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A Common Fixed Point Theorem for *R*-Weakly Commuting Maps Satisfying Property (*E.A.*) in Fuzzy Metric Spaces Using Implicit Relation

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***Abstract*—** The purpose of this paper is to prove a common fixed theorem for *R*-weakly commuting mappings via an implicit relation in fuzzy metric space. While proving our result, we utilize the idea of property (*E.A.*) due to Aamri and El. Moutawakil [1] together with common (*E.A.*) property due to Liu, Wu and Li [2].

***Index Terms*—** Fuzzy metric space, *R*-weakly commuting mappings, Property (*E.A.*), Common (*E.A*) property, Implicit relation

I. INTRODUCTION

In 1986, Jungck [3] introduced the notion of compatible maps for a pair of self mappings. However, the study of common fixed points of non-compatible maps is also very interesting. Aamri and El. Moutawakil [1] generalized the concept of non-compatibility by defining the notion of property (*E.A*) and in 2005, Liu, Wu and Li [2] defined common (*E.A*) property in metric spaces and proved common fixed point theorems under strict contractive conditions. Jungck and Rhoades [4] initiated the study of weakly compatible maps in metric space and showed that every pair of compatible maps is weakly compatible but reverse is not true. In the literature, many results have been proved for contraction maps satisfying property (*E.A.*) in different settings such as probabilistic metric spaces [5, 6]; fuzzy metric spaces [7, 8, 9, 10, 11].

In this paper, employing the common (*E.A*) property, we prove a common fixed theorem for *R*-weakly commuting mappings via an implicit relation in fuzzy metric space.

II. PRELIMINARIES

Definition 2.1. [12] A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is continuous *t*-norm if $*$ satisfies the following conditions:

(i) * is commutative and associative;
(ii) * is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Kramosil I and Michalek J.[13] introduced the concept of fuzzy metric spaces as follows:

*Definition 2.2[13]***:** The 3-tuple $(X, M, *)$ is called a fuzzy metric space (shortly, FM-space) if X is an arbitrary set, * is a continuous *t*-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s)

(Triangular inequality) and

(FM-5) *M(x, y, .)* : [0, 1) → [0, 1] is left continuous for all $x, y, z \in X$ and $s, t > 0$.

Note that *M(x, y, t)* can be thought of as the degree of nearness between x and y with respect to *t*.

We can fuzzify examples of metric spaces into fuzzy metric spaces in a natural way:

Let *(X, d)* be a metric space. Define *a * b = a + b* for all *a, b* in *X*. Define *M(x, y, t) = t /(t + d(x, y))* for all *x, y* in *X* and *t > 0*.Then *(X, M, *)* is a fuzzy metric space and this fuzzy metric induced by a metric *d* is called the *Standard fuzzy metric.*

Consider M to be a fuzzy metric space with the following condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all *x, y* in *X* and $t > 0$.

Definition 2.3[13]: Let *(X, M, *)* be fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in *X* is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$,

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$$

and

(b) a sequence $\{x_n\}$ in *X* is said to be convergent to a point $x \in X$ if, for all $t > 0$,

$$\lim_{n\to\infty} M(x_n, x, t) = 1.$$

Definition 2.4[13]: A fuzzy metric space *(X, M, *)* is said to be complete if and only if every Cauchy sequence in *X* is convergent.

Example 2.1[13]: Let $X = \{1/n : n \in N\} \cup \{0\}$ and let * be the continuous t-norm and defined by $a*b = ab$ for all $a, b \in [0,1]$. For each $t > 0$ and $x, y \in X$, define M, by

$$M(x,y,t) = \begin{cases} \frac{\mathrm{t}}{\mathrm{t}+|\mathrm{x}-\mathrm{y}|}, & \mathrm{t} > 0, \\ 0 & \mathrm{t} = 0 \end{cases}$$

Clearly, *(X, M, *)* is complete fuzzy metric space.

Definition 2.5[11]: A pair of self mappings (*A, S*) of a fuzzy metric space *(X, M, *)* is said to be commuting if

M(ASx , SAx , t) = 1 for all $x \in X$.

Definition 2.6[11]: A pair of self mappings (*A, S*) of a fuzzy metric space *(X, M, *)* is said to be weakly commuting if *M(ASx , SAx , t)* ≥ *M(Ax, Sx, t)* for all $x \in X$ and *t > 0*.

Definition 2.7[11]: A pair of self mappings (*A, S*) of a fuzzy metric space *(X, M, *)* is said to be compatible if $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ for all *t > 0*, whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u$ for some *u* in *X*.

Definition 2.8[10]: Let *(X, M, *)* be a fuzzy metric space. *A* and *S* be self maps on *X*. A point *x* in *X* is called a coincidence point of *A* and *S* iff *Ax = Sx*. In this case, *w = Ax = Sx* is called a point of coincidence of *A* and *S*.

Definition 2.9[10]: A pair of self mappings (*A, S*) of a fuzzy metric space *(X, M, *)* is said to be weakly compatible if they commute at the coincidence points i.e., if *Au = Su* for some $u \in X$, then *ASu = SAu.*

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.10 [1]: A pair of self mappings (*A, S*) of a fuzzy metric space *(X, M, *)* is said to satisfy the property (*E.A*) if there exist a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Example 2.2[1]: Let X =[0,∞). Consider *(X, M, *)* be a fuzzy metric space as in Example 2.1.

Define *A*, *S* : *X* →*X* by $Ax = \frac{x}{5}$ and $Sx = \frac{2x}{5}$ for all $x \in X$. Clearly, for sequence $\{x_n\} = \{1/n\}$, *A* and *S* satisfies property (*E.A*).

Definition 2.11 [2]: Two pairs (*A*, *S*) and (*B*, *T*) of self mappings of a fuzzy metric space *(X, M, *)* are said to satisfy the common (*E.A*) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

for some $z \in X$.

Example 2.3[2]: Let *X* = [-1, 1]. Consider *(X, M, *)* be a fuzzy metric space as in Example 2.1. Define self mappings *A*, *B*, *S* and *T* on *X* as $Ax = x/3, Bx = -x/3, Sx = x, Tx = -x$ for all $x \in X$. Then, with sequences $\{x_n\} = \{1/n\}$ and $\{y_n\} = \{-1/n\}$ in *X*, one can easily verify that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 0$. Therefore, pairs (*A*, *S*) and (*B*, *T*) satisfies the common (*E.A.*) property.

Definition 2.12[11]: A pair of self mappings (*A*, *S*) of a fuzzy metric space *(X, M, *)* is said to be *R*-weakly commuting if there exist *R* > 0 such that

$$M(ASu, SAu, t) \ge M(Au, Su, t/R).$$

III. Main Results

Implicit relations play important role in establishing of common fixed point results.

Let M_4 be the set of all real continuous functions $\phi : [0,1]^4 \to$ R, non-decreasing in the first argument and satisfying the following conditions:

(A) $\phi(u,1,u,1) \ge 0 \Rightarrow u \ge 1$,

(B) $\phi(u,1,1,u) \ge 0 \Rightarrow u \ge 1$

(C) $\phi(u,u,1,1) \ge 0 \Rightarrow u \ge 1$

Example 3.1: Define $\phi : [0,1]^4 \to R$ as $\phi(t_1,t_2,t_3,t_4) = 14t_1 - 12t_2 + 6t_3 - 8t_4$. Clearly, ϕ satisfies all conditions (A), (B) and (C). Therefore, $\phi \in M_4$.

We begin with following observation:

Lemma 3.1: Let $\{A_i\}$, S and T be self mappings of a fuzzy metric space *(X, M, *)* satisfying the following:

(3.1) the pair (A_0, T) satisfies the *E.A.* property;

(3.2) for any x, y in X, ϕ in M_4 and and for all $t > 0$ there exists $k \in (0,1)$ such that,

$$\phi\begin{pmatrix} M(A_i x, A_0 y, kt), M(Sx, Ty, t), \\ M(Sx, A_i x, t), M(Ty, A_0 y, kt) \end{pmatrix} \ge 0;$$

(3.3) $A_i(X) \subset T(X)$ or $A_0(X) \subset S(X)$.

Then the pairs (A_i, S) and (A_0, T) share the common (E.A.) property.

Proof: As the pair (A_0, T) satisfies E.A. property, then there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} A_0 x_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Since $A_0(X) \subset S(X)$, hence for each $\{x_n\}$, there exist $\{y_n\}$ in X such that $A_0 x_n = Sy_n$. Therefore, $\lim_{n\to\infty} A_0 x_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = z$. Now, we claim that $\lim_{n\to\infty} A_i y_n = z$.

Suppose that $\lim_{n\to\infty} A_i y_n \ne z$, then applying inequality (3.2), we obtain

$$\phi\begin{pmatrix} M(A_i y_n, A_0 x_n, kt), M(Sy_n, Tx_n, t), \\ M(Sy_n, A_i y_n, t), M(Tx_n, A_0 x_n, kt) \end{pmatrix} \ge 0$$

which on making $n \to \infty$ reduces to

$$\phi\begin{pmatrix} M(\lim_{n\to\infty} A_i y_n, z, kt), M(z,z,t), \\ M(\lim_{n\to\infty} A_i y_n, z, t), M(z,z,kt) \end{pmatrix} \geq 0$$

As ϕ is non-decreasing in the first argument, we have

$$\phi\begin{pmatrix} M(\lim_{n\to\infty} A_i y_n, z, t), 1, \\ M(\lim_{n\to\infty} A_i y_n, z, t), 1 \end{pmatrix} \geq 0$$

Using (B), we get $M(\lim_{n\to\infty} A_i y_n, z, t) \geq 1$. Hence $M(\lim_{n\to\infty} A_i y_n, z, t) = 1$. Therefore, $\lim_{n\to\infty} A_i y_n = z$. Hence, the pairs (A_i, S) and (A_0, T) share the common (E.A.) property.

Theorem 3.1: Let $\{A_i\}$, *S* and *T* be self mappings of a fuzzy metric space *(X, M, *)* satisfying the conditions (3.1), (3.2), (3.3) of lemma 3.1 and the pairs (A_i, S) and (A_0, T) are *R*-weakly commuting. If range of one of *S* and *T* is closed subspace of *X* then $\{A_i\}$, *S* and *T* have a unique common fixed point.

Proof: By lemma 3.1, the pairs (A_i, S) and (A_0, T) share the common *(E.A.)* property, i.e. there exist two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ for some $z \in X$. Suppose that *S(X)* is a closed subset of *X*, therefore, there exists a point $u \in X$ such that $z = Su$. We claim that $A_i u = z$. If $A_i u \neq z$, then by (3.2), take $x = u, y = x_n$,

$$\phi\begin{pmatrix} M(A_i u, A_0 x_n, kt), M(Su, Tx_n, t), \\ M(Su, A_i u, t), M(Tx_n, A_0 x_n, kt) \end{pmatrix} \geq 0$$

$n \to \infty$

$$\phi\begin{pmatrix} M(A_i u, z, kt), M(z,z,t), \\ M(z, A_i u, t), M(z,z,kt) \end{pmatrix} \geq 0$$

$$\phi\left(M(A_i u, z, kt), 1, M(A_i u, z, t), 1\right) \geq 0$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(M(A_i u, z, t), 1, M(A_i u, z, t), 1) \geq 0$$

and using (A), we get $M(A_i u, z, t) \geq 1$.

Hence $M(A_i u, z, t) = 1$. Therefore, $A_i u = z = Su$ which shows that u is a coincidence point of the pair *(A, S).*

Since, $A_i(X) \subset T(X)$, there exists $v \in X$ such that $Tv = z = A_i u = Su$. Now, we show that $A_0 v = z$.

If $A_0 v \neq z$, then by using inequality (3.2), take $x = y_n$, $y = v$, we have

$$\phi\begin{pmatrix} M(A_i y_n, A_0 v, kt), M(Sy_n, Tv, t), \\ M(Sy_n, A_i y_n, t), M(Tv, A_0 v, kt) \end{pmatrix} \geq 0;$$

$n \to \infty$

$$\phi\begin{pmatrix} M(z, A_0 v, kt), M(z,z,t), M(z,z,t), \\ M(z, A_0 v, kt) \end{pmatrix} \geq 0;$$

Using (B), we get $M(z, A_0 v, kt) \geq 1$.

Hence $M(z, A_0 v, kt) = 1$. Therefore, $A_0 v = z = Tv$ which shows that *v* is a coincidence point of the pair *(A₀, T)*. Since, A_i and *S* are pointwise *R*-weakly commuting, there exist $R > 0$ such that

$$M(A_i Su, SA_i u, t) \geq M(A_i u, Su, t/R) = 1.$$

This gives, $A_i Su = SA_i u = AA_i u = SSu$.

Similarly, as A_0 and *T* are pointwise *R*-weakly commuting, we have $A_0 Tv = TA_0 v$ and $A_0 Tv = TA_0 v = A_0 A_0 v = TTv$.

Take $x = A_i u$, $y = v$ in (3.2), we have

$$\phi\begin{pmatrix} M(A_iA_iu, A_0v, kt), M(SA_iu, Tv, t), \\ M(SA_iu, A_iA_iu, t), M(Tv, A_0v, kt) \end{pmatrix} \geq 0;$$
$$\phi\left(M(A_iA_iu, A_iu, kt), M(SSu, A_iu, t), 1, 1\right) \geq 0;$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(M(A_iA_iu, A_iu, t), M(SSu, A_iu, t), 1, 1) \geq 0;$$

Using (C), we get $M(A_iA_iu, A_iu, t) \geq 1$.

This gives, $M(A_iA_iu, A_iu, t) = 1$.

Therefore, $A_iA_iu = A_iu$. Hence, $A_iz = z = Sz$.

Similarly, by putting, $y = A_0v$, $x = u$ in (3.2), we get

$$\phi\begin{pmatrix} M(A_iu, A_0A_0v, kt), M(Su, TA_0v, t), \\ M(Su, A_iu, t), M(TA_0v, A_0A_0v, kt) \end{pmatrix} \geq 0;$$
$$\phi\left(M(A_0v, A_0A_0v, kt), M(A_0v, A_0A_0v, t), 1, 1\right) \geq 0;$$

As ϕ is non-decreasing in the first argument, we hav

$$\phi(M(A_0v, A_0A_0v, t), M(A_0v, A_0A_0v, t), 1, 1) \geq 0;$$
$$M(A_0v, A_0A_0v, t) \geq 1;$$

therefore, $M(A_0v, A_0A_0v, t) = 1;$

This gives, $A_0v = A_0A_0v$.

$A_0z = z = Tz$.

Hence, $A_iz = A_0z = Sz = Tz$, and z is common fixed point of A_i, A_0, S and T.

Uniqueness: Let z and w be two common fixed points of A_i, A_0, S and T. If $z \neq w$, then by using inequality (3.2), we have

$$\phi\begin{pmatrix} M(A_iz, A_0w, kt), M(Sz, Tw, t), \\ M(Sz, A_iz, t), M(Tw, A_0w, kt) \end{pmatrix} \geq 0,$$
$$\phi\begin{pmatrix} M(z, w, kt), M(z, w, t), \\ M(z, z, t), M(w, w, kt) \end{pmatrix} \geq 0,$$
$$\phi\left(M(z, w, kt), M(z, w, t), 1, 1\right) \geq 0,$$
$$\phi\left(M(z, w, t), M(z, w, t), 1, 1\right) \geq 0.$$

Using (C) and (F), we have $M(z, w, t) \geq 1$.

Hence, $M(z, w, t) = 1$.

Therefore, $z = w$

By choosing A_i, A_0, S and T suitably, one can derive corollaries involving two or more mappings. As a sample, we deduce the following natural result for a pair of self mappings by setting $A_i = A_0$ and $T = S$ in above theorem:

Corollary 3.1. Let $\{A_i\}$ and S be self mappings of a fuzzy metric space *(X, M, *)* satisfying the following:

(3.4) the pair (A_i, S) satisfies the *E.A.* property;

(3.5) for any x, y in X, ϕ in M_4 and for all $t > 0$,

$$\phi\begin{pmatrix} M(A_ix, A_iy, kt), M(Sx, Sy, t), \\ M(Sx, A_ix, t), M(Sy, A_iy, t) \end{pmatrix} \geq 0;$$

(3.6) *S(X)* is a closed subset of *X*.

Then, $\{A_i\}$ and S have a point of coincidence each. Moreover, if the pairs (A_i, S) is weakly compatible, then $\{A_i\}$and S have a unique common fixed point.

By taking $A_i = A_0 = A$ and $T = S$ in theorem 3.1, we get

Corollary 3.2. Let A and S be self mappings of a fuzzy metric space *(X, M, *)* satisfying the following:

(3.7) the pair (A, S) satisfies the *E.A.* property;

(3.8) for any x, y in X, ϕ in M_4 and for all $t > 0$,

$$\phi\begin{pmatrix} M(Ax, Ay, kt), M(Sx, Sy, t), \\ M(Sx, Ax, t), M(Sy, Ay, t) \end{pmatrix} \geq 0;$$

(3.8) *S(X)* is a closed subset of *X*.

Then, A and S have a point of coincidence each. Moreover, if the pairs (A, S) is weakly compatible, then A and S have a unique common fixed point.

The following example illustrates Theorem 3.1.

Example 3.2. Let *(X, M, *)* be a fuzzy metric space where *X = [0, 2)* and define $\phi:[0,1]^4 \to$ R as $\phi(t_1,t_2,t_3,t_4)=14t_1-12t_2+6t_3-8t_4$. Clearly, ϕ satisfies all conditions (A), (B) and (C). Therefore, $\phi \in M_4$. Define *A, B, S* and *T* by

$A_i x = A_0 x = 1$,

$$S(x)=\begin{cases}1 & x\in Q\\ \frac{2}{3} & x\notin Q,\end{cases},\quad T(x)=\begin{cases}1 & x\in Q\\ \frac{1}{3} & x\notin Q\end{cases}$$

and $M(x,y,t)=\dfrac{t}{t+|x-y|}$ for all *x, y* in *X = [0, 2)* and $t>0$ .Then with sequences $\left\{x_n=\frac{1}{n}\right\}$ and $\left\{y_n=\frac{-1}{n}\right\}$ in X, we have

$$\lim_{n\to\infty} A_i x_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} A_0 y_n = \lim_{n\to\infty} Ty_n = 1 \text{ in } X$$

which shows that pairs (A_i, S) and (A_0, T) share the common *(E.A.)* property. By a routine calculation, one can verify the condition (3.2). Thus, all the conditions of Theorem 3.1 are satisfied and *x = 1* is the unique common fixed point of A_i, A_0, *S* and *T*.

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