 1/2$ as $n \rightarrow \infty$, and $q = 3$ is fixed, thus we obtain

$$|T^n p - q| \rightarrow |1/2 - 3| = 2.5 \Rightarrow M_F(T^n p, q, s) \rightarrow \frac{s}{s + 2.5} \rightarrow 1 \text{ as } s \rightarrow \infty.$$

Therefore, all the assumptions of Theorem 3.3 are satisfied, and T has a unique fixed point. Finally, solving $T(p) = p$ gives

$$\frac{p+1}{3} = p \Rightarrow p = \frac{1}{2}.$$

So, the unique fixed point is $p = \frac{1}{2}$.

Theorem 3.5. Suppose that a self-mapping $T : X \rightarrow X$ is given and $(X, M_F, *)$ is a complete fuzzy multiplicative metric space. Assume that there exist $\sigma \in (0, 1/2)$ and $\phi \in \Phi$ such that

$$\phi(M_F(Tp, Tq, s)) \geq [\phi(M_F(p, Tq, s)) \cdot \phi(M_F(q, Tp, s))]^\sigma \quad (26)$$

for each p, q in X . Then, T has a unique fixed point.

Proof: Let $p_0 \in X$ be arbitrary, and define a sequence $\{p_n\}$ by $p_n = T^n p_0$ for $n \in \mathbb{N}$. We prove that $\{p_n\}$ converges to a unique fixed point of T in X . Applying the contractive condition (26) with $p = p_n$ and $q = p_{n-1}$, we get

$$\phi(M_F(p_{n+1}, p_n, s)) = \phi(M_F(Tp_n, Tp_{n-1}, s)) \geq [\phi(M_F(p_n, Tp_{n-1}, s)) \cdot \phi(M_F(p_{n-1}, Tp_n, s))]^\sigma. \quad (27)$$

Note that $Tp_n = p_{n+1}$ and $Tp_{n-1} = p_n$, so the inequality (27) becomes

$$\phi(M_F(p_{n+1}, p_n, s)) \geq [\phi(M_F(p_n, p_n, s)) \cdot \phi(M_F(p_{n-1}, p_{n+1}, s))]^\sigma \quad (28)$$

Since $M_F(p_n, p_n, s) = 1$, we have $\phi(M_F(p_n, p_n, s)) = \phi(1) = 1$. Thus, (28) becomes

$$\phi(M_F(p_{n+1}, p_n, s)) \geq [\phi(M_F(p_{n-1}, p_{n+1}, s))]^\sigma \quad (29)$$

By applying (M_F4) , we use the triangular-type inequality in fuzzy multiplicative metrics

$$M_F(p_{n-1}, p_{n+1}, s) \geq M_F(p_{n-1}, p_n, s^{1/2}) \cdot M_F(p_n, p_{n+1}, s^{1/2}). \quad (30)$$

By applying the function ϕ to both sides and using the fact that ϕ is monotone non-decreasing, we obtain

$$\phi(M_F(p_{n-1}, p_{n+1}, s)) \geq \phi(M_F(p_{n-1}, p_n, s^{1/2})) \cdot \phi(M_F(p_n, p_{n+1}, s^{1/2})). \quad (31)$$

Now, returning to inequality (29) and substituting the result from (31), we get

$$\phi(M_F(p_{n+1}, p_n, s)) \geq [\phi(M_F(p_{n-1}, p_n, s^{1/2})) \cdot \phi(M_F(p_n, p_{n+1}, s^{1/2}))]^\sigma \quad (32)$$

Rewriting as

$$\phi(M_F(p_{n+1}, p_n, s)) \geq [\phi(M_F(p_{n-1}, p_n, s^{1/2}))]^\sigma \cdot [\phi(M_F(p_n, p_{n+1}, s^{1/2}))]^\sigma \quad (33)$$

Let us set $A_n = \phi(M_F(p_n, p_{n+1}, s))$ for simplicity. Then (33) becomes

$$A_{n+1} \geq A_{n-1}^\sigma \cdot A_n^\sigma.$$

This recursive inequality leads to

$$A_{n+1}^{1-\sigma} \geq A_{n-1}^{\sigma} \Rightarrow A_{n+1} \geq A_{n-1}^{\sigma/(1-\sigma)} = A_{n-1}^t, \text{ where } t = \sigma/(1-\sigma) \in (0, 1).$$

Hence, we have the recursive inequality

$$\phi(M_F(p_{n+1}, p_n, s)) \geq [\phi(M_F(p_n, p_{n-1}, s))]^t. \quad (34)$$

By induction, inequality (34) implies the sequence $\{\phi(M_F(p_n, p_{n+1}, s))\}$ is non-decreasing and bounded above by 1. So, we obtain

$$\lim_{n \rightarrow \infty} \phi(M_F(p_n, p_{n+1}, s)) = 1 \Rightarrow \lim_{n \rightarrow \infty} M_F(p_n, p_{n+1}, s) = 1.$$

Next, we show that $\{p_n\}$ is a Cauchy sequence. Let $n > m$. Applying the recursive inequality repeatedly with (M_F4), we get

$$\begin{aligned} \phi(M_F(p_n, p_m, s)) &\geq \prod_{k=m}^{n-1} [\phi(M_F(p_k, p_{k+1}, s))]^t \\ &= \prod_{k=m}^{n-1} (\phi(M_F(p_1, p_0, s)))^t \\ &= (\phi(M_F(p_1, p_0, s)))^{\sum_{k=m}^{n-1} t^k}. \end{aligned} \quad (35)$$

As $t \in (0, 1)$, we have

$$\sum_{k=m}^{n-1} t^k \leq \frac{t^m}{1-t} \Rightarrow \phi(M_F(p_n, p_m, s)) \geq (\phi(M_F(p_1, p_0, s)))^{t^m/(1-t)}. \quad (36)$$

Taking $m \rightarrow \infty$, the right-hand side of (36) tends to 1. Hence, we obtain

$$\lim_{n, m \rightarrow \infty} \phi(M_F(p_n, p_m, s)) = 1 \Rightarrow \lim_{n, m \rightarrow \infty} M_F(p_n, p_m, s) = 1.$$

Therefore, $\{p_n\}$ is a Cauchy sequence. Since $(X, M_F, *)$ is complete fuzzy multiplicative metric space, there exists $p \in X$ such that $p_n \rightarrow p$. Next, we show that T has a fixed point. Applying the contractive condition (26) and using (M_F4), we have

$$\begin{aligned} \phi(M_F(Tp, p, s)) &\geq [\phi(M_F(Tp, p_n, s^{1/2})) \cdot \phi(M_F(p_n, p, s^{1/2}))]^{\sigma} \\ &\geq [\phi(M_F(p, p_n, s^{1/2})) \cdot \phi(M_F(p_n, p, s^{1/2}))]^{\sigma} \rightarrow 1. \end{aligned} \quad (37)$$

Hence, $\phi(M_F(Tp, p, s)) = 1 \Rightarrow M_F(Tp, p, s) = 1$ for all $s > 1$, so $Tp = p$. Therefore, p is a fixed point. Finally, we show the uniqueness. Let q be another fixed point of T . Then, using (26) and (M_F4) again, we obtain

$$\begin{aligned} \phi(M_F(p, q, s)) &= \phi(M_F(Tp, Tq, s)) \\ &\geq [\phi(M_F(p, Tq, s^{1/2})) \cdot \phi(M_F(q, Tp, s^{1/2}))]^{\sigma} \\ &= [\phi(M_F(p, q, s^{1/2})) \cdot \phi(M_F(q, p, s^{1/2}))]^{\sigma}. \end{aligned} \quad (38)$$

Since $M_F(p, q, s) = [\phi(M_F(p, q, s^{1/2}))]^2$, and iterating this gives

$$\phi(M_F(p, q, s)) \geq \phi(M_F(p, q, s^{1/\sigma^n}))^{\sigma^n} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Thus, $M_F(p, q, s) = 1$ for all $s > 1$, which implies $p = q$ by (M_F2). So, the fixed point is unique.

Remark 3.3. Theorem 3.5 is a fuzzy multiplicative analogue of the Chatterjea Fixed Point Theorem (Theorem 3.6 of [8]). By extending the classical result to fuzzy multiplicative metric spaces, the theorem accommodates imprecise distance evaluations via fuzzy membership functions.

Corollary 3.6. Let $(X, M_F, *)$ be a complete fuzzy multiplicative metric space, and let $T : X \rightarrow X$ be a self-mapping. Suppose there exist constants $\sigma \in (0, 1/2)$ and $\alpha \in (0, 1)$ such that for all $p, q \in X$ and $s > 1$,

$$[\phi(M_F(Tp, Tq, s))]^{\alpha} \geq ([\phi(M_F(p, Tq, s))]^{\alpha} \cdot [\phi(M_F(q, Tp, s))]^{\alpha})^{\sigma}.$$

Then, T has a unique fixed point.

Proof: Define $\phi(s) = s^{\alpha}$ for $s \in [0, 1]$. Clearly, ϕ is continuous, strictly increasing, and satisfies $\phi(s) > 0$ for all $s > 1$. Thus, $\phi \in \Phi$, the class of admissible control functions. Now, applying ϕ to both sides of the inequality in Theorem 3.5, the given contractive condition becomes

$$\phi(M_F(Tp, Tq, s)) \geq [\phi(M_F(p, Tq, s)) * \phi(M_F(q, Tp, s))]^\sigma \quad (39)$$

This is exactly the form required in Theorem 3.5. Since all the assumptions of Theorem 3.5 are satisfied, it follows that T has a unique fixed point in X .

Example 3.3. Let $X = [0, 1]$ and define a fuzzy multiplicative metric $M_F : X \times X \times (1, \infty) \rightarrow [0, 1]$ by

$$M_F(p, q, s) = \frac{1}{1 + |p - q| \log s}, \text{ for all } p, q \in X, s > 1.$$

Then, $(X, M_F, *)$ is a fuzzy multiplicative metric under the standard t-norm $a * b = a \cdot b$. Consider the function $T : X \rightarrow X$ defined by

$$T(p) = \frac{1}{2}p.$$

We show that T satisfies the contractive condition of Theorem 3.5. Suppose that $\phi(s) = \sqrt{s}$, which belongs to the class Φ (monotone, continuous, and $\phi(s) > 0$ for $s > 1$). Let $\sigma = 0.3 \in (0, 1/2)$. Now, we verify the contractive condition of Theorem 3.5, i.e.,

$$\phi(M_F(Tp, Tq, s)) \geq [\phi(M_F(p, Tq, s)) \cdot \phi(M_F(q, Tp, s))]^\sigma.$$

Choose $p = 0.8, q = 0.6$, and $s = 2$. First, we compute for all metric values as follows

$$\begin{aligned} \log s &= \log 2 \approx 0.6931, T(p) = 0.4, T(q) = 0.3, \\ M_F(Tp, Tq, s) &= \frac{1}{1 + |0.4 - 0.3| \cdot (0.6931)} = \frac{1}{1 + 0.06931} \approx 0.9352, \\ M_F(p, Tq, s) &= \frac{1}{1 + |0.8 - 0.3| \cdot (0.6931)} = \frac{1}{1 + 0.3466} \approx 0.7427, \\ M_F(q, Tp, s) &= \frac{1}{1 + |0.6 - 0.4| \cdot (0.6931)} = \frac{1}{1 + 0.1386} \approx 0.8783. \end{aligned}$$

Next, we apply the function $\phi(s) = \sqrt{s}$

$$\begin{aligned} \phi(M_F(Tp, Tq, s)) &= \sqrt{0.9352} \approx 0.9671, \\ \phi(M_F(p, Tq, s)) &= \sqrt{0.7427} \approx 0.8618, \\ \phi(M_F(q, Tp, s)) &= \sqrt{0.8783} \approx 0.9372. \end{aligned}$$

Then, we evaluate the right-hand side, yields

$$[\phi(M_F(p, Tq, s)) \cdot \phi(M_F(q, Tp, s))]^\sigma = (0.8618 \times 0.9372)^{0.3} \approx (0.8079)^{0.3} \approx 0.9334.$$

Hence, since

$$\phi(M_F(Tp, Tq, s)) \approx 0.9671 \geq 0.9334,$$

the contractive condition of Theorem 3.5 is satisfied for the chosen values. Therefore, the mapping $T(p) = \frac{1}{2}p$ with the defined fuzzy multiplicative metric and $(s) = \sqrt{s}$, verifies the applicability of the theorem in a non-trivial case. Furthermore, solving the fixed-point equation $T(p) = p$ gives

$$\frac{1}{2} = p \Rightarrow p = 0.$$

Thus, the mapping T admits a unique fixed point at $p = 0$.

4. Application to Nonlinear Integral Equations

The concept of fuzzy multiplicative metric spaces is particularly useful in modeling problems involving uncertainty, vagueness, or multiplicative type error propagation common in control systems, biological models, and engineering processes. Unlike classical or fuzzy additive metrics, fuzzy multiplicative metrics are better suited for capturing the nonlinear growth or decay behavior often observed in such systems.

As a concrete application, consider the following nonlinear Volterra-type integral equation:

$$p(u) = f(u) + \int_0^u F(u, v, p(v))dv, \quad u, v \in [0, 1], \quad (40)$$

where $f \in C([0, 1], \mathbb{R})$ is a given continuous function and $F : [0, 1]^2 \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies a Lipschitz-type condition, which is described below.

Let $C(I, \mathbb{R})$ denote the Banach space of continuous real-valued functions on $I = [0, 1]$, and define a fuzzy multiplicative metric M_F on $C(I, \mathbb{R})$ by

$$M_F(p, q, s) = e^{-\frac{\sup_{u \in I} |p(u) - q(u)|^2}{s}}, \quad s > 1. \quad (41)$$

Then $(C(I, \mathbb{R}), M_F, *)$ becomes a complete fuzzy multiplicative metric space under the standard product t -norm.

Theorem 4.1. Let $T : C(I, \mathbb{R}) \rightarrow C(I, \mathbb{R})$ be the operator defined by

$$Tp(u) = f(u) + \int_0^u F(u, v, p(v))dv. \quad (42)$$

Suppose there exists a function $f : [0, 1]^2 \rightarrow \mathbb{R}_+$ such that:

(i) For all $p, q \in C(I, \mathbb{R})$ and $u, v \in I$,

$$|F(u, v, p(v)) - F(u, v, q(v))|^2 \leq f^2(u, v)|p(v) - q(v)|^2,$$

(ii) and

$$\sup_{u \in I} \int_0^u f^2(u, v)dv \leq s^\sigma < 1 \text{ for some } \sigma \in (0, 1).$$

Then the integral equation (40) admits a unique solution in $C(I, \mathbb{R})$.

Proof: Let $p, q \in C(I, \mathbb{R})$ and $s > 1$. Then, using the definition of T and applying the assumed inequality

$$\begin{aligned} M_F(Tp, Tq, s^\sigma) &= e^{-\frac{\sup_{u \in I} |Tp(u) - Tq(u)|^2}{s^\sigma}} \\ &= e^{-\frac{\sup_{u \in I} \left| \int_0^u F(u, v, p(v)) - F(u, v, q(v))dv \right|^2}{s^\sigma}} \\ &\geq e^{-\frac{\sup_{u \in I} \left(\int_0^u |F(u, v, p(v)) - F(u, v, q(v))|dv \right)^2}{s^\sigma}} \\ &\geq e^{-\frac{\sup_{u \in I} \int_0^u f^2(u, v)|p(v) - q(v)|^2 dv}{s^\sigma}} \\ &\geq e^{-\frac{\sup_{u \in I} |p(v) - q(v)|^2 \int_0^u f^2(u, v)dv}{s^\sigma}} \\ &\geq e^{-\frac{\sup_{u \in I} |p(u) - q(u)|^2 s^\sigma}{s^\sigma}} \\ &= M_F(p, q, s). \end{aligned}$$

Hence, the contraction condition of Theorem 3.1 is satisfied with $\phi(s) = s$, which belongs to the class Φ . Since $(C(I, \mathbb{R}), M_F, *)$ is complete, Theorem 3.1 guarantees that T has a unique fixed point. Thus, the integral equation (40) has a unique solution.

5. Conclusion

In this manuscript, we have developed novel fixed-point theorems within the framework of fuzzy multiplicative metric spaces. By introducing generalized contractive conditions involving auxiliary functions ϕ and control parameters such as σ and α , we have extended classical results such as those of Banach, Kannan, and Chatterjea to a fuzzy multiplicative context. These extensions capture more nuanced behaviors in systems where uncertainty and multiplicative type distortions are inherent.

Unlike traditional fuzzy or multiplicative spaces alone, the fuzzy multiplicative structure enables a richer modeling of problems characterized by non-additive uncertainties and exponential decay in similarity. The nontrivial corollaries and carefully constructed examples demonstrate that our results are not mere restatements of existing theorems but rather provide a robust generalization that accommodates broader classes of mappings.

Furthermore, the application to nonlinear integral equations highlights the utility of our approach in analyzing real-world systems governed by integral dynamics under imprecise conditions. This not only validates the theoretical developments but also opens the door for future studies in applied analysis, particularly in differential and integral equations, fuzzy control, and decision-making models.

Future work will explore multi-valued mappings, generalized hybrid contractions, or applications in fuzzy optimization and neural computation under the fuzzy multiplicative framework.

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