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Transmission Dynamics of Malware in Networks Using Caputo Fractional Order Derivative

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Abstract: Fractional calculus plays a crucial role in the representation of various natural and physical phenomena by incorporating the inherent non-locality and long-term memory effect of fractional operators. These models offer a more precise and systematic depiction of the underlying phenomena. The focus of this research paper is on the utilization of fractional calculus in the context of the epidemic model. Specifically, the model considers a fractional order ρ, where $0<\rho\leq1$, and employs the Caputo fractional order derivative to describe the transmission of malware in both wireless and wired networks. The basic reproduction number, along with the fractional order ρ, is identified as the threshold parameter in this model. The stability of the system is analysed at different stages of the reproduction number, considering both local and global asymptotic stability. Additionally, sensitivity analysis is conducted on the model parameters to determine the direction of change in the reproduction number. This analysis aids in understanding whether the reproduction number will increase or decrease under different scenarios. To obtain numerical results, the Fractional Forward Euler Method is utilized for simulation purposes. This method enables the computation of the model's dynamics and offers insights into the behaviour of the system. While the Caputo fractional order derivative offers a promising framework for modelling epidemic dynamics, they often entail significant computational overhead, limiting the scalability and practical utility of fractional calculus-based epidemic models, especially in real-time simulation and forecasting scenarios.

Index Terms: Fractional calculus, Caputo-fractional order derivative, Basic reproduction number, Sensitivity analysis, Fractional forward Euler method.

1. Introduction

The field of calculus encompasses various branches, one of which is fractional calculus, aimed at solving problems related to derivatives and integrals with fractional order. Fractional calculus, despite its misleading name, presents an intriguing and initially perplexing concept [1]. It has been established those chaotic systems with fractional time exhibit more intricate dynamics and degrees of freedom, primarily due to their dependence on the fractional order [2]. Consequently, fractional calculus has been employed as a valuable tool in representing physical processes across domains such as physics, chemistry, and engineering [3]. In contrast to the conventional derivative, which operates locally, the memory effect serves as a defining characteristic of the fractional order derivative. Specifically, the subsequent state of the fractional derivative for a given function not only relies on its current state but also on its preceding states, significantly influencing its behaviour [4].

1.1 Relevance of Epidemic model on cyber-attacks in Wireless networks:

In recent times, the adoption of the epidemic model in various domains, including epidemiology, sociology, and computer science, has garnered significant attention. As cybersecurity threats continue to evolve and proliferate, exploring innovative approaches becomes imperative. The application of the epidemic model in cybersecurity measures presents a promising avenue worth considering. The epidemic model, inspired by the spread of infectious diseases, can be adapted to understand, and mitigate cyber threats. Its fundamental principles of propagation, containment, and immunization resonate well with cybersecurity challenges. By modeling cyber threats as "digital infections," we can better comprehend their spread dynamics and devise effective countermeasures. One key advantage of leveraging the epidemic model in cybersecurity is its ability to capture the dynamics of threat propagation. Similar to how diseases spread through vulnerable populations, cyber threats propagate through interconnected systems and networks. By simulating these propagation patterns, cybersecurity professionals can anticipate and intercept threats more effectively. Furthermore, the concept of immunization in the epidemic model aligns with preventive measures in cybersecurity. Just as vaccinations confer immunity against diseases, proactive security measures such as patching vulnerabilities and implementing access controls can immunize systems against cyber threats. By strategically deploying these measures, organizations can bolster their resilience to attacks. However, implementing the epidemic model in practical cybersecurity measures comes with its challenges. One such challenge is accurately modeling the complex interactions within digital ecosystems. Cyber threats often exhibit nonlinear behaviors influenced by factors such as network topology, user behavior, and attacker tactics. Capturing these nuances requires sophisticated modeling techniques and real-time data analytics. Despite these challenges, the feasibility of integrating the epidemic model into cybersecurity strategies is promising. Advances in machine learning, network analysis, and threat intelligence provide the necessary tools and frameworks to operationalize this approach. By embracing interdisciplinary collaboration and leveraging cutting-edge technologies, we can harness the power of the epidemic model to enhance cyber resilience and safeguard digital assets. The feasibility of implementing the epidemic model in practical cybersecurity measures holds immense potential for mitigating emerging threats and fortifying defenses. By combining theoretical insights with empirical data and technological innovations, we can pave the way for a more robust and adaptive cybersecurity landscape. However, recent advancements have illuminated epidemic models potential for broader applications, particularly in the analysis of network traffic patterns. By adopting a novel perspective, we can harness the intrinsic principles of the epidemic model to glean valuable insights into network behavior and dynamics. Network traffic analysis plays a pivotal role in cybersecurity, enabling the detection of anomalies, identification of malicious activities, and optimization of network performance. By conceptualizing data flows as "information contagions," we can leverage the model's principles to elucidate the propagation, diffusion, and clustering of network activities. This holistic perspective transcends traditional packet-level analysis, providing a deeper understanding of emergent phenomena within the network. One compelling application of the epidemic model in network traffic analysis is anomaly detection. Just as infectious diseases manifest as deviations from baseline health patterns, malicious activities and network anomalies manifest as deviations from normal traffic behavior. By modeling normal traffic patterns as a "healthy state" and deviations as "anomalies," we can effectively detect and mitigate potential threats in real-time.

Mathematical epidemiology serves as the foundation for examining the propagation of malware in various types of device networks, including wireless sensor networks (WSNs) and wired networks. Thorough surveys and research in this field are imperative as they contribute to the development of security solutions for WSNs and wired networks, which find extensive applications in areas such as agriculture, healthcare, and smart cities. WSNs typically consist of several hundred to thousands of sensors, depending on the specific application and environment. The equipment within a sensor node typically includes a radio transceiver, an antenna, an interface electronic circuit, a microcontroller, and a power source, commonly in the form of a battery.Wireless Sensor Networks (WSNs) acquires data to construct information and communication systems that will greatly optimize the efficiency of infrastructure systems. The research on WSNs has gained significant attention due to their widespread utilization. These networks are considered a revolutionary approach to gather information and establish communication systems, with the potential to greatly improve the dependability and effectiveness of infrastructure systems. However, WSNs are vulnerable to hacking and easily targeted by attackers. Additionally, the sensor nodes within these networks have limited resources, making them susceptible to defense breaches and attractive targets for worms. The injection of malware into a few nodes poses a significant threat. In recent times, there has been an increase in the emergence of harmful codes that specifically target wireless devices. These codes can propagate effortlessly from one device to another through wireless communication technologies such as Wi-Fi and Bluetooth.

1.2 The efficacy of cyber-attack models incorporating fractional derivatives

Throughout the years, the design and development of networks and systems have faced consistent testing from various security disasters. Failure to incorporate network security into the network design renders existing networks vulnerable to cyber-attacks, posing risks to individuals, organizations, the state, and society. The emergence of cyber-attacks in various domains such as smart cities, automotive bus systems, internet of medical things, e-commerce, and space technology has prompted computer scientists and mathematicians to collaborate in order to create a robust defense framework. The objective is to minimize the occurrence of these attacks, which have a history of causing significant damage and spreading rapidly. In this regard, fractional derivatives epidemic models will be instrumental in comprehending the transmission patterns of malicious codes. Additionally, simulating parameters will aid in the development of a secure defense network. The study of fractional differential equations holds significant implications for our research due to several reasons. The adoption of fractional derivative epidemic models provides a notable advantage in capturing the memory and hereditary characteristics of systems, unlike integer-order models which either overlook or face challenges in incorporating these effects. The nature of these evolutionary equations enables accurate modeling of various universal phenomena. In contrast to local operators such as integer-order differential operators, fractional-order differential operators are non-local. This implies that they consider not only the present state but also the history of previous states when determining the future state.

The fractional SIR epidemic model equations are derived by replacing the first-order derivatives in the conventional SIR epidemic equations with fractional derivatives of order ρ ($0 < \rho \leq 1$) within the context of mathematical modeling. Consequently, there has been a growing interest in studying the dynamic behaviours of fractional-order differential systems in recent years. The literature extensively explores the existence of solutions for initial value problems related to fractional order differential equations, along with the references cited within [5].

1.3 Fractional epidemic model and cyber security

The fractional epidemic model, traditionally applied in epidemiology to understand the spread of diseases, has recently found a promising application in the cybersecurity domain. In essence, the fractional epidemic model offers a unique lens through which we can comprehend the propagation of cyber threats and vulnerabilities within complex networks. By adopting fractional calculus principles, this model accounts for the non-local and memory-dependent characteristics inherent in cyber-attacks, providing a more nuanced understanding of their dynamics. Traditional epidemiological models often fall short in capturing the long-range interactions and persistent effects characteristic of cyber threats. By incorporating fractional calculus, our model can better predict the evolution of cyber-attacks, enabling proactive defense strategies and risk mitigation measures. Cyber-attacks frequently exploit the interconnectedness and vulnerabilities present in complex networks. The fractional epidemic model offers a refined framework for assessing network resilience by accurately assessing the impact of attacks across various network topologies, including scale-free and small-world networks. Understanding the underlying mechanisms driving the spread of cyber threats is crucial for devising effective mitigation strategies. The fractional epidemic model provides valuable insights into the efficacy of different intervention approaches, ranging from patch management and network segmentation to behavioral analytics and incident response protocols.The integration of the fractional epidemic model into cybersecurity research not only expands our analytical toolkit but also deepens our understanding of the intricate dynamics underlying cyber-attacks. By elucidating the propagation mechanisms of threats within digital ecosystems, our study contributes to the advancement of cybersecurity practices and fortifies defenses against emerging cyber threats.

Kermack-Mckendrick's classical epidemic SIR model, as described by [6], has been extensively used to analyze the spread of contagious illnesses in closed populations over time. This model, rooted in mathematical principles, has traditionally been employed in the study of infectious diseases and their control. Interestingly, the propagation of computer worms and viruses bears similarities to epidemic diseases. Consequently, epidemiological models, such as the SIR model, can be utilized to investigate the behaviour of malicious objects within a network [7]. The introduction of Riemann-Liouville type fractional differentiation has significantly contributed to the advancement of fractional derivatives and integrals theory. This development has found applications in various areas of pure mathematics, including the solution of differential equations with integer orders, the exploration of new function categories, and the computation of series summation [1]. In the context of epidemic modeling, stability analysis and Hopf bifurcation have been examined in the fractional order SEIRV model by Mahata et al. [8]. Additionally, Swati and Neelam [9] have endeavoured to study a non-linear, non-integer SIR model specifically tailored for COVID-19. Notably, Caputo's approach offers a notable advantage by preserving the same initial conditions for fractional differential equations as those for integer-order differential equations. This characteristic ensures that the limit values of integer-order derivatives of unknown functions at a lower terminal $t=a$ are included [1]. Furthermore, Mahata et al. [10] have investigated the dynamics of the Caputo fractional order SEIRV model, incorporating optimal control and stability analysis.

This paper explores the formulation of fractional differential equations (FDEs) that govern the dynamics of the susceptible, infectious, and recovered (SIR) compartments in the fractional epidemic model. These FDEs encapsulate the fractional-order derivatives, which capture the memory-dependent behavior inherent in epidemic spreading processes. Transitioning from the traditional SIR model, we extend our analysis to encompass the susceptible-infectious-recovered-susceptible (SIRS) dynamics, which accounts for the temporary immunity acquired by recovered individuals before becoming susceptible again. The incorporation of fractional calculus into the SIRS model introduces additional complexities, necessitating novel analytical techniques for solution. With the mathematical foundations laid, we delve into the numerical solutions and simulations of the fractional SIRS model. The roadmap culminates in a detailed analysis of the fractional SIRS dynamics, unravelling the intricate interplay between fractional-order derivatives, network structure, and epidemiological parameters. Through sensitivity analysis, and stability analysis, we uncover the underlying mechanisms shaping the propagation and control of infectious diseases in complex networks.

2. Preliminaries on Fractional Order Derivatives

2.1. Definition

[1] "The Caputo's fractional derivative of order ρ can be defined as

$$ {}_{0}^{c}D_{t}^{\rho} = \frac{1}{\Gamma(\rho-n)} \int_{a}^{t} \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\rho+1-n}}, (n-1) < \rho < n. $$

Under natural conditions on the function f(t), for ρ →n the Caputo derivative becomes a conventional n-th derivative of the function f(t)".

2.2. Beddington- De Angelis functional response

[11] "They proposed a predator-dependent functional response in the form

$$p(X,Y) = \frac{aX}{bY + ahX + 1} = \frac{\omega X}{\alpha + \beta Y + \gamma X}$$

2.3. Holling type II trophic function [11]

$$p(X) = \frac{X}{X + D}$$

The attack rate increases at a decreasing rate with prey density until it becomes satiated.

2.4. Generalized mean value theorem [12]*:*

Suppose that $f(x) \epsilon\, C[a,b]$ and $D_a^\rho f(x) \in C[a,b]$, for $0 < \rho \leq 1$, then we have,

$$f(x) = f(a) + \frac{1}{\Gamma(\rho)} \left(D_a^\rho f\right)(\xi).(x-a)^\rho$$

With $a \leq \xi \leq x, \forall x \in (a,b]$.

2.5. Lemma 1[8] *:*

Consider the following fractional order system,

$$^{C}D_t^\rho\big(Y(t)\big) = \phi(Y), Y_{t_0} = (y_{t_0}^1, y_{t_0}^2, \dots\dots\dots\dots\dots, y_{t_0}^n),\ y_{t_0}^j, \text{ j=1,2,3,.........,n}$$

with $0 < \rho < 1, Y(t) = \big(y^1(t), y^2(t), \dots\dots\dots\dots, y^n(t)\big)$ *and* $\phi(Y): [t_0, \infty) \to \mathbb{R}^{n\times n}$ .For $\phi(Y) = 0$, we get all the equilibrium points are locally asymptotically stable iff each eigenvalue λ_j of the jacobian matrix $J(Y) = \frac{\partial(\phi_1, \phi_2, \dots\dots\dots, \phi_n)}{\partial(y^1, y^2, \dots\dots\dots y^n)}$ calculated at the equilibrium points satisfies $|\arg(\lambda_j)| > \frac{\rho\pi}{2}$.

2.6. Lemma 2[8]*:*

Let $h(t) \in \mathbb{R}^+$ be a differential function. Then, for any t > 0,

$$^{C}D_t^\rho\left[h(t) - h^* - h^* \ln\frac{h(t)}{h^*}\right] \leq \left(1 - \frac{h^*}{h(t)}\right) {}^{C}D_t^\rho\big(h(t)\big),$$
$$h^* \in \mathbb{R}^+, \forall \rho \in (0,1).$$

3. Formulation of Fractional Order Epidemic Model

Assuming that recovery is not permanent, we are proposing an epidemic model, SIRS(Susceptible-Infectious-Recovered-Susceptible) of fractional order ρ, where $0 < \rho \leq 1$, using the Caputo fractional order derivative of order ρ for studying the transmission dynamics of malware in networks. In epidemiological compartmental models, the fractional order ρ is related to the memory or history of the most biological systems[9]. The total population N(t) is categorized into three classes namely susceptible (S(t)), infectious (I(t)) and recovered (R(t)).A Caputo-fractional order SIRS model is represented here by the following non-linear fractional differential equations:

$$\begin{gathered} {}_0^C D_t^\rho S(t) = \mu - \frac{\beta S(t) I(t)}{1 + \beta_1 S(t) + \beta_2 I(t)} - \lambda S(t) + \epsilon R(t) \\ {}_0^C D_t^\rho I(t) = \frac{\beta S(t) I(t)}{1 + \beta_1 S(t) + \beta_2 I(t)} - (\lambda + \alpha + \gamma) I(t) - \frac{\sigma I(t)}{1 + \delta I(t)} \\ {}_0^C D_t^\rho R(t) = \frac{\sigma I(t)}{1 + \delta I(t)} + \gamma I(t) - (\lambda + \epsilon) R(t) \end{gathered} \tag{1}$$

With initial conditions $S(0) = S_0 > 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0.\ {}_0^C D_t^\rho$ is the Caputo fractional operator of order $0 < \rho \leq 1$.

Parameters of the model:

- μ =*Constant rate of recruitment rate of susceptibles*
- β =*Coefficient of transmission between susceptible S(t) and infectious class I(t)*

- β_1 =*Measure of inhibition (Measures taken by susceptibles like running anti-malicious software, use a VPN or encrypt your network through the control panel settings)*
- β_2 =*Measure of inhibition (Measures taken by infectious class like maintaining firewall settings, disable remote access)*
- λ =*Natural mortality rate*
- ϵ =*Rate at which recovered population goes to susceptible class.*
- α =*Infection mortality rate in infectious class*
- γ =*Rate of recovery*
- σ =*Treatment rate of disease (attack)*
- δ =*Rate of limitation in treatment availability*

Total population is given by

$$N(t) = S(t) + I(t) + R(t) \tag{2}$$

Then,

$${}_0^C D_t^\rho N(t) = \mu - \lambda N(t) - \alpha I(t)$$

In absence of disease, I(t)=0, we have,

$${}_0^C D_t^\rho N(t) = \mu - \lambda N(t) \tag{3}$$

Which implies $N(t) \to \frac{\mu}{\lambda}\ as\ t \to \infty$.

We study the dynamics of the fractional order SIRS model in the biologically feasible set

$$\Omega = \left\{(S, I, R) \in \mathbb{R}^3 \middle| N(t) \leq \frac{\mu}{\lambda}\right\}. \tag{4}$$

Considering (3) as Initial Value Problem(IVP) with initial condition $N(t)|_{t=0}$ =N(0). Applying Laplace transform[1] to (3), we get,

$$\text{L } [{}_0^C D_t^\rho N(t)] = L[\mu - \lambda N(t)]$$

$$or, \quad s^\rho L[N(t)] - s^{\rho-1}N(0) = \frac{\mu}{s} - \lambda L[N(t)]$$

$$or, \quad L[N(t)] = \frac{s^{\rho-1}}{s^\rho + \lambda}N(0) + \frac{\mu s^{-1}}{s^\rho + \lambda}$$

Applying inverse Laplace transform[1] to the above equation, we get,

$$N(t) = N(0).E_{\rho,1}(-\lambda t^\rho) + \mu t^\rho E_{\rho,\rho+1}(-\lambda t^\rho) \tag{5}$$

According to the properties of Mittag-Leffler function,

$$E_{\rho,\alpha}(z) = z.E_{\rho,\rho+\alpha}(z) + \frac{1}{\Gamma(\alpha)}$$

We get from (4),

$$N(t) = \left(N(0) - \frac{\mu}{\lambda}\right)E_{\rho,1}(-\lambda t^\rho) + \frac{\mu}{\lambda}$$

Thus,

$$\lim_{t\to\infty} Sup\ N(t) \leq \frac{\mu}{\lambda}$$

Hence, the model is bounded above and $S(t), I(t), R(t)$ are all non-negative and the model is non-negative invariant.

4. Reproduction Number, Equilibrium Points and Stability

4.1. Reproduction number

Reproduction number is a non-dimensional variable that determines the nature of epidemiological spread. The basic reproductive number holds significant importance for epidemiologists as it provides a quantification of the anticipated number of secondary infections generated by an individual during their whole infectious period when introduced into a group of susceptible individuals [13].Because it is difficult to define mathematically what constitutes a "typical" infectious individual in populations with significant levels of variation, this non-dimensional quantity cannot typically be estimated explicitly[14].

The Basic Reproduction Number is given by R_0, which is obtained for (1) as

$$R_0 = \frac{\beta\mu}{(\lambda+\beta_1\mu)(\lambda+\alpha+\gamma+\sigma)}$$

4.2. Disease-free equilibrium and the endemic equilibrium

4.2.1. Disease-free equilibrium:

The disease free equilibrium(DFE) is found by equating all equations of system (1) to zero and $S(0) = S_0,\ \ I(0) = 0,\ \ R(0) = 0$, we get,

$$\text{DFE:}\quad (S_0, 0, 0) = \left(\frac{\mu}{\lambda}, 0, 0\right)$$

Showing no infection in the environment, nodes are susceptibles only.

4.2.2. Endemic equilibrium:

Endemic equilibrium (EE) is given by equating all equations of system (1) to zero and

$$S(t)=S^*,\quad I(t)=I^*,\quad R(t)=R^*, \text{ where } S^*, I^*, R^* \in \mathbb{R}^+.$$

$$\text{EE: } S^* = \frac{\beta[\sigma+(\lambda+\alpha+\gamma)(1+\delta I^*)](1+\beta_2 I^*)}{\beta(1+\delta I^*)-\beta_1[\sigma+(\lambda+\alpha+\gamma)(1+\delta I^*)]}$$

$$I^* = \frac{\mu-\lambda N^*}{\alpha};\ N^*=S^*+I^*+R^*$$

$$R^* = \frac{I^*}{\lambda+\alpha}\left[\gamma+\frac{\sigma}{1+\delta I^*}\right]$$

4.2.3. Theorem:

The disease free equilibrium (DFE) is locally asymptotically stable if R_0<1,otherwise not stable.

Proof: For the disease free equilibrium(DFE), $\left(\frac{\mu}{\lambda}, 0, 0\right)$, the Jacobian matrix for the system (1) is given as-

$$J_{DFE} = \begin{pmatrix} -\lambda & \dfrac{-\beta\mu}{\lambda+\beta_1\mu} & \varepsilon \\ 0 & (\lambda+\alpha+\gamma+\sigma)(R_0-1) & 0 \\ 0 & (\sigma+\gamma) & -(\lambda+\varepsilon) \end{pmatrix}$$

The eigenvalues are here: $q_1 = -\lambda;\ q_2 = (\lambda+\alpha+\gamma+\sigma)(R_0-1);\ and\ q_3 = -(\lambda+\varepsilon)$.

When R_0<1, then all the above eigenvalues are negative. Hence, by Fractional Routh-Hurwitz criteria [15], all the roots follow-

$$|\arg(q_i)| > \frac{\rho\pi}{2};\ i = 1,2,3\ \ and\ \ 0<\rho<1.$$

Hence, the disease free equilibrium is locally asymptotically stable when R_0<1.

4.2.4. Theorem:

If $R_0 > 1$, then the endemic equilibrium is locally asymptotically stable.

Proof: For the endemic equilibrium, the Jacobian for the system (1) is given as

$$J=\begin{pmatrix} -\lambda-\dfrac{\beta I^*+\beta\beta_2 {I^*}^2}{(1+\beta_1 S^*+\beta_2 I^*)^2} & -\dfrac{\beta S^*+\beta\beta_1 {S^*}^2}{(1+\beta_1 S^*+\beta_2 I^*)^2} & \varepsilon \\ \dfrac{\beta I^*+\beta\beta_2 {I^*}^2}{(1+\beta_1 S^*+\beta_2 I^*)^2} & \dfrac{\beta S^*+\beta\beta_1 {S^*}^2}{(1+\beta_1 S^*+\beta_2 I^*)^2}-(\lambda+\alpha+\gamma)-\dfrac{\sigma}{(1+\delta I^*)^2} & 0 \\ 0 & \dfrac{\sigma}{(1+\delta I^*)^2}+\gamma & -(\lambda+\varepsilon) \end{pmatrix}$$

Let $A=\frac{\beta I^*+\beta\beta_2 {I^*}^2}{(1+\beta_1 S^*+\beta_2 I^*)^2}$;

$B=\frac{\beta S^*+\beta\beta_1 {S^*}^2}{(1+\beta_1 S^*+\beta_2 I^*)^2}$;

$C=\frac{\sigma}{(1+\delta I^*)^2}$, then

$$J=\begin{pmatrix} -\lambda-A & -B & \varepsilon \\ A & B-C-(\lambda+\alpha+\gamma) & 0 \\ 0 & C+\gamma & -(\lambda+\varepsilon) \end{pmatrix}$$

Which gives rise to characteristic equation as

$$x^3+A_1x^2+A_2x+A_3=0 \tag{6}$$

Where,

$$A_1=(A-B+C)+3\lambda+\alpha+\gamma$$
$$A_2=(2\lambda+\varepsilon)(A-B+C)+3\lambda^2+2\alpha\lambda+2\gamma\lambda+2\lambda\varepsilon+(A+\varepsilon)(\alpha+\gamma)+AC$$
$$A_3=(\lambda+\varepsilon)[\lambda(A-B+C)+A(\alpha+\gamma)+\lambda(\lambda+\alpha+\gamma)](7)$$

Let us denote its discriminant

$$\Delta=18A_1A_2A_3+(A_1A_2)^2-4{A_2}^3-4{A_1}^3A_3-27{A_3}^2 \tag{7}$$

The following lemma will complete our proof of the theorem.

4.2.5. Lemma 3[2]:

Assume that $R_0>1$ and one of the following conditions are satisfied

(i). $\Delta>0, A_1>0, A_2>0, and\ A_1A_2-A_3>0.$

(ii). $\Delta<0, \rho\in\left(0,\frac{2}{3}\right], A_1\geq 0, A_2\geq 0, and\ A_3>0.$

Then, endemic equilibrium of the fractional order model with Caputo derivative is locally asymptotically stable.

Proof: The detailed proof of this lemma is similar to that of [15].

4.2.6. Global stability of endemic equilibrium:

Let us construct a Lyapunov function as

$$L(t)=I(t)-I^*-I^*\ln\frac{I(t)}{I^*}; \tag{8}$$

Using fractional derivative in (7) and using lemma (2), we get

$${}_0^cD_t^\rho L(t)\leq\frac{I(t)-I^*}{I(t)}\,{}_0^cD_t^\rho I(t) \qquad ; \quad \rho\in(0,1)$$

Using endemic conditions in above equation, we have

$$or, \qquad {}_0^cD_t^\rho L(t)\leq\frac{I(t)-I^*}{I(t)}\left[\beta\left(\frac{S(t)I(t)}{1+\beta_1 S(t)+\beta_2 I(t)}-\frac{S^*I^*}{1+\beta_1 S^*+\beta_2 I^*}\right)-(\lambda+\alpha+\gamma)(I(t)-I^*)-\frac{\sigma((I(t)-I^*)}{(1+\delta I(t))(1+\delta I^*)}\right]$$

$$or, \quad {}_0^cD_t^\rho L(t) \leq -\frac{(I(t)-I^*)^2}{I(t)}\left[(\lambda+\alpha+\gamma)+\frac{\sigma}{(1+\delta I(t))(1+\delta I^*)} - \frac{\beta}{(I(t)-I^*)}\left(\frac{S(t)I(t)}{1+\beta_1 S(t)+\beta_2 I(t)}-\frac{S^*I^*}{1+\beta_1 S^*+\beta_2 I^*}\right)\right]$$

Hence, ${}_0^cD_t^\rho L(t) \leq 0, for\ R_0 > 1$. Therefore, $E^*(S^*, I^*, R^*)$ is globally asymptotically stable, according to LaSalle's invariance principle[16-19].

5. Sensitivity Analysis of Model Parameter

Sensitivity analysis is a way to determine the importance of each parameter for the worms transmission. The sensitivity index of R_0 with respect to x is defined as

$$\Gamma_x^{R_0} = \frac{\partial R_0}{\partial x}\frac{x}{R_0}$$

The sign of each index makes it possible to know whether the parameter increases (positive sign) or decreases (negative sign) the value of R_0 [2]. The parameters for this model are:

$\beta, \mu, \beta_1, \beta_2, \lambda, \varepsilon, \alpha, \gamma, \sigma, \delta$. We have,
$\Gamma_\beta^{R_0} = 1;$
$\Gamma_\mu^{R_0} = -\frac{\beta_1}{(\lambda+\beta_1\mu)^2};$
$\Gamma_\lambda^{R_0} = -\frac{\lambda(2\lambda+\alpha+\gamma+\sigma+\mu\beta_1)}{(\lambda+\mu\beta_1)(\lambda+\alpha+\gamma+\sigma)};$
$\Gamma_{\beta_1}^{R_0} = -\frac{\beta_1\mu}{(\lambda+\beta_1\mu)};$
$\Gamma_{\beta_2}^{R_0} = 0;$
$\Gamma_\varepsilon^{R_0} = 0;$
$\Gamma_\alpha^{R_0} = -\frac{\alpha}{(\lambda+\alpha+\gamma+\sigma)};$
$\Gamma_\gamma^{R_0} = -\frac{\gamma}{(\lambda+\alpha+\gamma+\sigma)};$
$\Gamma_\sigma^{R_0} = -\frac{\sigma}{(\lambda+\alpha+\gamma+\sigma)};$
$\Gamma_\delta^{R_0} = 0;$

Note that R_0 does not depend upon $\beta_2, \varepsilon, \delta$, so,

$\Gamma_{\beta_2}^{R_0} = 0$, $\Gamma_\varepsilon^{R_0} = 0$, $\Gamma_\delta^{R_0} = 0$. We observe here that $\Gamma_\mu^{R_0}, \Gamma_{\beta_1}^{R_0}, \Gamma_\alpha^{R_0}, \Gamma_\gamma^{R_0}, \Gamma_\sigma^{R_0} < 0$, which means that an increment in $\mu, \beta_1, \alpha, \gamma, \sigma$ will cause R_0 to decrease. Also, $\Gamma_\beta^{R_0} > 0$, cause R_0 to increase.

6. Fractional forward Euler Method

Consider the Caputo type differential equation,

$$\begin{gathered} {}_0^cD_t^\rho\ u(t) = f(t, u(t)), \quad m-1 < \rho < m \in \mathbb{Z}^+ \\ u^{(j)}(0) = u_0^j, \quad j = 1,2,3, \dots\dots\dots, (m-1). \end{gathered} \tag{9}$$

If we apply ${}_0^cD_t^{-\rho}$ on the both sides of (8), we can obtain the following equivalent Volterra integral equation-

$$u(t) = \sum_{j=0}^{m-1}\frac{t^j}{j!}u_0^{(j)} + \frac{1}{\Gamma(\rho)}.\int_0^t (t-s)^{(\rho-1)}.f(s, u(s))ds$$

$$or, \quad u(t) = \sum_{j=0}^{m-1}\frac{t^j}{j!}u_0^{(j)} + {}_0^cD_t^{-\rho}f(t, u(t)) \tag{10}$$

In the above equation (9), the approximation of $\left[{}_0^cD_t^{-\rho}f(t, u(t))\right]_{t=t_{n+1}}$ is done by Left fractional rectangular formula [20]

$$u_{n+1} = \sum_{j=0}^{m-1} \frac{t_{n+1}^{j}}{j!} u_0^{(j)} + (\Delta t)^{\rho} \sum_{j=0}^{n} b_{j,n+1}\ f(t_j, u_j),$$

Where,

$$b_{j,n+1} = \frac{1}{\Gamma(\rho+1)}[(n-j+1)^{\rho} - (n-j)^{\rho}].$$

We have by fractional forward Euler method [20] for the system (1),

$$S_{n+1} = S_0 + \frac{h^{\rho}}{\Gamma(\rho+1)} \sum_{j=1}^{n} [(n-j+1)^{\rho} - (n-j)^{\rho}] \left[\mu - \beta \frac{S_j I_j}{1+\beta_1 S_j + \beta_2 I_j} - \lambda S_j + \varepsilon R_j\right]$$

$$I_{n+1} = I_0 + \frac{h^{\rho}}{\Gamma(\rho+1)} \sum_{j=1}^{n} [(n-j+1)^{\rho} - (n-j)^{\rho}] \times \left[\frac{\beta S_j I_j}{1+\beta_1 S_j + \beta_2 I_j} - (\lambda+\alpha+\gamma) I_j - \frac{\sigma I_j}{1+\delta I_j}\right]$$

$$R_{n+1} = R_0 + \frac{h^{\rho}}{\Gamma(\rho+1)} \sum_{j=1}^{n} [(n-j+1)^{\rho} - (n-j)^{\rho}] \left[\frac{\sigma I_j}{1+\delta I_j} + \gamma I_j - (\lambda+\varepsilon) R_j\right]$$

7. Numerical Simulation

The analysis conducted in this study involved a quantitative examination of the obtained results. To validate these findings, numerical simulations were performed in this section. The numerical solution to evaluate and verify the model's result was accomplished through the utilization of the fractional forward Euler method, employing MATLAB software. It is important to note that the investigation for the numerical simulation was based solely on assumptions. Within this section, two specific cases, namely Case I and Case II, were considered. In the first case, Figure 1 presented a population-time graph that depicted the dynamic behaviour of all nodes for a fractional order of ρ=0.45. From figure 1, it was evident that a lower value of ρ corresponded to higher infectivity and a decrease in recovery. Additionally, Figure 2 showcased a time-series analysis of the infectious class and recovered class in relation to time. Moving on to the second case, Figure 3 displayed a population-time graph that illustrated the dynamic behaviour of all nodes for a fractional order of ρ=0.90. It was observed that a higher value of ρ resulted in lower infectivity and an increase in recovery. Furthermore, Figure 4 presented a time-series analysis of the infectious class and recovered class with respect to time. The results obtained from the model simulations, along with their associated findings, are provided in the subsequent sections.

Case I- When $\rho = 0.45$

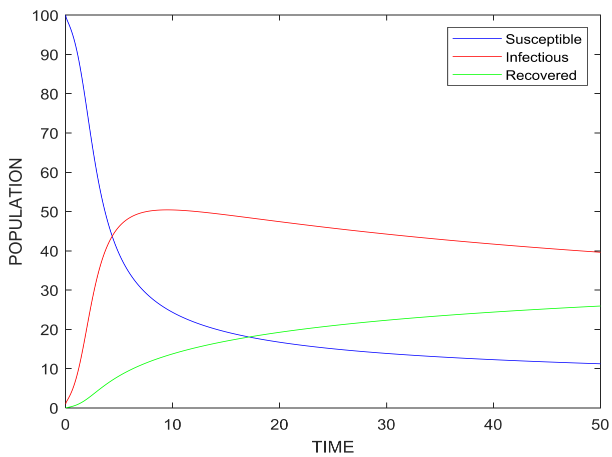

Fig. 1. Dynamical behavior of nodes with respect to time, ρ=0.45

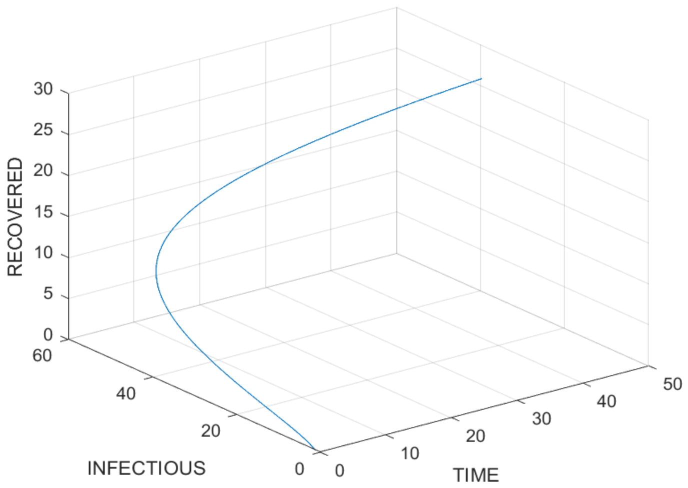

Fig. 2. Time -series analysis of Infectious class versus Recovered class

Case II- When $\rho = 0.9$

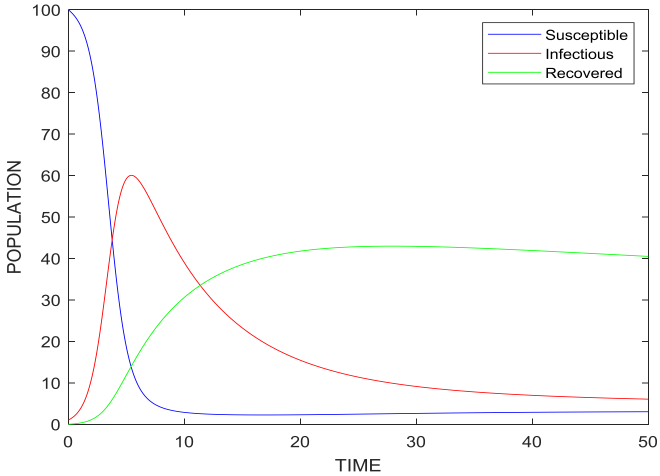

Fig. 3. Dynamical behavior of computer population with respect to time, ρ=0.9

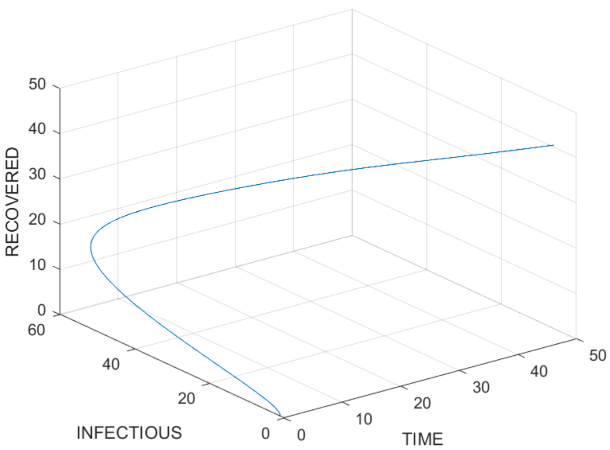

Fig. 4. Time-series analysis of Infectious class versus Recovery class

8. Conclusion

The susceptible-infectious-recovered-susceptible (SIRS) epidemic model of fractional order using Caputo derivative has been proposed in this paper. The local asymptotic stability of both disease-free equilibrium and endemic equilibrium has been established by determining the basic reproduction number. Furthermore, the global asymptotic stability of endemic equilibrium has been established through the construction of Lyapunov function. To capture the dynamics of the epidemic, the model incorporates a Beddington-De-Angelis incidence rate and a Holling type II recovery rate. These rates allow for the examination of the impact of measures taken by the susceptible and infectious populations to inhibit the spread of the malware. Sensitivity analysis is performed to determine the importance of each parameter for the transmission of attacks in the network. To approximate the solutions of the model, the Fractional forward Euler method has been employed. The accuracy of the results has been enhanced by analyzing the solutions of the model through numerical simulations using MATLAB. Simulation results demonstrate that a decrease in the Caputo derivative ρ is associated with increased infectivity and reduced recovery, while an increase in ρ leads to decreased infectivity and increased recovery. It is crucial to carefully manage the Caputo derivative parameter to ensure effective control.

The proposed SIRS fractional epidemic model can help us to understand the assessment of cybersecurity risks posed by potential malware propagation or denial-of-service attack in smart cars. Similarly, in the realm of the Internet of Medical Things (IoMT), where interconnected medical devices and systems exchange sensitive patient data, the fractional SIRS model aids in identifying vulnerabilities and designing robust security protocols to mitigate the risk of unauthorized access or data breaches. Furthermore, in the context of smart warfare, where networked military systems and autonomous weapons are deployed, the fractional SIRS model offers a framework for analyzing the propagation of cyber threats and devising resilient defense mechanisms against adversarial attacks targeting critical infrastructure and communication networks. By leveraging the fractional SIRS epidemic model, stakeholders in these domains can proactively identify and address cybersecurity challenges, thereby enhancing the resilience and security posture of networked systems in an increasingly interconnected world.

One significant limitation of the Caputo fractional order derivative in our model lies in its inability to account for initial conditions effectively. The non-local nature of fractional derivatives poses challenges in interpreting and validating model outcomes, whereas potential limitation is the computational complexity associated with solving fractional differential equations. Future research efforts will focus on refining computational techniques, enhancing model interpretability, and validating model outcomes against real-world data to unlock the full potential of fractional calculus in epidemic modeling. The validation of our developed model with real world data and predictive analytics will be our future work.The forthcoming research will prioritize the investigation of the time delay fractional derivative epidemic model, with a specific focus on incorporating the various time lags associated with the transition between different compartments.

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