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On the Noisy Four-Parameter Fisher's Z-Distribution of Bayesian Mixture Autoregressive (FZBMAR) Process via Mode as a Stable Location Parameter

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Abstract: This paper aims at providing in-depth refinement to switching time-variant autoregressive processes via the mode as a stable location parameter in adopted noisy Fisher's z-distribution that was impelled in a Bayesian setting. Explicitly, a four-parameter Fisher's z-distribution of Bayesian Mixture Autoregressive (FZBMAR) process was proposed to congruous k-mixture components of Fisher's z-switching mixture autoregressive processes that was based on shifting number of modes in the marginal density of any switching time-variant series of interest. The proposed FZBMAR process was not only used to seize what is term "most likely mode value" of the present conditional modal distribution given the immediate past but was also used to capture the conditional modal distribution of the observations given the immediate past that can either be perceived as an asymmetric or symmetric distributed varieties. The proposed FZBMAR process was compared with the existing Student-*t* Mixture Autoregressive (StMAR) and Gaussian Mixture Autoregressive (GMAR) processes with the demonstration of monthly average share prices (stock prices) of sixteen (16) swaying European economies. Based on the findings, the FZBMAR process outperformed the existing S*t*MAR and GMAR processes in explaining the sixteen (16) swaying European economies share prices via a minimum Pareto-Smoothed Important Sampling Leave-One-Out Cross-Validation (PSIS-LOO) error process performance in comparison with AIC, HQIC by the latters. The same singly truncated student-*t* prior distribution was adopted for the noisy adoption of Fisher's **z** hyper-parameters and the embedded autoregressive coefficients in the proposed FZBMAR process; such that their resulting posterior distributions gave the same singly truncated student-*t* distribution (conjugate) with an embedded Gamma variate.

Index Terms: Autoregressive, Bayesian, FZBMAR Process, Share Price, Singly Truncated Student-*t* Distribution, Switching Time-Variant

1. Introduction

Few nonlinear time series processes hold contributive answers to switching time-variant events. Uniform or non-uniform events that are usually contained in financial returns, climate change series, physiological monitoring, control and genetic sequence analysis, e-commerce, social media time-stamped data, etc. Notably, Bayesian methods do present the cutting edge in exploiting unified time series generalizations to present these switching time-variant events in time-variant processes (contained in either time domain or frequency domain process). In other words, the emergence of treating these time-variant based processes in Bayesian time series techniques is the leading twenty-first innovative research and statistical learning [1, 2].

The ambitious interest of nonlinearity time-variant processes obscures is the commonalities in the designing of stochastic processes and techniques that truly describe the process in generality [3]. The point of interest is the time-variant stochastic switching process with the assumption that the underlying time-variant events changes from one parameter regime to another parameter regime over time, and affirmation that the design or adopted process changes with time–variant environment [4, 5]. Alternatively, change point processes are restricted or unrestricted switching processes (emanated from random walk process) that are usually used to detect abrupt changes in time series analyzes, as well as to carve-out new process insights [6]. The unified focus of this write-up is to drive the switching time-variant autoregressive processes by knowledge-based Bayesian methods.

In light of the needed stochastic time-variant switching process, [7 - 9] linked-up some mingling probabilistic mixture processes via the emergence of autoregressive processes called Mixture Autoregressive Model (MAR) with embedded traits, like regime-shifting, outburst, change-point like behavior, flat stretches, full range shape changing predictive distributions (multimodalities), and ability to handle cycles. Notably, the MAR process still leaves the lacuna of short-tailed data patterns to be captured, as well as providing convincing solution(s) to instability of embedded distributional location parameter(s). It is because of these shortcomings that some notable extensions were made, extensions like Logistic Mixture Autoregressive with Exogenous Variables (LMARX) process by [10] in order to incorporate additional logistic-type time-variant and independent variable. In addition, [11] proposed a Gaussian Mixture Transition Distribution (GMTD) process as a special form of generalization to Gaussian Mixture Autoregressive (GMAR). In [8], a Student-t Mixture Autoregressive (StMAR) process was proposed for proper capturing of heteroscedasticity and multimodal conditional distribution. Additionally, [12, 13] adopted a Laplace MAR noise and Scale Mixture of Skew-Normal Distributions (SMSN-MAR) processes for proper sub-setting of symmetric observational content; and observational content that belongs to the class of Scale Mixture of Skew-Normal (SMSN) distributions respectively. Another notable extension of MAR process is the refinement of its random noise to Fréchet distribution by [14] as one of the leading noisy distributions of the Extreme-Value-Distributions (EVDs) for absolving extreme value measurements in switching time-variant events.

In [15], a modification on MAR was based on finite mixture of autoregressive processes with the assumption that the random noise belongs to the class of Scale Mixture of Skew-Normal (SMSN) distributions. The noise innovation employed for the process was able to capture outliers, fat-tails, and outliers simultaneously. Among notable advantages of Bayesian paradigm to MAR process is the ability to incorporate uncertainty into the process for improved predictive distribution and some h -step ahead predictions [16]. In [16], a new sampling technique was introduced for the posterior distribution; such that all the coefficients that make-up the parameter space was proper subset of the space. They further introduced a re-labelling algorithm for the improvement of the posteriori distribution via label switching of priors. The considered algorithm for the re-labelling was the redefined Markov Chain Monte Carlo (MCMC). However, [17, 18] pinpointed that it is not statistically ideal to use improper priors to drive the MCMC algorithm that involve μ_k and σ^{2k} (where k is the number of components that make-up the whole process) of any designed mixture stochastic process, in order to avoid much higher risk of being trapped by some local modes in the designed parameter estimation algorithm. Consequently, some notable priors like Zellner's "g" prior, Zellner-Siow Cauchy priors, and Liang hyper "g" priors have been used to correct this higher risk of being trapped by some local modes in some algorithms [19, 20].

Thereby, notable full-scale Bayesian paradigm of mixture autoregressive process was by [21] via a proposition of reduced constraints placed on Metropolis–Hastings algorithm with the use of truncated Gaussian random noise. The imposition was on the autoregressive coefficients, to be drawn from the choice of boundaries of the truncated Gaussian random noise kernel. The merit of the imposition was that it keeps the constrained coefficients within the stationary region, and this makes it leads to a conservative autoregressive order via a reversible jump. Relatively, [22] spelt-out the importance of Monte Carlo Simulation via the use of refined descriptive sampling in dealing with the parameter estimation of the involved autoregressive coefficients. The random noise adopted was the exponential distribution, such that Bayesian Autoregressive Adaptive Refined Descriptive Sampling (B2ARDS) algorithm was used to estimate the involved ϕ_1, the AR (1) coefficient. Inferentially, it was deduced that B2ARDS algorithm provides accurate and efficient point estimate of the involved ϕ_1.

Notably, a more stable location parameter (whether symmetric or skewed contained location parameter) is needed to redefine switching time-variant autoregressive process. It is because of this need that Fisher's z-distribution with stable mode location parameter (as stated by [23]) will be adopted as an alternative stable location and shape parameters (with their associated degrees of freedom). Fisher's z distribution is the statistical distribution of half of the logarithm of an $F-$ distribution variate: That is, $z = \frac{1}{2}\log F$, where, F is the F-distribution with support $x \in (-\infty, \infty)$.

Thereby, a four-parameter Fisher's $\mathbf{z}$-distribution will be adopted as a random noise to drive, ameliorate, and redefine the switching time-variant of mixture autoregressive processes. According to [24], Fisher's z-distributional innovation belongs to the family of inverse proportion of Gaussian to χ^2-variates. In light of this refinement, a random noise of four-parameter Fisher's z-distribution of the MAR process will be fully impelled into a Bayesian paradigm. Inferentially, the noisy four-parameter Fisher's z to be incorporated into the Bayesian MAR process will not only uses the noise in each component of the associated regime-switching (denotes observed stochastic behavior of a specific time series by two or more states with different underlying stochastic processes) process to seize what is term ''most likely mode value" (not via the median neither the mean) of the present conditional modal distribution given the immediate past, but the process will also be of merit for capturing the conditional modal distribution of the observations given the immediate past that can either be perceived as an asymmetric or asymmetric distributed variates. Explicitly, a four-parameter Fisher's z-distribution of Bayesian Mixture Autoregressive (FZBMAR) process that consist of k-mixture components of Fisher's z regime-switching autoregressive processes will be based on shifting number of modes in the marginal density of any switching time-variant series of interest. The prior distribution of the involved Fisher's z hyper-parameters and embedded autoregressive coefficients in the proposed FZBMAR process will be via the singly truncated student-*t* distribution, such that their corresponding posterior distributions will be the same singly truncated student-*t* distribution (conjugate posterior) with an embedded Gamma variate. A singly truncated Student-t distribution is a probability distribution derived from the standard Student-t distribution by restricting the support (the range of possible values) to a specific interval, either below a certain value or above a certain value, or both. However, the involved latent variable ($\square_{kt}$) and mixing weights of each regime (that is, α_k; $k = 1, \dots, g$) prior distributions will be via Gamma and Dirchet distributions respectively, such that their derived associated posterior distributions will be via their conjugates. The mean of all coefficients and precision components via ξ and $\lambda_k = \frac{1}{\sigma_k}$ of the Fisher's z hyper-parameters respectively will follow a Gaussian random noise for all. Identifiable constraints that usually caused label switching of priors, as well as the technique of choosing optimal autoregressive orders for the FZBMAR embedded components will be explicitly expounded. The strength of the FZBMAR model relative to other existing processes like GMAR, MAR, StMAR, and Gaussian Transition Mixture Distribution (GTMD) is that its consists of a mixture of K-component Fisher's z autoregressive models, where the numbers of components are based on the number of modes in the marginal density. FZBMAR employs stable central tendency value as mode to estimate involved coefficients that are symmetric or skewed affected in Bayesian paradigm. In contrast to the FZBMAR process, the classical GMAR, MAR, StMAR, and GTMD could not take into consideration prior information about the switching stochastic times series from the marginal density of mode as a location parameter from a Bayesian perspective. Additionally, the demonstration of the FZBMAR process is applied to monthly average share prices (stock prices) of the sixteen (16) influential European economies, such that their error process evaluations via the proposed FZBMAR process is compared with the existing ones of GMAR and S*t*MAR processes. The sixteen (16) influential selected European countries' stock prices (share prices) directly or indirectly drive the developing countries' economies and the strength of their currencies. The selected European countries includes Australia, Canada, Austria, Czech Republic, Germany, Denmark, Finland, France, Ireland, Japan, Italy, Mexico, Korea, Poland, Norway, and United State of America (USA).

2. Methodology

Assuming Z is a random distributed variable from a standardized Fisher's z-distribution with mode location parameter, say μ_k and scale parameter, say σ_k. The Cumulative Distribution Function (CDF) of X_t, a switching time-variant variate, $X_t = \sigma_k Z + \mu_k$ could be defined as,

$$F_{x_t}(x_t|m_{1k}, m_{2k}, \sigma_k, \mu_k) = Pr(X_t \le x_t) = Pr(\sigma_k Z + \mu_k \le x_t) = Pr\left(Z \le \frac{x_t - \mu_k}{\sigma_k}\right) \quad (1)$$

A stochastic process, say $\{x_t\}_{t=1,2,\dots,T}$ of a real-valued switching time-variant, that is $\{x_t\}\epsilon\mathbb{R}^{-,+}$, is said to follow a standardized Fisher's z-distribution Mixture Autoregressive (FZMAR) process, if its distribution function, conditioned on the immediate previous information condition on $t-1$ could be written as,

$$F_{x_t}(x_t|\sigma\xi_{t-1}) = \sum_{k=1}^{g} \alpha_k Z_{(m_{1k}, m_{2k})}\left(\frac{x_t}{\sigma_k}\right) \quad (2)$$

Its unstandardized process is,

$$F_{x_t}(x_t|\sigma\xi_{t-1}) = \sum_{k=1}^{g} \alpha_k Z_{(m_{1k}, m_{2k})}\left(\frac{x_t - \mu_{k,t}}{\sigma_k}\right) \quad (3)$$

Alternatively, equation (3) could be re-written as,

$$X_t = \begin{cases} \mu_{1,t} + \sigma_1 e_{1,t} & \ni \quad \alpha_1 \\ \mu_{2,t} + \sigma_2 e_{2,t} & \ni \quad \alpha_2 \\ \vdots & \vdots \quad \vdots \\ \mu_{K,t} + \sigma_K e_{K,t} & \ni \quad \alpha_K \end{cases} \quad (4)$$

$\sigma\zeta_{t-1}$ denote the information set-up at time $t-1$, that is the sigma-field generated by the process up to $t-1$. $F_{X_t}(x_t|\sigma\zeta_{t-1})$ $\forall k = 1, \dots, K$ is the conditional CDF of X_t given immediate previous information examined at x_t. g is the totality of the autoregressive components; $\alpha_k > 0$ is the mixing weights $\ni \sum_{k=1}^{g} \alpha_k \approx 1$, $1 - \sum_{k=1}^{g} \alpha_k$ and $\alpha = (\alpha_1, \dots, \alpha_g)$. $\mu_{k,t} = \phi_{k,0} + \sum_{i=1}^{p_k} \phi_{k,i} x_{t-i}$; for $k = (1,2, \dots, K)$. $e_{k,t}$ is the k^{th}-component noise for i^{th} lag, such that $i = 1,2, \dots, p_k$.

Where

$$Z_{(m_{1k}, m_{2k})}(z_t) = \left(\frac{m_{1k}}{2}, \frac{m_{2k}}{2}\right) \times \mathrm{I}_{z^{\otimes}} = \frac{\int_0^{z^{\otimes}} t^{\frac{2(m_{2k}-1)}{t}} (1-t)^{\frac{2(m_{2k}-1)}{t}} \partial t}{B\left(\left(\frac{m_{1k}}{2}, \frac{m_{2k}}{2}\right)\right)} \quad (5)$$

According to [25], $Z_{(m_{1k},m_{2k})}(z)$ is the CDF of the four-parameter Fisher's z-distribution**,** such that the corresponding Probability Density Function (PDF) of the distributed random variable is an F-distribution with m_{1k} and m_{2k} degrees of freedom is

$$f_{z_t}(z_t|m_{1k},m_{2k}) = \frac{2e^{-zm_{2k}}\left(\frac{m_{1k}}{m_{2k}}\right)^{\frac{m_{2k}}{2}}}{B\left(\frac{m_{1k}}{2},\frac{m_{2k}}{2}\right)\left(1+e^{\ln\frac{m_{2k}}{m_{1k}}+\ln(2z_t)^{-1}}\right)^{\frac{(m_{1k}+m_{2k})}{2}}} \quad (6)$$

Where the variate$=\frac{\ln X}{2}$, $I_{z^{\otimes}}$ is the incomplete Beta function ratio; $z^{\otimes} = \frac{e^{-2z}m_{2k}}{e^{-2z}m_{2k}+m_{1k}}$. $B(.)$ is the Beta function of $0 < m_{1k}, m_{2k} < 1$ and $-\infty < z < \infty$. $\sigma_k > 0$ is the scale coefficient of the k^{th}-component, such that $\sigma = \{\sigma_1, \ldots, \sigma_g\}$ is the vector of the scale coefficients. The precision of the k^{th}-component could be specified as $\lambda_k = \frac{1}{\sigma_k}$. The four-parameter Fisher's z distribution was built upon the standardized Fisher's z distribution, extends the origin by adding location parameter μ (mode) and dispersion parameter σ, allowing a wider range of modeling capabilities, that includes shifting the mode from zero central tendency. It is a flexible distribution suitable for modeling scenarios where the mode needs to be located away from zero, or where the distribution needs to be asymmetric. It is related to Chi-squared, Student-*t* and Normal distribution. It simultaneously determines the mode, standard deviation, skewedness and kurtosis. It provides better fit than the normal distribution, Pareto distribution, Extreme-Value-Distribution, and it is ideal as prior random noise when modeling mixture models in a Bayesian context. The four embedded measures of location and dispersion, ability to navigates through shifting modes and being a special case of the distributions mentioned makes the Four-parameter Fisher's z distribution superior and chosen.

The error term at k^{th}-component at time "t" could be defined

$$e_{k,t} = x_t - \phi_{k,0} - \sum_{i=1}^{p_k} \phi_{k,i} x_{t-i} = x_t - \mu_{k,t} \quad (7)$$

Its corrected mean is of the form

$$\mu_{k,t} = \sum_{i=1}^{p_k} \phi_{k,i}\,(x_{t-i} - \mu_k) + \mu_k \quad (8)$$

Comparing the coefficients of equation (6) and (7), we have

$$\phi_{k,0} = \mu_k\left(1 - \sum_{i=1}^{p_k} \phi_{k,i}\right) \quad (9)$$

If $\sum_{i=1}^{p_k} \phi_{k,i} \in \{\mathbb{R} - 0\}$, then equation (8) becomes $\mu_k = \frac{\phi_{k,0}}{\left(1-\sum_{i=1}^{p_k}\phi_{k,i}\right)}$ for $k = \{1,2,\ldots,g\}$

The necessary and sufficient condition for the stationarity of either equation (3) or (4) could be written as

$$\Theta = \{\phi_k \in \mathbb{R}^{\rho_k} : \phi_k(v) \neq 0, v \in \mathbb{C}, -1 \leq |v| \leq 1\} \quad (10)$$

Where $\phi(v)$ is the characteristic polynomial, such that:

$$\phi(v) = 1 - \phi_{1k}v^{1k} - \phi_{2k}v^{2k} \ldots \phi_{\rho k}v^{\rho k} \quad (11)$$

2.1. Likelihood Function and Latent Variable Formulation of the FZBMAR Process

Assuming that $x_1, \ldots, x_n$ is the switching time-variant observations, then the likelihood function for the FZMAR process could be explicitly specified as

$$L_{1x_t}(m_{1k}, m_{2k}, \phi_k, \alpha_k|x_t) = \prod_{t=p+1}^{n} \sum_{k=1}^{g} \alpha_k Z_{(m_{1k},m_{2k})} \left[\frac{2e^{-zm_{2k}}\left(\frac{m_{1k}}{m_{2k}}\right)^{\frac{m_{2k}}{2}}}{B\left(\frac{m_{1k}}{2},\frac{m_{2k}}{2}\right)\left(1+e^{\ln\frac{m_{2k}}{m_{1k}}+\ln[2(z_t)]^{-1}}\right)^{\frac{(m_{1k}+m_{2k})}{2}}}\right] \quad (12)$$

Where $\phi = \{\phi_{k0}, \phi_{k1}, \ldots, \phi_{k\rho_k}\}$ is the vector of Autoregressive (AR) coefficients of the k^{th}-mixture components, p_k is the AR order, such that $p = max(p_k)$ is the largest autoregressive order of the process. Conventionally, $p_k < j \leq p$ for $\phi_{kj} = 0$.

The likelihood function of equation (9) would not be tractable. This connotes that a recursive standard approach will be required in order to cater for the regime-switching observations via a latent variable (otherwise known as the recursive unobserved variable of a missing data formulation) [26].

Assuming Υ is an unobserved random variable, where Υ_t is a $k-$dimensional vector, such that $\Upsilon = \{\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n\}$ with entry k equal to one if x_t is generated from k^{th} -mixture components, and zero otherwise. In other words, $\Upsilon_t's$ are Bernoulli discretized random variables that could be independently drawn from

$$\Upsilon_{i,t} = \begin{cases} 1 & if\ X_t\ emanated\ from\ j^{th}\ state \\ 0, & otherwise \end{cases} \tag{13}$$

For $1 \leq j \leq K$, that is,

$$Pr(\Upsilon_{kt} = 1|\alpha, g) = \alpha_k \tag{14}$$

This connotes that,

$Pr(\Upsilon_t = [1,0,\ldots,0]^T) = \alpha_1; Pr(\Upsilon_t = [0,1,\ldots,0]^T) = \alpha_2$;... ; $Pr(\Upsilon_t = [0,0,\ldots,1]^T) = \alpha_k$. Then the tractable likelihood function of equation (12) could then be written as,

$$L_{2x_t}(m_{1k}, m_{2k}, \phi_k, \alpha_k|x_t) =$$

$$\prod_{t=p+1}^{n} \sum_{k=1}^{g} \alpha_k \left(Z_{(m_{1k}, m_{2k})} \frac{2e^{-(x_t-\phi_{k,0}+\sum_{i=1}^{p+k}\phi_{k,i}x_{t-i})m_{2k}} \left(\frac{m_{1k}}{m_{2k}}\right)^{\frac{m_{2k}}{2}}}{B\left(\frac{m_{1k}}{2}, \frac{m_{2k}}{2}\right)\left(1+e^{\ln\frac{m_{2k}}{m_{1k}}+\ln[2(x_t-\phi_{k,0}+\sum_{i=1}^{p+k}\phi_{k,i}x_{t-i})]^{-1}}\right)^{\frac{(m_{1k}+m_{2k})}{2}}}\right)^{\Upsilon_{kt}} \tag{15}$$

This makes it easy to handle when $\Upsilon_t = 1$, at each time t, the supplemented likelihood will be a product.

2.2. Bayesian Framework of the Fisher's z-Distribution Mixture Autoregressive (FZBMAR) Process

In practice, $\Upsilon_t's$ are not usually available. Bayesian paradigm will deal with the unavailability of $\Upsilon_t's$. Setting-up appropriate prior distributions for the latent variables formulation, mixing weights, and other involved coefficients of the FZBMAR process will make-up the development of posterior distributions for the involved parameters and other lacuna that might have arisen in the model build-up. However, non-informative priors have been the usual priors use for AR coefficients, mixing weights, and scale coefficients. However, [27] affirmed that prior distribution suitable for latent variables in latent components of autoregressive process to be uniform Dirichlet distribution, while Gaussian distribution is the ideal prior distribution for AR coefficients in a Markov switching process. In [28], [28] makes it known that the singly truncated Student-*t* distribution will be ideal as prior distributions for m_{1k}, $m_{2,k}$, and σ_k with their corresponding degrees of freedom v_{1k}, v_{2k}, v_{3k} $\ni$ their location parameters are μ_{1k}, μ_{2k}, and μ_{3k}; their corresponding scale parameters are s_{1k}^2, s_{2k}^2, and s_{3k}^2.

Mathematically, this means that x_t cannot be observed directly. Consequently, the latent variable $\Upsilon = \{\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n\}$ is a label component, $\ni$, $\Upsilon_i = \mathrm{j}$ if i^{th}-observation emanated from j^{th} state of nature as pinpointed in equation (13), for $\mathrm{i} = 1,2,\ldots,\mathrm{n}$ $\ni$

$$Pr(\Upsilon_i|\alpha) = \sum_{j=1}^{k} \alpha_j I_{(\Upsilon_i=j)} \quad \Upsilon_i = 1,2,\ldots,k \tag{16}$$

$I_{(\Upsilon_i=j)}$ is the $\Upsilon_{i,t}$ specification of equation (13)

$$Pr(\alpha|\phi_k, \Upsilon, x_t) \propto Pr(\alpha, c, \Upsilon, x_t) \tag{17}$$

$$\propto Pr(\Upsilon|\alpha,)Pr(\alpha) \tag{18}$$

$$\propto \prod_{i=1}^{n} \sum_{j=1}^{k} \alpha_j I_{(\Upsilon_i=j)} \prod_{j=1}^{k} \alpha_j^{-(1-\delta_j)} \quad j = 1,\ldots,k \tag{19}$$

$$\alpha_1, \alpha_2, \ldots, \alpha_g \sim Dir(\delta_1, \delta_2, \ldots, \delta_g), \ \delta_1 = \ldots = \delta_g = 1 \tag{20}$$

The prior distribution of each of $Z_t = \frac{\ln X_t}{2}$ depends on the corresponding Υ_t. Then, for $\Upsilon_t = 1$, the prior distribution on $Z_t = \frac{\ln X_t}{2}$ follows a Gamma prior of the form,

$$Z_t \mid \Upsilon_t \sim \Gamma\left(\frac{v_{1k}}{2}, \frac{v_{2k}}{2}\right) \tag{21}$$

The means component prior distribution follows a Gaussian distribution with mean and precision hyper-parameters ξ and $\lambda_k = \frac{1}{\sigma_k}$ respectively, ∋ ,

$$m_{ik} \sim N(\xi, \lambda^{-1}) \text{ for } k = 1, \dots, g \;\; i = 1,2 \tag{22}$$

Prior distributions for m_{1k}, m_{2k}, and σ_k are

$$Tt_{m_{1k}} \sim (\mu_{1k}, s_{1k}^2, v_{1k}) I_{(0,\infty)} \tag{23}$$

$$Tt_{m_{2k}} \sim (\mu_{2k}, s_{2k}^2, v_{2k}) I_{(0,\infty)} \tag{24}$$

$$Tt_{\sigma_k} \sim (\mu_{3k}, s_{3k}^2, v_{3k}) I_{(0,\infty)} \tag{25}$$

While prior distributions for ϕ_{k0} and ϕ_{ki} follows a Gaussian distribution with mode location parameters μ_{k0}, μ_{ki} and scale parameters ϕ_{k0}^2 and ϕ_{ki}^2, thus,

$$\phi_{k0}^2 \sim N(\mu_{k0}, \phi_{k0}^2) \tag{26}$$

$$\phi_{ki}^2 \sim N(\mu_{ki}, \phi_{ki}^2) \tag{27}$$

The corresponding singly truncated student-*t* distribution PDF is

$$Tt(Z_{t.}) = \frac{1}{\tau\sigma_k} \frac{\left(\frac{Z_t - \mu_k}{\sigma_k}\right)}{T\left(\frac{Z_{t1} - \mu_k}{\sigma_k}\right) - T\left(\frac{Z_{t2} - \mu_k}{\sigma_k}\right)} \sim Tt_{(\mu_k, \sigma_k, v_k)} \tag{28}$$

∋ $Z_{t1} \leq Z_t \leq Z_{t2}$, and zero otherwise. T and τ are the CDF and PDF of the conventional student-*t* distribution respectively. μ_k and σ_k are the mode location and scale parameters of the distribution respectively, while v_k is the degree of freedom embedded in the conventional student-*t* distribution.

2.3. Posterior Distributions and Simulation of Latent Variables

We shall affirm the corresponding posterior distribution of the latent variable γ_t and other FZBMAR process coefficients in this section. From the first-step (E-step) of the conventional EM-algorithm, the probability of the latent variable across the switching regime is

$$\Pr(\gamma_{tk} = 1 | \alpha_k, \phi_k, \mu_k, \sigma_k, \lambda_k, x_t) = \frac{\alpha_k Z_{(m_{1k}, m_{2k})}\left(\frac{e_{kt}}{\sigma_k}\right)}{\sum_{l=1}^{g} \alpha_l Z_{(m_{1k}, m_{2k})}\left(\frac{e_{kl}}{\sigma_l}\right)} \tag{29}$$

$$\alpha_k | \phi_k, \mu_k, \sigma_k, \lambda_k, x_t \sim Dir(1 + n_1, 1 + n_2, \cdots, 1 + n_g) \tag{30}$$

According to [29], it is of paramount to specify some hyper-parameters for standard setup of mixture models initialization. Let $\Re_{x_t} = \max(x_t) - \min(x_t)$ be the length of the interval variation of the observations.

The other hyper-parameters are

$$a = 0.2, c = 2, \psi = \frac{\Re_{x_t}}{2} + \min(x_t), \kappa = {}^{1}\!/_{\Re_{x_t}}, b = \frac{10}{\Re_{x_t}^2}.$$

$$m_{1k} | \phi_k, \lambda_k, \gamma, z_t, \alpha_k, m_{-m_k} \sim Tt\left(\frac{\lambda_k\psi + b_k\mu_{1k}n_k\bar{e}_k\Gamma\left(\frac{v_{1k}}{2}, \frac{v_{1k}}{2}\right)}{\mu_{1k}n_k b_k^2 \bar{e}_k \Gamma\left(\frac{v_{1k}}{2}, \frac{v_{1k}}{2}\right) + \psi}, \frac{1}{\mu_{1k}n_k b_k^2 \Gamma\left(\frac{v_{1k}}{2}, \frac{v_{1k}}{2}\right) + \psi}, v_{1k}\right) \tag{31}$$

$$m_{2k} | \phi_k, \lambda_k, \gamma, z_t, \alpha_k, m_{-m_k} \sim Tt\left(\frac{\lambda_k\psi + b_k\mu_{1k}n_k\bar{e}_k\Gamma\left(\frac{v_{2k}}{2}, \frac{v_{2k}}{2}\right)}{\mu_{1k}n_k b_k^2 \bar{e}_k \Gamma\left(\frac{v_{2k}}{2}, \frac{v_{2k}}{2}\right) + \psi}, \frac{1}{\mu_{1k}n_k b_k^2 \Gamma\left(\frac{v_{2k}}{2}, \frac{v_{2k}}{2}\right) + \psi}, v_{2k}\right) \tag{32}$$

$$\sigma_k | \phi_k, \lambda_k, \gamma, z_t, \alpha_k, m_{-m_k} \sim Tt\left(\frac{\lambda_k\psi + b_k\mu_{1k}n_k\bar{e}_k\Gamma\left(\frac{v_{3k}}{2}, \frac{v_{3k}}{2}\right)}{\mu_{1k}n_k b_k^2 \bar{e}_k \Gamma\left(\frac{v_{3k}}{2}, \frac{v_{3k}}{2}\right) + \psi}, \frac{1}{\mu_{1k}n_k b_k^2 \Gamma\left(\frac{v_{3k}}{2}, \frac{v_{3k}}{2}\right) + \psi}, v_{3k}\right) \tag{33}$$

$$\lambda_k|\phi_k,\sigma_k,\gamma,z_t,\alpha_k,m_{-m_k} \sim \Gamma\left(gc+a,\ \sum_{k=1}^{g} z_t + b\right) \tag{34}$$

$$z_k|\phi_k,\sigma_k,\gamma,z_{-zk},m_{-m_k},\alpha_k \sim \Gamma\left(\frac{n_k}{2}+c,\ \frac{\lambda}{2}\sum_{t=p+1}^{n} e_{tk}^2\gamma_{tk}\right) \tag{35}$$

For $k = 1,\cdots,g$. It is to be noted that $\Gamma\left(\frac{1}{h},\ \frac{1}{h}\right) \approx \Gamma\,(1);\ \forall\, h \in \mathbb{R}^+$
$n_k = \sum_{t=p+1}^{n}\gamma_{tk}\,;\quad e_{tk} = -(\nu_{tki} - x_t);\quad b_k = 1 - \sum_{i=1}^{\rho_k}\phi_{ki}\,;\quad \bar{e}_k = \frac{1}{n_k}\sum_{t=p+1}^{n} e_{tk}\gamma_{tk}$, in the context of ν_{tki}, $i = 1,2,3$.

It is to be noted that the posterior distribution of $\phi_k = \{\phi_{k0},\phi_{k1},\phi_{kp_k}\}$ and ν_{ki} do not follow a standard form distribution. Metropolis-Hastings concurrent method of simulation technique will be ideal for $k = 1,\cdots,g$ components. Having assumed that ϕ_k is from a concurrent state of Markov Chain to be drawn. However, we can simulate the ϕ_k^* candidacy from a proposed distribution of the Multivariate Normal Distribution, that is, $\mathrm{MVN}(\phi_k,\ \eta_k I_{\rho_k})$, ∍ η_k is the regularization parameter, while I_{ρ_k} is the identity matrix of size ρ_k. The probability of acceptance for the proposed ϕ_k^* goes for

$$\wp[\phi_k,\phi_k^*] = \min\left[1,\ \frac{\dfrac{e^{-\left(x_t-\phi_{k0}^*+\sum_{i=1}^{p+k}\phi_{ki}^* x_{t-i}\right)m_{2k}}}{e^{\ln\frac{m_{2k}}{m_{1k}}+\ln\left[2\left(x_t-\phi_{k0}^*+\sum_{i=1}^{p+k}\phi_{ki}^* x_{t-i}\right)\right]^{-\frac{(m_{1k}+m_{2k})}{2}}}}}{\dfrac{e^{-\left(x_t-\phi_{k0}+\sum_{i=1}^{p+k}\phi_{ki} x_{t-i}\right)m_{2k}}}{e^{\ln\frac{m_{2k}}{m_{1k}}+\ln\left[2\left(x_t-\phi_{k0}+\sum_{i=1}^{p+k}\phi_{ki} x_{t-i}\right)\right]^{-\frac{(m_{1k}+m_{2k})}{2}}}}}\right] \tag{36}$$

It is to be noted that the ratio of $\wp[\phi_k,\phi_k^*]$ of equation (36) above is usually equal to one (1) because of the flat prior associated to the autoregressive coefficients. This connotes that the ratio's likelihood of k^{th}-components is independently present in the coefficients of the remaining components. This makes it feasible to estimate likelihood ratios of the process separately for each component. This routine builds-up the candidacy process with updated proportion (α_k), the mode location and scale parameters; and the switching autoregressive coefficients in the FZBMAR process.

In a similar vein, the probability of acceptance for the proposed m_{1k}^* and m_{2k}^* degrees of freedom goes for

$$\wp[m_{1k}^*,m_{2k}^*] = \min\left[1,\ \frac{\dfrac{e^{-\left(x_t-\phi_{k,0}+\sum_{i=1}^{p+k}\phi_{k,i}x_{t-i}\right)m_{2k}^*\left(\frac{m_{1k}^*}{m_{2k}^*}\right)^{\frac{m_{2k}^*}{2}}}}{B\left(\frac{m_{1k}^*}{2},\frac{m_{2k}^*}{2}\right)\left(1+e^{\ln\frac{m_{2k}^*}{m_{1k}^*}+\ln\left[2\left(x_t-\phi_{k,0}+\sum_{i=1}^{p+k}\phi_{k,i}x_{t-i}\right)\right]^{-1}}\right)^{\frac{(m_{1k}^*+m_{2k}^*)}{2}}}}{\dfrac{e^{-\left(x_t-\phi_{k,0}+\sum_{i=1}^{p+k}\phi_{k,i}x_{t-i}\right)m_{2k}\left(\frac{m_{1k}}{m_{2k}}\right)^{\frac{m_{2k}}{2}}}}{B\left(\frac{m_{1k}}{2},\frac{m_{2k}}{2}\right)\left(1+e^{\ln\frac{m_{2k}}{m_{1k}}+\ln\left[2\left(x_t-\phi_{k,0}+\sum_{i=1}^{p+k}\phi_{k,i}x_{t-i}\right)\right]^{-1}}\right)^{\frac{(m_{1k}+m_{2k})}{2}}}}\right] \tag{37}$$

2.4. Choosing Autoregressive Orders for the FZBMAR

Having ascertained that ρ_k is the present order of a particular order. If $\rho_{\max}$ is the largest possible autoregressive order. Consequently, the template for incremental autoregressive order for the FZBMAR process is $\rho_k^* = 1 + \rho_k$ with probability ↑ $\Pr b(\rho_k)$, or decrement $\rho_k^* = \rho_k - 1$ with probability ↓ $\Pr d(\rho_k)$. $b(\rho_k)$ and $d(\rho_k)$ are defined function on $[0, 1]$. That is, $b(\rho_k): \mathbb{R} \to [0,1]$ and $d(\rho_k): \mathbb{R}^{(-,+)} \to [0,1]$ a 1−1 mapping for all, ∍, $b(\rho_{\max}) = 0$ and $d(\rho_k) = 1 - b(\rho_k)$. The probability of acceptance for the proposed template of $\rho_k^* = \rho_k - 1$ is the product of the proposed ratio and likelihood ∍,

$$\wp[\rho_k,\rho_k^*]=\min\left\{1,\frac{\dfrac{e^{-\left(x_t-\phi_{k0}^{\rho_k^*}+\sum_{i=1}^{p+k}\phi_{ki}^{\rho_k^*}x_{t-i}\right)m_{2k}}}{e^{\ln\frac{m_{2k}}{m_{1k}}+\ln\left[2\left(x_t-\phi_{k0}^{\rho_k^*}+\sum_{i=1}^{p+k}\phi_{ki}^{\rho_k^*}x_{t-i}\right)\right]^{\frac{(m_{1k}+m_{2k})}{2}}}}}{\dfrac{e^{-\left(x_t-\phi_{k0}^{\rho_k^*}+\sum_{i=1}^{p+k}\phi_{ki}^{\rho_k^*}x_{t-i}\right)m_{2k}}}{e^{\ln\frac{m_{2k}}{m_{1k}}+\ln\left[2\left(x_t-\phi_{k0}^{\rho_k^*}+\sum_{i=1}^{p+k}\phi_{ki}^{\rho_k^*}x_{t-i}\right)\right]^{\frac{(m_{1k}+m_{2k})}{2}}}}}\times\frac{b(\rho_k^*)}{d(\rho_k)}\times Z_{(m_{1k},m_{2k})}\left(\frac{\phi_{\rho_k}^*-\phi_{\rho_k}}{1/\sqrt{\sigma_k}}\right)\right\} \tag{38}$$

For the proposed template of $\rho_k^*=\rho_k+1$. Since the autoregressive coefficients of $\phi_{k_{\rho_k}}$ can be simulated from $U[-1.5,\ 1.5]$ distribution (because of the constraint that autoregressive coefficients must always be within the unit circle) , it literally means that estimates closer to zero are likely to be taken into account, as well as estimates far from zero. Similarly to that of $\rho_k^*=\rho_k-1$, the probability of acceptance for the proposed template of $\rho_k^*=\rho_k+1$ is the product of the proposed ratio and likelihood such that,

$$\wp[\rho_k,\rho_k^*]=\min\left\{1,\ \frac{\dfrac{e^{-\left(x_t-\phi_{k0}^{\rho_k^*}+\sum_{i=1}^{p+k}\phi_{ki}^{\rho_k^*}x_{t-i}\right)m_{2k}}}{e^{\ln\frac{m_{2k}}{m_{1k}}+\ln\left[2\left(x_t-\phi_{k0}^{\rho_k^*}+\sum_{i=1}^{p+k}\phi_{ki}^{\rho_k^*}x_{t-i}\right)\right]^{-\frac{(m_{1k}+m_{2k})}{2}}}}}{\dfrac{e^{-\left(x_t-\phi_{k0}^{\rho_k}+\sum_{i=1}^{p+k}\phi_{ki}^{\rho_k}x_{t-i}\right)m_{2k}}}{e^{\ln\frac{m_{2k}}{m_{1k}}+\ln\left[2\left(x_t-\phi_{k0}^{\rho_k}+\sum_{i=1}^{p+k}\phi_{ki}^{\rho_k}x_{t-i}\right)\right]^{-\frac{(m_{1k}+m_{2k})}{2}}}}}\times\frac{d(\rho_k^*)}{b(\rho_k)}\times 3\right\} \tag{39}$$

Where the magnitude three (3) is the inverse density of $\phi_{k_{\rho_k}}$ usually under uniform interval of $U[x_1=-1.5,\ x_2=1.5]$ proposed distribution, that is the maximum length of addition of x_1 and x_2 is three (3). It is to be noted that in both scenarios if the candidate process fails to honor the stability condition, then it shall be automatically rejected. (See Appendix 1 for stability condition of the FZBMAR process).

2.5. Dealing with Identifiable Label Switching Priors

Identification of mixture constraints, such as $\alpha_1>\alpha_2>\cdots>\alpha_g$ is a well-known problem in mixture models. It is with this that [21] demonstrated that applying identifiable constraints, such as $\alpha_1>\alpha_2>\cdots>\alpha_g$; $\alpha_1<\alpha_2<\cdots<\alpha_g$ or mixture of inequalities of these weights would surely affect convergence of chain, especially in MCMC algorithm or its variants. Moreover, there is likelihood that two or more of the mixing weights might be equal, in such scenario(s); proper switching label is needed to deal with the identifiability constraints.

Assuming c is a label switching of unknown value, but large enough to be able to reliably calculate via some initial values of clusters and their respective variances. Let $\vartheta=(\vartheta_1,\vartheta_2,\cdots,\vartheta_g)$ be the subset of the process's coefficients of size "g", such that "N" is the size of the converged sample. It is of paramount that sub-setting of the associated coefficients of different mixture components must be selected, for example, $\vartheta=(\alpha_1,\alpha_2,\cdots,\alpha_g)$or $\vartheta=(\mu_1,\mu_2,\cdots,\mu_g)$ among others. The mean and variance, based on the first "c" simulated values via some centralized coordinates, say ϑ_i, $i=1,\cdots,q$ are;

$$\bar{\vartheta}_i=\frac{\sum_{j=1}^{c}\vartheta_i^{(j)}}{c};\qquad \bar{s}_{\vartheta i}^2=\frac{1}{c}\sum_{j=1}^{c}\left(\vartheta_i^{(j)}-\bar{\vartheta}_i\right)$$

Setting the real permutation of the components means setting the initials to be mean$=\bar{\vartheta}^{(0)}$ with variances$=\bar{s}_{\vartheta_i}^{(0)}$, $i=1,\cdots,q$ for $g!-1$ permutations. The initial estimates can be estimated via h^{th}- iteration of $(h=1,\cdots,N-c)$ of the two procedural steps:

1. Normalized squared distance of coefficients is,

$$\vartheta^{(c+h)}=\sum_{i=1}^{g}\frac{\left(\vartheta_i^{(c+h)}-\bar{\vartheta}_i^{(c+h-1)}\right)^2}{\left(s_i^{(c+h-1)}\right)^2} \tag{40}$$

2. The centralized coordinates and their corresponding variances are as follows:

$$\bar{\vartheta}_i^{(h+c)} = \frac{1}{h+c}\vartheta_i^{(h+c)} + \frac{h+c-1}{h+c}\bar{\vartheta}_i^{(h+c-1)}$$

$$\left(s_i^{(h+c)}\right)^2 = \frac{(h+c-1)\left(s_i^{(h+c-1)}\right)^2}{h+c} + \frac{h+c-1\left(\bar{\vartheta}_i^{(h+c-1)}-\bar{\vartheta}_i^{(h+c)}\right)^2}{h+c} + \frac{\left(\vartheta_i^{(h+c)}-\bar{\vartheta}_i^{(h+c)}\right)^2}{h+c} \quad (41)$$

$for\ \ i = 1, \cdots, g$

It is to be noted that once the selected proper subset has been re-labelled, labels for the remaining coefficients could be switched accordingly.

2.6. Algorithm for Dealing with Label Switching

1. Label each simulation from 1 to N such that $\vartheta^{(i)},\ i = 1, \cdots, N$.
2. Calculate density forecast $f^{(i)}\left(x_{t+h}|F_t,\ \vartheta^{(i)}\right)$
3. Estimate the density forecast $\hat{f}(x_{t+h}|F_t) = \frac{1}{N}\sum_{i=1}^{N} f^{(i)}\left(x_{t+h}|F_t,\ \vartheta^{(i)}\right)$

The density forecast estimate uses the highest posterior density value(s) (i.e. the peak of the posterior distribution). Where "h" is the h-step forecast horizon ahead density forecast of observations of interest.

3. Application

Truth to be told that price of shares is not fixed, such that price of these shares fluctuates according to market conditions, retail and wholesale price stability, economy behavior; economy and currency stability or instability of the involved economy/country. It will likely increase if the economy, firm, or organization is perceived to be stable or making profit, whereas decrease when expectations are not met. It is usually denoted by "SHPRICE" if it is to be quoted from the internationally amalgamation. Alternatively, is the price of a single share-out of saleable equity share of a country, firm, or organization. In other words, it is the highest amount a buyer is willingly to pay for a unit stock, or the lowest amount that it can be bought for. In financial price derivatives and econometric theories, analysts usually use random walk technique to model behavior of share prices on stock markets that are peculiar to companies, firms, or countries. Empirical results have shown that price of shares do not usually follow the conventional random walks, which are associated with low serial correlations for short-term assessment, while slightly stronger correlations are attached to longer-term. The signs and strength of the random influence of share prices on stock markets depend on the mixture of random works characterized within. In light of the stronger correlations that are usually attached to longer-term mixture random walk, the FZBMAR process shall be applied to monthly average share prices (stock prices) of sixteen (16) influential European economies. It is to be noted that the stock prices of these countries are the average accumulated price of each economy (that is, the accumulated average share prices of all share firms, and companies as compiled by the country's centralized stockbroker). The sixteen (16) influential selected European countries' ameliorate and drive the world economies' stock prices. The selected European economies are: Australia, Canada, Austria, Czech Republic, Germany, Denmark, Finland, France, Ireland, Japan, Italy, Mexico, Korea, Poland, Norway, and United State of America (USA). These European's economies possessed the largest Europe economies in terms of nominal GDP. The European's economies were selected on the basis of being the world's biggest economies. These mentioned countries' economies are regarded as the swaying world stock prices and dominancy of developing countries' currencies. The monthly datasets of the financial returns to be considered will be from 2011-01 (January, 2012) to 2023-10 (December, 2023). Application wise, the stock prices of these sixteen (16) mentioned countries' economies will be applied to the proposed FZBMAR process, such that its error process evaluation will be juxtaposed with the existing StMAR and GMAR processes.

Table 1. Error Evaluation Performances of the GMAR, StMAR, and FZBMAR Processes of the Sixteen (16) Swaying European Economies' Stock Prices.

Countries/Model	Australia	Austria	Canada	Czech Republic	Germany	Denmark
StMAR: $StMAR(4{:}\ 4,4,4,4)$	log-likelihood: -546.13, AIC: 1154.25, HQIC: 1192.33, BIC: 1247.99	log-likelihood: -592.73, AIC: 1247.46, HQIC: 1285.54, BIC: 1341.20	log-likelihood: -569.64, AIC: 1201.28, HQIC: 1239.37, BIC: 1295.03	log-likelihood: -563.53, AIC: 1189.06, HQIC: 1227.14, BIC: 1282.80	log-likelihood: -588.10, AIC: 1238.19, HQIC: 1276.27, BIC: 1331.93	log-likelihood: -630.22, AIC: 1322.44, HQIC: 1360.53, BIC: 1416.18
GMAR: $MAR(4{:}\ 4,4,4,4$	log-likelihood: -540.64, AIC: 1135.27, HQIC: 1168.44, BIC: 1216.92	log-likelihood: -593.15, AIC: 1240.30, HQIC: 1273.47, BIC: 1321.95	log-likelihood: -557.73, AIC: 1169.46, HQIC: 1202.63, BIC: 1251.11	log-likelihood: -562.39, AIC: 1178.77, HQIC: 1211.94, BIC: 1260.42	log-likelihood: -573.24, AIC: 1200.48, HQIC: 1233.65, BIC: 1282.13	log-likelihood: -643.12, AIC: 1340.24, HQIC: 1373.41, BIC: 1421.89
FZBMAR $FBZMAR(4{:}\ 4,4,4,4)$	log-likelihood: -580.13, PSIS-LOO= 870.642*	log-likelihood: -599.15, PSIS-LOO= 954.534*	log-likelihood: -578.67, PSIS-LOO= 990.77*	log-likelihood: -563.53, PSIS-LOO= 932.05*	log-likelihood: -588.10, PSIS-LOO= 924.03*	log-likelihood: -643.12, PSIS-LOO= 892.89*

Countries/Model	Finland	France	Ireland	Japan	Italy	Mexico
StMAR: $StMAR(4{:}\,4,4,4,4)$	log-likelihood: -594.43, AIC: 1250.87, HQIC: 1288.95, BIC: 1344.61	log-likelihood: -581.11, AIC: 1224.22, HQIC: 1262.30, BIC: 1317.96	log-likelihood: -565.54, AIC: 1193.08, HQIC: 1231.16, BIC: 1286.82	log-likelihood: -594.47, AIC: 1250.93, HQIC: 1289.01, BIC: 1344.67	log-likelihood: -599.42, AIC: 1260.85, HQIC: 1298.93, BIC: 1354.59	log-likelihood: -541.62, AIC: 1145.23, HQIC: 1183.31, BIC: 1238.97
GMAR: $MAR(4{:}\,4,4,4,4)$	log-likelihood: -586.50, AIC: 1227.00, HQIC: 1260.16, BIC: 1308.64	log-likelihood: -584.48, AIC: 1222.95, HQIC: 1256.12, BIC: 1304.60	log-likelihood: -576.47, AIC: 1206.94, HQIC: 1240.10, BIC: 1288.58	log-likelihood: -593.72, AIC: 1241.44, HQIC: 1274.61, BIC: 1323.08	log-likelihood: -591.18, AIC: 1236.35, HQIC: 1269.52, BIC: 1318.00	log-likelihood: -523.23, AIC: 1100.46, HQIC: 1133.63, BIC: 1182.10
FZBMAR $FBZMAR(4{:}\,4,4,4,4)$	log-likelihood: -594.43, PSIS-LOO= 975.32*	log-likelihood: -600.42, PSIS-LOO= 920.186*	log-likelihood: -590.23, PSIS-LOO= 905.23*	log-likelihood: -600.29, PSIS-LOO= 912.56*	log-likelihood: -607.43, PSIS-LOO= 907.32*	log-likelihood: -570.97, PSIS-LOO= 829.05*

Countries/Model	Korea	Poland	Norway	USA
StMAR: $StMAR(4{:}\,4,4,4,4)$	log-likelihood: -589.65, AIC: 1241.30, HQIC: 1279.38, BIC: 1335.04	log-likelihood: -580.42, AIC: 1222.84, HQIC: 1260.92, BIC: 1316.58	log-likelihood: -650.09, AIC: 1362.18, HQIC: 1400.26, BIC: 1455.92	log-likelihood: -577.55, AIC: 1217.09, HQIC: 1255.17, BIC: 1310.83
GMAR: $MAR(4{:}\,4,4,4,4)$	log-likelihood: -589.76, AIC: 1233.51, HQIC: 1266.68, BIC: 1315.16	log-likelihood: -578.89, AIC: 1211.79, HQIC: 1244.95, BIC: 1293.43	log-likelihood: -671.48, AIC: 1396.96, HQIC: 1430.12, BIC: 1478.60	log-likelihood: -585.04, AIC: 1224.09, HQIC: 1257.25, BIC: 1305.73
FZBMAR $FBZMAR(4{:}\,4,4,4,4)_t$	log-likelihood: -617.09, PSIS-LOO= 953.09*	log-likelihood: -594.33, PSIS-LOO= 967.036*	log-likelihood: -590.23, PSIS-LOO= 905.23*	log-likelihood: -630.89, PSIS-LOO= 940.34*

Keys: AIC= Alkaike Information Criteria; HQIC= Hannan-Quinn Information Criteria; PSIS-LOO=Pareto-Smoothed Important Sampling Leave-One-Out Cross-Validation.
Source: Authors' Computation (2025)

3.1 Results and Discussion

Figure 1, 2, and 3 below showed that stock prices of Australia, Denmark, and Mexico series has a four modal marginal distribution, where the estimated locations via mode are at -1000, 0, 1000, and 2000 points thereabouts. This connotes that a four-component regime switching (for GMAR, StMAR, and FZBMAR processes) for the share price series, that is, $k = 1,2,3,4$ is peculiar to the sixteen (16) influential European countries' share price series. Setting-up the warming-up stage for the prior distributions and posterior density plots of the FZBMAR process. Table 1 above is the result of the process error evaluation of the FZBMAR process via a 2000 iteration that was accompanied by four (4) chains with 6000 sampling iterations and two (2) thins. Interestingly, the optimal order per each component ($k = 1,2,3,4$) is four (4), that is ρ_{14}, ρ_{24}, ρ_{34}, ρ_{44} and it is the largest optimal per each component, that is $\rho_{\max}$. The optimal order was used for each of $MAR(4{:}\,4,4,4,4)$, $StMAR(4{:}\,4,4,4,4)$, and $FZBMAR(4{:}\,4,4,4,4)_{tv_k}$ processes respectively. In affirmation, orders of the autoregressive components were chosen alternatively for the FZBMAR process via minimum Pareto-Smoothed Important Sampling Leave-One-Out Cross-Validation (PSIS-LOO) criterion. PSIS-LOO is a method in Bayesian statistics used for efficient approximate leave-one-out cross-validation (LOO-CV) to assess model fit and compare models in Bayesian models, FZBMAR process in this context. It is a method used to estimate the log-likelihood of the leave-one-out predictive densities, which are used in LOO-CV. Akaike Information Criterion (AIC) is a metric for evaluating and comparing the accuracy of predictions and, by extension, the validity of statistical models' performance for a given set of dataset. Models in this context are classical time series of models of StMAR and GMAR processes. Hannan-Quinn Information Criteria (HQIC) and Bayesian Information Criterion (BIC) are similar variant of AIC criterion.

The PSIS-LOO index was carved-out from the posterior distribution of the Metropolis-Hastings concurrent method in equation (37). Notably, $FZBMAR(4{:}\,4,4,4,4)_{tv_k}$ process was carved-out from the share prices of all the considered countries. However, Australia, Czech Republic, Ireland, and Mexico economies gave minimum classical process error evaluations of (StMAR: AIC=1154.25; GMAR: AIC = 1135.27); (*StMAR*: AIC=1189.06; GMAR: AIC = 1178.77); (StMAR: AIC=1193.08; GMAR: AIC = 1206.94); and (StMAR: AIC=1145.23; GMAR: AIC = 1100.46) respectively of their share prices over the thirteen (13) years of study compare to other countries. Furthermore, the four-component FZBMAR process with intercept was mathematically represented by FZBMAR(4:4,4,4,4) for the countries of studied share prices gave minimum PSIS-LOOs of 870.642, 932.05, 905.23, and 829.05 for Australia, Czech Republic, Ireland, and Mexico respectively in comparison to a larger classical error process evaluation indexes of AIC, HQIC, and BIC by StMAR and GMAR processes. Thus far, the FZBMAR process is preferred over the StMAR and GMAR processes by the indication of the miniature PSIS-LOO values than the larger error process evaluations of AICs, HQICs, and BICs produced for the Australia, Czech Republic, Ireland, and Mexico by GMAR and StMAR processes. It is to be noted that m_{1k} and m_{2k} are shape parameters, such that they are define to meaningfully interpret both the fatness of tails and skewedness associated to the variate process. For $m_{1k} \neq m_{2k}$, it implies asymmetrically distributed mixture variates; If

$m_{1k} = m_{2k}$, it connotes symmetrically distributed mixture varieties, while larger m_{1k} and m_{2k} connotes thin tails. Interpretably, all the $m_{1k}'s$ and $m_{2k}'s$ valued at 105.91, 72.78, 90.92, 85.45, 104.86, 107.86, 103.37, 75.67, and so and so forth per each regime of the four-component of each of the Australia, Czech Republic, Ireland, and Mexico's FZBMAR processes in Table 2 below are differently far large to imply thinner tails of their mode distributional marginal densities (See Figure 1, 2, 3 below). In addition, since $m_{1k} \neq m_{2k}$ for all the sixteen (16) countries considered, it can be deduced that the fluctuating series of all the stock prices of the countries considered are asymmetrically distributed mixture variates.

The Gelman-Rubin Convergence (Rhat) Statistic for convergence diagnoses could be perceived as potential scale reduction factor, and its effective sample size, denoted by n_{eff} is usually used by the Metropolis-Hastings concurrent method chain to reach convergence in the iteration procedure. However, Since Rhat statistic is less than 1.01 (approximately equal to 1 for all) and the n_{eff} sample size that should be greater than 400 (but > 2000 estimated n_{eff} for all estimated coefficients) affirmed the robustness miniature error process evaluation by PSIS-LOO of the FZBMAR process for all [30].

Table 2. FZBMAR Coefficients of the Highlighted Australia, Czech Republic, Ireland, and Mexico Economies' Stock Prices

FZBMAR	Australia				Denmark			
Regime 1	**Mode**	**2.5%**	**97.5%**	**Rhat**	**Mode**	**2.5%**	**97.5%**	**Rhat**
$\phi_{1,0}$; $\phi_{1,1}$	304.18 -0.32 (3.90) (0.009)	299.34 -0.39 (3.97) (0.023)	307.21 -0.28 (3.85) (0.007)	1	313.53 -0.27 (9.55) (0.05)	307.04 -0.31 (9.05) (0.03)	313.53 -0.25 (8.78) (0.04)	1
$\phi_{1,2}$; $\phi_{1,3}$	-0.22 -0.39 (0.03) (0.01)	-0.27 -0.45 (0.05) (0.03)	-0.20 0.37 (0.028) (0.02)	1	-0.09 -0.28 (0.06) (0.05)	-0.094 -0.29 (0.08) (0.07)	-0.097 -0.24 (0.04) (0.03)	1
$\phi_{1,4}$; α_1	-0.95 0.82 (0.00)	-0.99 0.82 (0.00)	-0.92 0.82 (0.00)	1	-0.99 1.00 (0.00)	-0.99 1.00 (0.00)	-0.93 1.00 (0.00)	1
m_{1k}; m_{12}; $s^2_{\sigma_{1k}}$; v_{1k}	105.91; 72.78; 19731.25; 10.45	102.06; 75.83; 19728.09; 09.77	107.47; 70.09; 19738.05;13.66	1	118.76;120.45; 1347.82;2.32	113.09; 115.86; 1340.89; 2.39	118.76; 120.60; 1347.82;2.30	1
Regime 2	**Mode**	**2.5%**	**97.5%**	**Rhat**	**Mode**	**2.5%**	**97.5%**	**Rhat**
$\phi_{2,0}$; $\phi_{2,1}$	144.20 0.41 (0.564) (0.020)	142.34 0.38 (0.569) (0.056)	142.34 0.38 (0.50) (0.01)	1	219.32 0.09 (0.202) (0.042)	215.74 0.06 (0.211) (0.041)	222.92 0.10 (0.200) (0.04)	1
$\phi_{2,2}$; $\phi_{2,3}$	-0.41 -0.59 (0.03) (0.01)	-0.37 -0.55 (0.04) (0.03)	-0.45 -0.63 (0.02) (0.01)	1	0.34 -0.19 (0.08) (0.08)	0.31 -0.22 0.09) (0.09)	0.38 -0.34 (0.06) (0.06)	1
$\phi_{2,4}$; α_2	-0.001 0.17 (0.000)	-0.001 0.17 (0.000)	-0.001 0.17 (0.000)	1	-0.79 0.00 (0.08) (0.08)	-0.84 0.00 (0.07) (0.08)	-0.77 0.00 (0.06) (0.08)	1
m_{1k}; m_{12}; $s^2_{\sigma_{1k}}$; v_{1k}	90.92; 85.45; 88114.37;13.78	87.75; 83.70; 88110.45;11.73	95.30; 88.67; 88118.43;14.04	1	150.69;140.34; 298.99; 7.77	145.77; 135.73; 296.67; 7.70	155.89; 145.56; 306.08; 9.87	1
Regime 3	**Mode**	**2.5%**	**97.5%**	**Rhat**	**Mode**	**2.5%**	**97.5%**	**Rhat**
$\phi_{3,0}$; $\phi_{3,1}$	144.20 0.41 (0.56) (0.02)	142.61 0.37 (0.51) (0.01)	142.61 0.37 (0.51) (0.01)	1	97.10 0.04 (9.55) (0.17)	95.66 0.03 (9.97) (0.2)	99.45 0.01 (9.55) (0.165)	1
$\phi_{3,2}$; $\phi_{3,3}$	-0.41 -0.59 (0.03) (0.01)	-0.40 -0.56 (0.02) (0.01)	-0.40 -0.56 (0.02) (0.01)	1	0.77 -0.30 (0.23) (0.21)	0.75 -0.26 (0.25) (0.24)	0.79 -0.39 (0.21) (0.19)	1
$\phi_{3,4}$; α_3	-0.001 0.00 (0.00)	-0.001 0.00 (0.00)	-0.001 0.00 (0.00)	1	-0.79 0.00 (0.11)	-0.65 0.00 (0.10)	-0.75 0.00 (0.10)	1
m_{1k}; m_{12}; $s^2_{\sigma_{1k}}$; v_{1k}	104.86;107.86; 225.80;11.56	101.67; 105.56; 222.53; 09.34	107.11; 109.86 226.05; 13.45	1	76.59; 64.59; 212.18; 3.20	72.90; 60.34; 210.20; 3.18	78.01; 68.67; 214.56; 3.22	1
Regime 4	**Mode**	**2.5%**	**97.5%**	**Rhat**	**Mode**	**2.5%**	**97.5%**	**Rhat**
$\phi_{4,0}$; $\phi_{4,1}$	93.08 0.13 (11.08) (0.05)	91.44 0.11 (11.19) (0.08)	95.63 0.15 (10.50) (0.03)	1	14.06 0.24 (15.97) (0.20)	15.00 0.21 (15.02) (0.20)	13.45 0.20 (15.04) (0.18)	1
$\phi_{4,2}$; $\phi_{4,3}$	-0.39 0.10 (0.06) (0.05)	-0.43 0.099 (0.08) (0.07)	-0.35 0.13 (0.05) (0.03)	1	1.42 -0.22 (0.25) (0.16)	1.45 -0.20 (0.27) (0.18)	1.40 -0.25 (0.23) (0.14)	1
$\phi_{4,4}$; α_4	-0.95 0.00 (0.03)	-0.91 0.00 (0.035)	-0.98 0.00 (0.025)	1	-0.60 0.00 (0.16)	-0.60 0.00 (0.16)	-0.60 0.00 (0.16)	1
m_{1k}; m_{12}; $s^2_{\sigma_{1k}}$; v_{1k}	103.37; 75.67; 177714.62; 09.39	101.44; 72.04; 177710.3;08.99	106.56; 78.93; 177709.34;09.87	1	93.02; 65.74; 1489.98; 2.11	91.78; 60.20; 1484.76; 2.06	91.78; 64.45 1480.23; 2.14	1

FZBMAR	Mexico			
Regime 1	**Mode**	**2.5%**	**97.5%**	**Rhat**
$\phi_{1,0}$; $\phi_{1,1}$	72.01 -1.84 (24.48) (0.15)	70.78 -1.81 (24.02) (0.12)	74.45 -1.88 (24.48) (0.15)	1
$\phi_{1,2}$; $\phi_{1,3}$	-0.27 1.45 (0.11) (0.16)	-0.25 1.42 (0.10) (0.14)	-0.29 1.47 (0.12) (0.17)	1
$\phi_{1,4}$; α_1	0.87 0.71 (0.03)	0.83 0.71 (0.05)	0.85 0.71 (0.02)	1
m_{1k}; m_{12}; $s^2_{\sigma_{1k}}$; v_{1k}	90.14; 83.14; 69.72; 15.76	88.45; 78.23; 67.52; 14.34	93.67; 86.23 71.53; 15.84	1
Regime 2	**Mode**	**2.5%**	**97.5%**	**Rhat**
$\phi_{2,0}$; $\phi_{2,1}$	30.62 0.45 (24.48) (0.15)	28.67 0.42 (25.081) (0.16)	33.07 0.48 (23.98) (0.18)	1
$\phi_{2,2}$; $\phi_{2,3}$	-0.05 -0.19 (0.11) (0.16)	-0.03 -0.17 (0.10) (0.14)	-0.06 -0.21 (0.10) (0.145)	1
$\phi_{2,4}$; α_2	0.62 0.22 (0.03)	0.60 0.22 (0.02)	0.64 0.22 (0.028)	1

m_{1k}; m_{12}; $s^2_{\sigma_{1k}}$; v_{1k}	186.00; 120.45; 5.19; 2.00	184.36; 115.67 5.04; 1.42	188.54; 125.79 5.97; 2.36	1
Regime 3	Mode	2.5%	97.5%	Rhat
$\phi_{3,0}$; $\phi_{3,1}$	202.35 -0.08 (0.92) (0.04)	200.14 -0.06 (0.3) (0.035)	205.14 -0.09 (0.68) (0.05)	1
$\phi_{3,2}$; $\phi_{3,3}$	-0.27 -0.71 (0.05) (0.05)	-0.25 -0.70 (0.04) (0.04)	-0.27 -0.71 (0.05) (0.05)	1
$\phi_{3,4}$; α_3	0.09 0.06 (0.05)	0.10 0.06 (0.04)	0.12 0.06 (0.06)	1
m_{1k}; m_{12}; $s^2_{\sigma_{1k}}$; v_{1k}	103.04; 80.67; 59.67; 99999.59	102.67; 76.07; 57.45; 99998.00	107.63; 79.12 60.21; 99999.98	1
Regime 4	**Mode**	**2.5%**	**97.5%**	**Rhat**
$\phi_{4,0}$; $\phi_{4,1}$	108.52 0.19 (0.00) (0.10)	110.49 0.21 (0.01) (0.14)	106.21 0.16 (0.00) (0.15)	1
$\phi_{4,2}$; $\phi_{4,3}$	0.33 0.06 (0.17) (0.14)	0.36 0.08 (0.19) (0.16)	0.39 0.08 (0.15) (0.125)	1
$\phi_{4,4}$; α_4	-0.77 0.01 (0.12)	-0.79 0.01 (0.11)	-0.755 0.01 (0.13)	1
m_{1k}; m_{12}; $s^2_{\sigma_{1k}}$; v_{1k}	90.96; 80.45; 71.04; 2.91	88.95; 77.07; 68.87; 2.98	92.51; 83.05; 73.41; 3.84	1

Keys: Rhat = Gelman-Rubin Convergence Statistic;
Source: Authors' Computation (2025)

Table 2 above is the FZBMAR process coefficients for the four-highlighted Australia, Czech Republic, Ireland, and Mexico economies' stock prices. In brackets are standard errors of estimates. Literally, the Fisher's z-distribution is always unimodal, such that its mode is usually $x_t = \mu$ but constrained to switching process of autoregressive process. The choice of Fisher's z-distribution as the random noise prior distribution added more advantage of estimating and knowing the mode per each regime at 2.5% and 97.5% quantiles in each of the considered countries as against the classical processes that could not provide such estimates. Additionally, the four-parameter Fisher's z-distribution incorporated simultaneously dispersion, mode-location, skewedness, and kurtosis coefficients per each with associated degree of freedoms.

However, since the intercept distances between components are far apart from regime 1 to regime 4 at modal level, as well at its 2.5% and 97.5% quantiles for the estimated FZBMAR coefficients' of Australia and Denmark stock prices in Table 2 above, it technically connotes that the unconditional kurtoses for Australia and Denmark stock prices' series are large enough in their observational variates. Dissimilarly, since the intercept distances between the components are miniature from regime 1 to regime 4 at modal level, as well at its 2.5% and 97.5% quantiles for Mexico's FZBMAR coefficients, this signifies little or low unconditional kurtosis among components. See equation 42, 43, and 44 below for the mathematically representation of the FZBMAR processes of Australia, Denmark, and Mexico stock prices. Issues concerning identifiable label switching were noticeably shown in the adopted Metropolis-Hastings concurrent algorithm output of the mixing weights of the first and third components (blue and deep green lines) of the Australia FZBMAR mixing weight plot of Figure 1 below. First, third, and fourth components (blue, deep green and red lines) of the label switching of Denmark FZBMAR mixing weight plot of figure 2 experienced deeper issue of identifiable label switching, and the re-labelled output after applying the algorithm. Only the first component (blue line) of the Mexico FZBMAR mixing weight plot of Figure 3 below experienced heavy-loaded label switching. The stochastic coefficients in each regime of each country change in accordance to the representative value (mode) per each regime. The changes in each regime of each country vary based on number of the time series switching peculiar to each country. Hence, the sensitivity of the results obtained per each stochastic model lines in the driving shifting number of modal density that yielded other coefficient accordingly.

$$FBZMAR_{Australia}(4{:}\,4,4,4,4)Tt_{v1k} = F(x_t|\sigma\zeta_{t-1})_{Aus} =$$
$$0.82Tt_{10.45}\left(\frac{\underset{(3.97)}{304.18} - \underset{(0.01)}{0.32x_{t-1}} - \underset{(0.03)}{0.22x_{t-2}} - \underset{(0.01)}{0.39x_{t-3}} - \underset{(0.00)}{0.95x_{t-4}}}{21.24(4.72)}\right)(m_{11} = 105.91; m_{21} = 72.78;\ s^2_{\sigma_{1k}} = 19731.25; v_{1k} = 10.45) +$$
$$0.17Tt_{13.78}\left(\frac{\underset{(0.564)}{144.20} + \underset{(0.020)}{0.41x_{t-1}} - \underset{(0.03)}{0.41x_{t-2}} - \underset{(0.01)}{0.59x_{t-3}} - \underset{(0.00)}{0.001x_{t-4}}}{36.59(17.24)}\right)(m_{12} = 90.92; m_{22} = 85.45; s^2_{\sigma_{2k}} = 88114.37; v_{2k} = 13.78) +$$
$$0.00Tt_{11.56}\left(\frac{\underset{(0.564)}{144.20} + \underset{(0.020)}{0.41x_{t-1}} - \underset{(0.03)}{0.41x_{t-2}} - \underset{(0.01)}{0.59x_{t-3}} - \underset{(0.00)}{0.001x_{t-4}}}{36.59(17.24)}\right)(m_{13} = 104.86; m_{23} = 107.86; s^2_{\sigma_{3k}} = 225.80; v_{3k} = 11.56) +$$
$$0.00Tt_{09.39}\left(\frac{\underset{(11.08)}{93.08} + \underset{(0.05)}{0.13x_{t-1}} - \underset{(0.06)}{0.39x_{t-2}} + \underset{(0.05)}{0.10x_{t-3}} - \underset{(0.03)}{0.95x_{t-4}}}{11.67(5.21)}\right)(m_{14} = 103.37; m_{24} = 75.67; s^2_{\sigma_{4k}} = 177714.62; v_{4k} = 09.39) \quad (42)$$

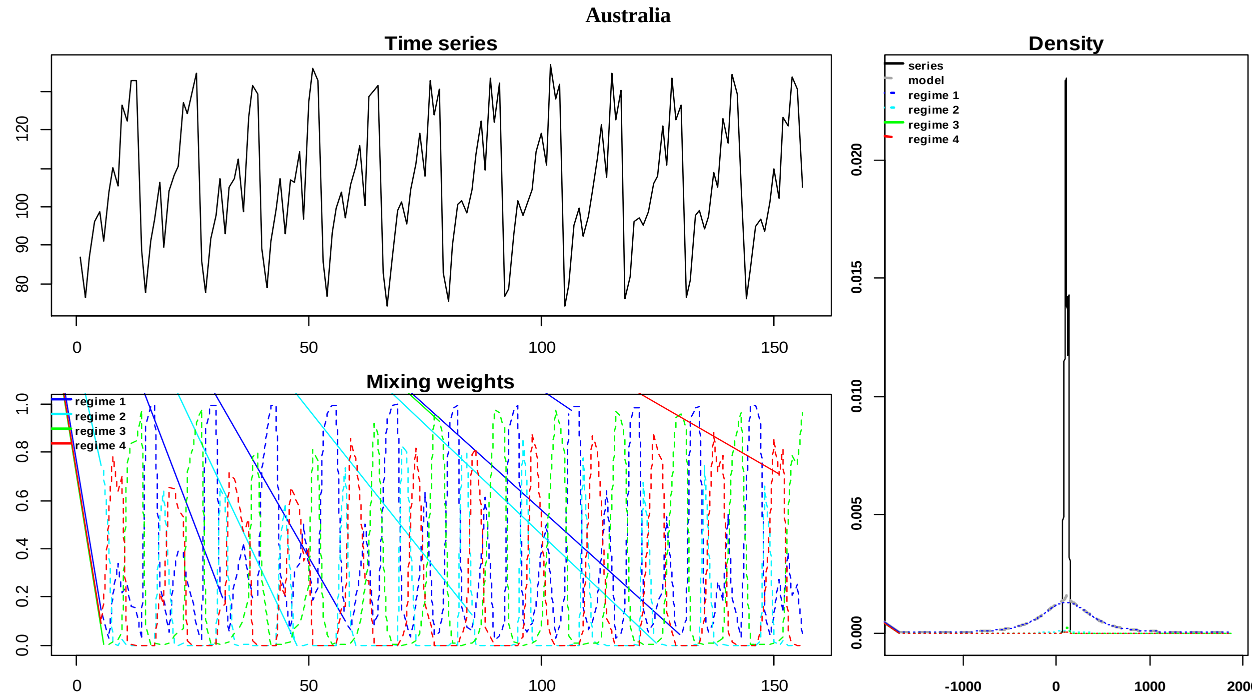
Process Mean: 105.93; Process Variance: 360801.04; First ρ autocors: 0.57, -0.33, -0.96, -0.78.

$$FBZMAR_{DNK}(4:4,4,4,4)Tt_{v1k} = F(x_t|\sigma\zeta_{t-1})_{DNK} =$$
$$1.00Tt_{0.09}\left(\frac{\underset{(9.55)}{313.53} - \underset{(0.05)}{0.27x_{t-1}} - \underset{(0.06)}{0.09x_{t-2}} - \underset{(0.05)}{0.28x_{t-3}} - \underset{(0.00)}{0.99x_{t-4}}}{31.29(03.45)}\right)\left(m_{11} = 118.76; m_{21} = 120.45; s^2_{\sigma_{1k}} = 1347.82; v_{1k} = 2.32\right) +$$
$$0.00Tt_{3.61}\left(\frac{\underset{(0.202)}{219.32} + \underset{(0.042)}{0.09x_{t-1}} + \underset{(0.12)}{0.43x_{t-2}} - \underset{(0.08)}{0.19x_{t-3}} - \underset{(0.08)}{0.79x_{t-4}}}{21.45(10.10)}\right)\left(m_{12} = 150.69; m_{22} = 140.34; s^2_{\sigma_{2k}} = 298.99; v_{2k} = 7.77\right) +$$
$$0.00Tt_{0.42}\left(\frac{\underset{(9.55)}{97.10} + \underset{(0.17)}{0.04x_{t-1}} + \underset{(0.23)}{0.77x_{t-2}} - \underset{(0.21)}{0.30x_{t-3}} - \underset{(0.11)}{0.79x_{t-4}}}{09.34(9.23)}\right)\left(m_{13} = 76.59; m_{23} = 640.59; s^2_{\sigma_{3k}} = 212.18; v_{3k} = 3.20\right) +$$
$$0.00Tt_{0.07}\left(\frac{\underset{(15.97)}{14.06} + \underset{(0.20)}{0.24x_{t-1}} + \underset{(0.25)}{1.42x_{t-2}} - \underset{(0.16)}{0.22x_{t-3}} - \underset{(0.16)}{0.60x_{t-4}}}{11.67(5.21)}\right)\left(m_{14} = 93.02; m_{24} = 65.74; s^2_{\sigma_{4k}} = 1489.98; v_{4k} = 2.11\right) \tag{43}$$

Process Mean: 118.68; Process Variance: 6516609.94; First ρ autocors: 0.58, -0.25, -0.89, -0.87.

$$FBZMAR_{MEX}(4:4,4,4,4)Tt_{v1k} = F(x_t|\sigma\zeta_{t-1})_{MEX} =$$
$$0.71Tt_{22.26}\left(\frac{\underset{(24.48)}{72.01} - \underset{(0.15)}{1.84x_{t-1}} - \underset{(0.11)}{0.27x_{t-2}} + \underset{(0.16)}{1.45x_{t-3}} + \underset{(0.03)}{0.87x_{t-4}}}{17.76(16.08)}\right)\left(m_{1k} = 90.14; m_{2k} = 83.14; s^2_{\sigma_{1k}} = 69.72; v_{1k} = 15.76\right) +$$
$$0.22Tt_{22.20}\left(\frac{\underset{(24.48)}{30.62} + \underset{(0.15)}{0.45x_{t-1}} - \underset{(0.11)}{0.05x_{t-2}} - \underset{(0.16)}{0.19x_{t-3}} + \underset{(0.03)}{0.62x_{t-4}}}{06.80(19.034)}\right)\left(m_{1k} = 186.00; m_{2k} = 120.45; s^2_{\sigma_{2k}} = 5.19; v_{2k} = 2.00\right) +$$
$$0.06Tt_{2.06}\left(\frac{\underset{(09.22)}{202.35} - \underset{(0.04)}{0.08x_{t-1}} - \underset{(0.05)}{0.27x_{t-2}} - \underset{(0.05)}{0.71x_{t-3}} + \underset{(0.11)}{0.09x_{t-4}}}{09.34(9.23)}\right)\left(m_{1k} = 103.04; m_{2k} = 80.67; s^2_{\sigma_{3k}} = 59.67; v_{3k} = 99999.59\right) +$$
$$0.01Tt_{0.07}\left(\frac{\underset{(00.00)}{108.52} + \underset{(0.10)}{0.19x_{t-1}} + \underset{(0.23)}{0.33x_{t-2}} + \underset{(0.14)}{0.06x_{t-3}} - \underset{(0.12)}{0.77x_{t-4}}}{08.43(6.78)}\right)\left(m_{1k} = 90.96; m_{2k} = 80.45; s^2_{\sigma_{4k}} = 71.04; v_{4k} = 2.91\right) \tag{44}$$

Process Mean: 112.29; Process Variance: 26895.14; First ρ autocors: -0.85, 0.88, -0.63, 0.62.



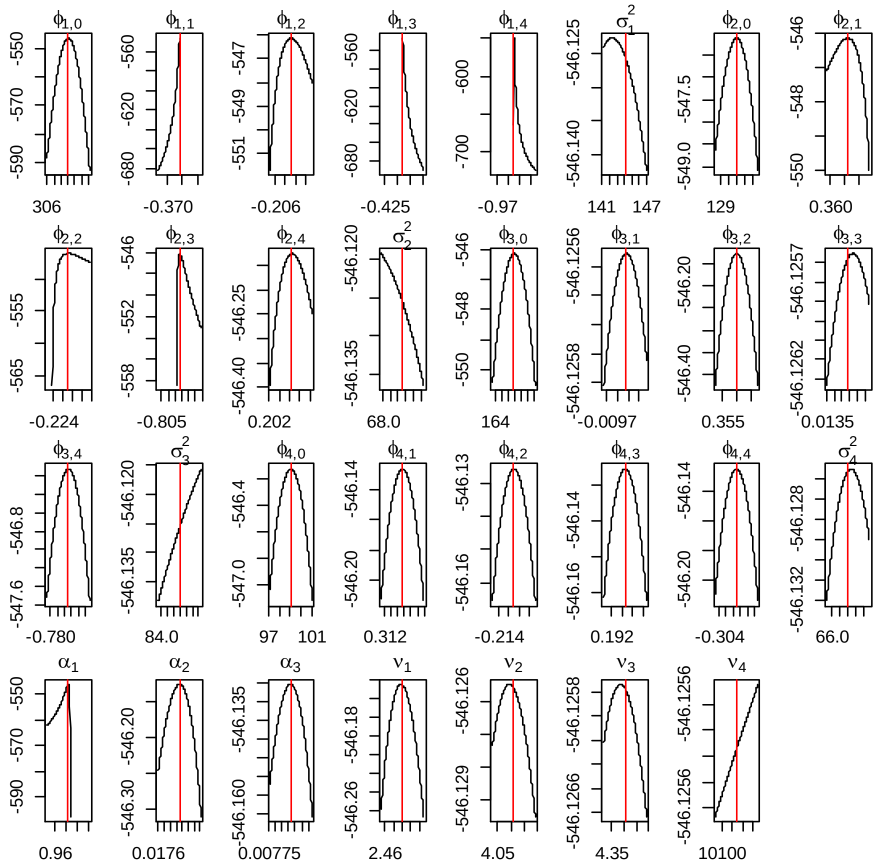
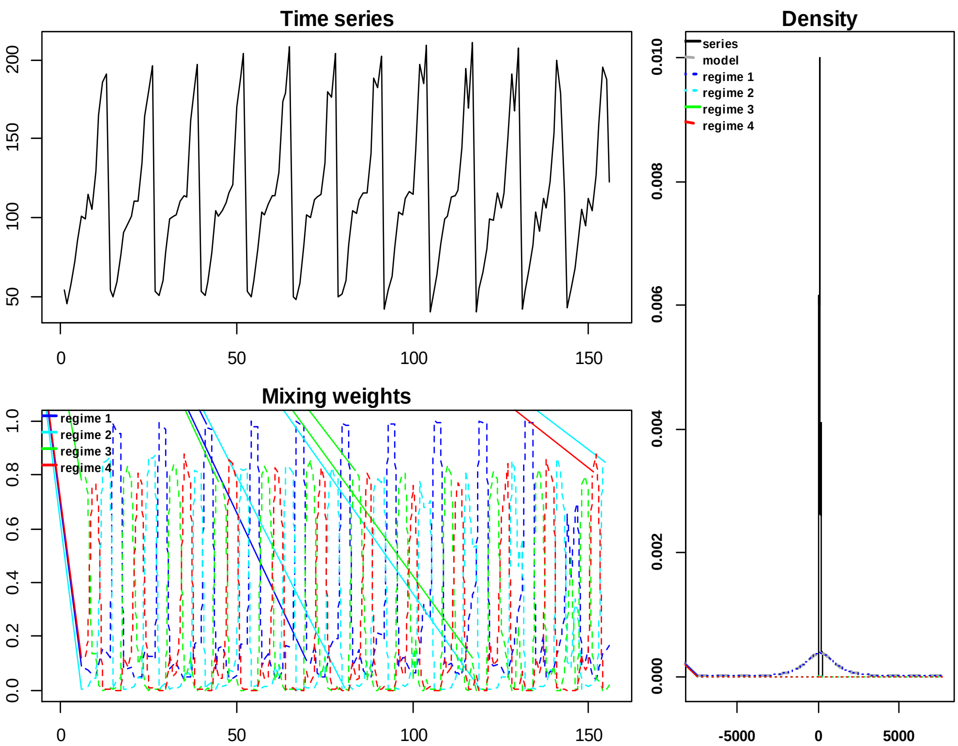


Fig. 1. The Time Series Plot, Marginal Density Plot, Mixing Weights, and Density Coefficients' Curve of the Australia Share Prices.
Source: Authors' Computation (2025)

Denmark



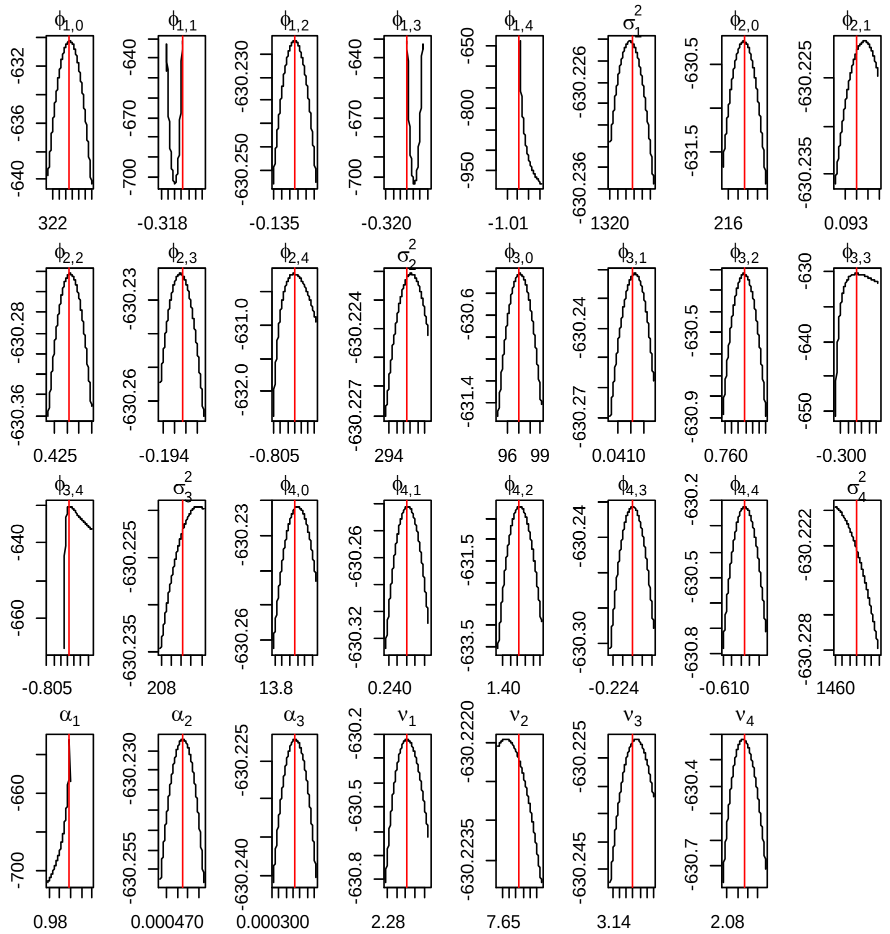
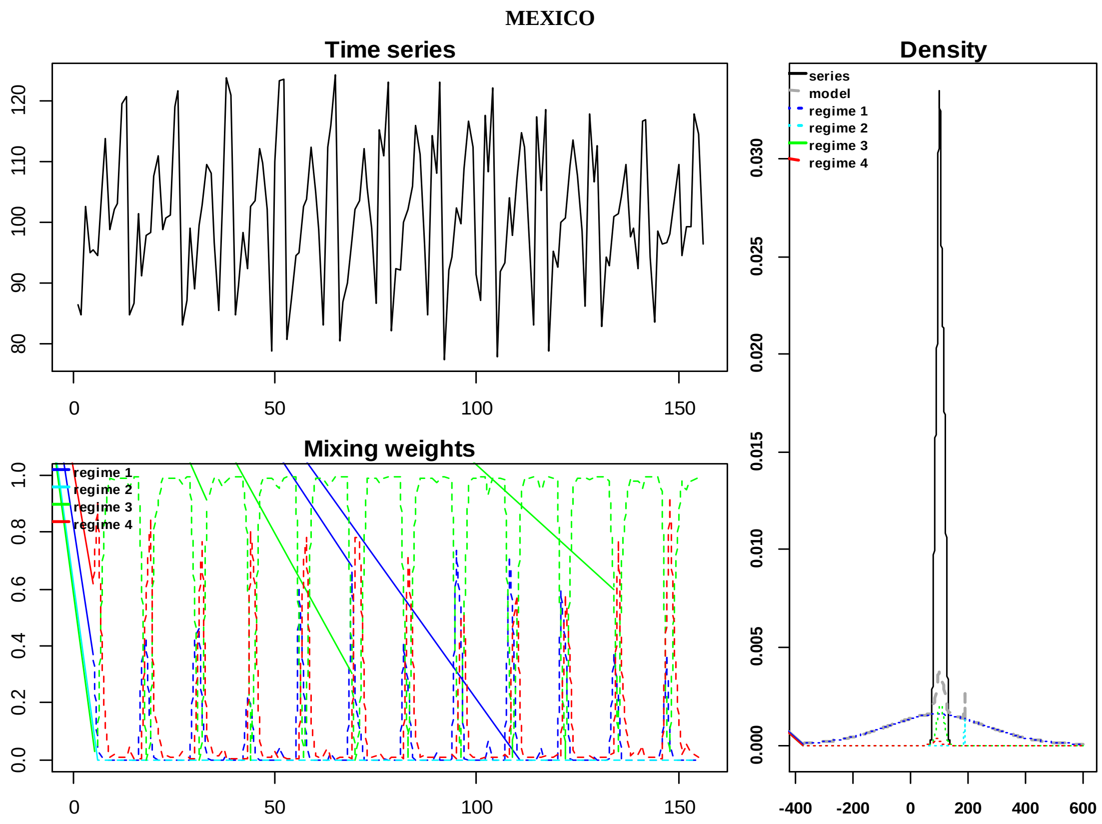


Fig. 2. The Time Series Plot, Marginal Density Plot, Mixing Weights, and Density Coefficients' Curve of the Denmark Share Prices.
Source: Authors' Computation (2025)



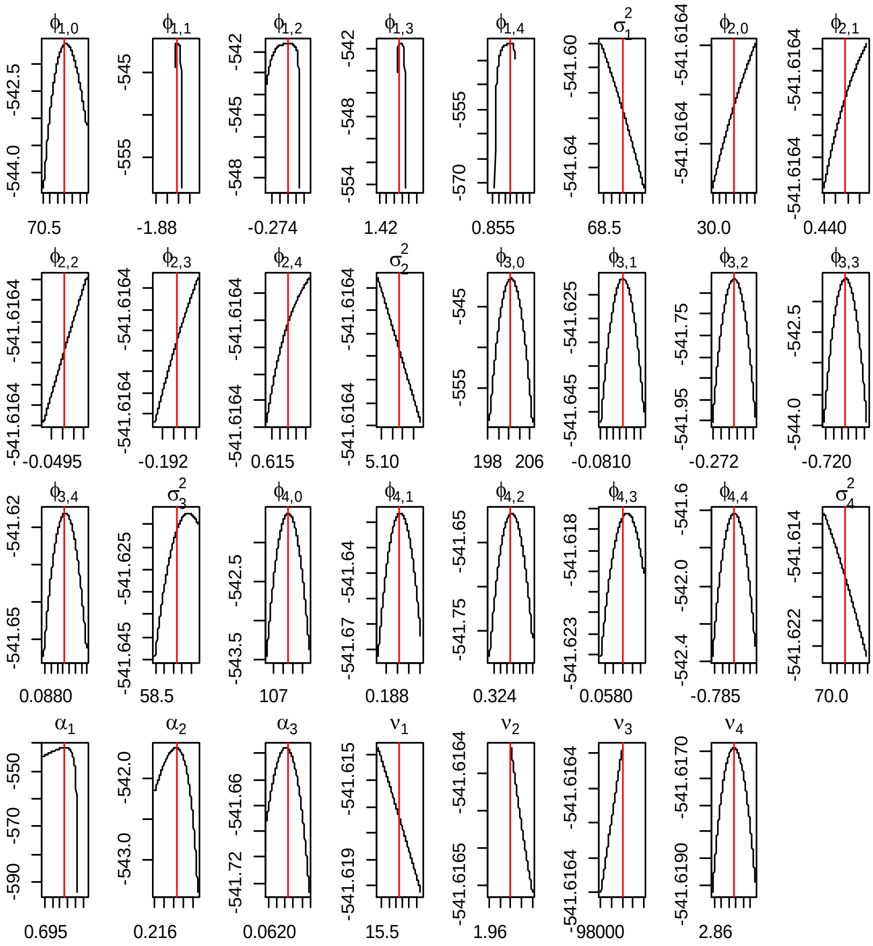


Fig. 3. The Time Series Plot, Marginal Density Plot, Mixing Weights, and Density Coefficients' Curve of the Mexico Share Prices.
Source: Authors' Computation (2025)

4. Conclusions and Future Works

In this work, we expounded a new class of nonlinearity time-variant framework of switching Mixture Autoregressive (MAR) process, and adopted mode as a location parameter for capturing jump phenomena, outburst, change-point like behavior, flat stretches, full range shape changing predictive (multimodal conditional distribution) via a noisy four-parameter Fisher's z distribution. Alternatively, this paper provides a full-detailed MAR process via a noisy four-parameter Fisher's z-distribution innovation in a full-scale Bayesian paradigm. The process was termed as the four-parameter Fisher's **z**-Distribution of Bayesian Mixture Autoregressive (FZBMAR) process, because it constitutes mixture of k -mixture components of Fisher's z regime switching autoregressive processes with the use of mode as a stable location parameter for the adopted random noise. The FZBMAR process offers great flexibility that other mixture processes like StMAR and GMAR could not, via the mode stability as a substitute location and shape parameters (including their associated degrees of freedom) for mean location parameter. Metropolis-Hastings concurrent method of Bayesian parameter estimation algorithm was adopted with options for dealing with identifiable constraints of label switching priors and effective selection of autoregressive optimal orders associated to the FZBMAR process.

Furthermore, the proposed FZBMAR process was compared with the existing StMAR and GMAR processes with the demonstration of monthly average share prices (stock prices) of sixteen (16) swaying European economies and concluded that the FZBMAR process outperformed the existing StMAR and GMAR processes via a minimum Pareto-Smoothed Important Sampling Leave-One-Out Cross-Validation (PSIS-LOO) error process performance in comparison with AIC, HQIC by the latters. Contributively, this study makes it possible to model time switching stochastic processes with prior information about the data via a noisy stable location parameter in order to cater for asymmetric or symmetric anomalies. Implicatively, this study via FZBMAR process might not be able to model volatile or variance process in a Bayesian paradigm without introducing a variance contained switching processes.

Thus, in recommendation, the noisy four-parameter Fisher's z-distribution in its full-scale Bayesian paradigm can be extended to Gaussian Mixture Vector Autoregressive (GMVAR) process. Alternatively, the multivariate skew-*t* distribution and multivariate Fisher's z-distribution can be substituted for the noisy four-parameter Fisher's z-distribution as a driver to redefine the MAR process from a Bayesian perspective. Lastly, other distributions that also have a stable mode in its location parameter are the MSNBurr and the log F-distributions. FZBMAR could be extended in application to cryptocurrency, stock market, exchange rates, inflation and some financial indexes.

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Appendix 1

1. P $\Longrightarrow$ Probability of a Function. For instant, the probability of an event, say A is P(A);
2. *Stability of the FZBMAR Process*

We say FZBMAR process is stable if and only if

$$B = \sum_{k=1}^{g} \alpha_k B_k \otimes B_k$$

$$B_k = \begin{bmatrix} \phi_{k1} & \phi_{k2} & \cdots & \phi_{(\rho-1)k} & \phi_{\rho_k} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Where, $\otimes$ is the Kronecker product or Tensor product. This literally means that if FZBMAR process is stable, the FZBMAR process is said to be a stationary time process. Alternatively, it is referred to as the stationary condition of the process. If this condition is satisfied, then the first and second order stationary is guaranteed. It is to be noted that FZBMAR (1) will be stable if and only if $B_1 \otimes B_1$ is stable, that is, when $g = 1$. This is the same as saying that AR(1) is stationary. For $g > 1$ stationary condition to be ascertained when $B_1, \cdots, B_g$; for $k = 1, \cdots, g$, then B must be stable.

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