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Introducing High School Students to the Mystery of Lobachevsky Geometry

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Abstract: This article primarily aims to introduce high school students to the mystery of Lobachevsky geometry, one of the cornerstones of non-Euclidean geometries. Lobachevsky geometry, often known as hyperbolic geometry, differs from Euclidean geometry in several basic ways. The concepts and figures of Lobachevsky geometry can appear in different plane models, such as the Klein and Poincaré disk models. It further examines students' general attitudes and behaviors toward non-Euclidean geometries. Lobachevsky's geometry has helped expand students' horizons and enriched their critical thinking skills by challenging traditional Euclidean paradigms. This study is supported by a mixed-method approach utilizing quantitative and qualitative data. The mock exam results obtained from students during the educational process were compared, and the study was further supported by the positive feedback received from the participating students. The intriguing lessons on Lobachevsky geometry were conducted over 4 weeks, with weekly 2-hour geometry classes involving 12th-grade students at Stirling Schools in Erbil. Throughout the study, we observed significant improvements in students' ability to adopt, understand, and apply advanced geometric concepts. This article also discusses findings and implications that address gaps in the literature and considers the potential for curriculum updates to enhance the future of geometry education.

Index Terms: Escher, GeoGebra, Hands-On Learning, Holonomy, Hyperbolic geometry, Lobachevsky geometry, Mathematics Education, Non-Euclidean geometry, Student Engagement, WebSketchpad

1. Introduction

In modern conditions, experts working in the field of education must deal with large amounts of information. Effectively processing this information, making timely and necessary decisions, and taking accurate and actionable measures will significantly determine success [1]. In this context, emphasizing the importance of strengthening schools to shape a more efficient and productive future and acknowledging the indispensable role of education in preparing the next generation to live at the desired level [2] is essential. [3] states that discoveries in mathematics often involve extraordinary challenges, intricate puzzles, complex mental games, paradoxes, and thought-provoking sophisms. Therefore, delving into fascinating topics and examples not only has the potential to enlighten and inspire but also fosters a desire to explore.

[4] states that, despite significant interest and research, the current state of mathematics education is unacceptable, and real change progresses unfortunately slowly. He laments that policymakers are not expected to grasp the nature of mathematics or mathematics education, which highlights the fact that what we need is not merely to acknowledge its existence and importance but to better understand the underlying issues. When you truly understand mathematics and possess the mental perspective to see it as a whole, an extraordinary mental compression occurs. This allows you to recall it quickly and completely when needed and to use it as just one step in another mental process. The insight that comes with this compression is one of the true pleasures of mathematics.

After mastering mathematical concepts, you may find yourself engaging in conversations in various settings with others who share the same feelings, setting sail into many mysterious worlds, even among the knowledge already available. One of these mysterious concepts is our new siblings in geometry, born in the 19th century, which we call non-Euclidean geometries. The key mathematicians who contributed to this birth include Gauss, Bolyai, Lobachevsky, Beltrami, Riemann, and Poincaré[5].

As is well known, Euclid built his geometry on five axioms, one of which has been a topic of debate throughout history: the fifth axiom (postulate). It states:

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"Given a point not on a line, there is exactly one line parallel to the given line that passes through the point within a plane."

Since this statement also provides the condition for parallelism between two lines, it is often referred to as the parallel axiom or the parallel postulate in some contexts [6]. Unfortunately, Euclid could not derive this axiom logically from the others, and it remained at almost the same level of foundational assumption [7].

The fifth axiom was also discussed during the Middle Ages by Islamic mathematicians, the most notable of whom was Nasir al-Din al-Tusi (1201–1274). In his work al-Risala al-Shafiya, al-Tusi examined Euclid's fifth postulate but was unable to prove it within the framework of Euclidean geometry. However, he demonstrated that in a quadrilateral ABCD, where AB and DC are equal and both perpendicular to BC, if A and D are acute angles, the sum of the angles of a triangle is less than 180 degrees. This is a property of Lobachevsky geometry, showing that al-Tusi, like al-Khayyam (1048–1131), proved some properties of what would later be known as non-Euclidean geometry [8,9].

Similarly, Bolyai (1802–1860), like his father, worked on Euclid's parallel postulate and rejected it, developing a new geometric system. He explained this new geometry in a letter written to his father in 1823, and the final version of his work was published in 1832 as an appendix to his father's book, commonly known as Tentamen [10].

The Italian mathematician Beltrami (1835–1899) published his first paper on non-Euclidean geometry in Milan in 1862. His most influential work, Essay on the Interpretation of Non-Euclidean Geometry, was published in 1868. In this paper, Beltrami proposed a theory of hyperbolic space, which provided an analytic foundation for the development of non-Euclidean geometry. He demonstrated that the concepts and formulas of Lobachevsky geometry apply to geodesics on surfaces of constant negative curvature. Additionally, he showed the existence of "pseudospherical" surfaces, created by rotation, and was the first to use the term pseudospherical [11,12].

Lobachevsky (1793–1856) was the first to explicitly state and prove that the fifth postulate cannot be derived from the other postulates of geometry. To this end, Lobachevsky retained Euclid's fundamental premises except for the parallel postulate, assuming instead that it is invalid. Based on this assumption, he constructed a logical system in which various propositions followed from the accepted premises. This system, free from logical contradictions like Euclidean geometry, established a new geometry, which he referred to as "imaginary." Lobachevsky elevated his "imaginary geometry" to the same status as Euclidean geometry.

While he encountered no logical inconsistencies during this process, Lobachevsky recognized that the absence of contradictions in his work did not conclusively prove that his new geometry was entirely free from them. He acknowledged that identifying any contradictions would depend on the stage of development at which they might arise. To address this, Lobachevsky conducted a deep algebraic analysis of his equations, resolving the issue as much as was possible in his time [7].

The results of Lobachevsky's research can be summarized as follows:

- The parallel postulate is not a necessary consequence of the other postulates of geometry.
- The fifth postulate cannot be derived from the other postulates, as there exists another "imaginary" geometry where this postulate does not hold, in contrast to Euclidean geometry, where it is valid.

The questions surrounding the study of Lobachevsky geometry are closely linked to issues in psychology and epistemology, particularly those related to the emergence of spatial imagination and intuition. Klein's work was a significant triumph in establishing Lobachevsky geometry as a logically consistent geometric system. This recognition affirmed that Lobachevsky geometry was as valid and real as Euclidean geometry. Moreover, A. Poincaŕs statement regarding the consideration of various models of Lobachevsky planar axiomatics provided additional support for its acceptance [13].

When comparing Euclidean and Lobachevsky geometries in terms of the sum of the angles of a triangle:

- In Euclidean geometry, the sum of the interior angles of a triangle equals two right angles.
- In Lobachevsky geometry, the sum of the interior angles of a triangle is less than two right angles.

If we classify these geometries by the condition of parallel lines:

- Euclidean geometry accepts that in a plane, for a given line and a point outside it, there is exactly one parallel line through the point.
- Lobachevsky geometry allows an infinite number of parallel lines to pass through a point outside the given line.

In terms of surface curvature:

- Euclidean geometry has zero curvature.
- Lobachevsky geometry has negative curvature.

While geometry cannot always be transferred to a tangible world, there is a need for learning approaches that help students visualize these axioms more concretely. When students see, touch, and manipulate shapes, they begin to develop spatial reasoning skills [14].

In this chain of generalizations, Lobachevsky's non-Euclidean geometries have emerged as the first and defining link. Although they have not yet fully found their place in high school-level mathematics curricula, they continue to be taught in university geometry courses, including master's and doctoral programs. This study aims to investigate high school students' awareness of Lobachevsky and other non-Euclidean geometries, and to restructure students' geometric cognitive structures through mental assessments at the end of engaging activities. It should be noted that the concepts we explore in this study are not directly included in current school geometry curricula. While conducting the study, it is assumed that students sincerely respond to our questions. Although there are certain limitations to our study, it is recognized that existing research on these topics is not at the desired level, and it is believed that this research will contribute to the literature and help future researchers with their studies. From this point, we aim to bring Lobachevsky geometry to a certain level by seeking answers to the research questions below.

1.1 Research Question

Q1. How can the mysterious concepts of Lobachevsky geometry be engagingly introduced to high school students?
Q2. How does this introduction affect students' attitudes and comprehension of non-Euclidean geometries?

2. Research Background

Geometry is essentially a global language that has been understood similarly across different times, countries, and cultures. The proof of a geometric theorem or construction is known more or less in the same way by everyone. This universality, among other factors, highlights the human face of geometric and mathematical achievements and fosters unity among people and civilizations. The pedagogical value of teaching geometry in secondary education is indisputable, as it helps students enhance their spatial perception skills and directly relates mathematics to the real world [15].

[16], in their literature review, investigated the impact of introducing non-Euclidean concepts into the high school Euclidean geometry curriculum to increase student engagement and deepen understanding. By examining student behaviors and their reactions to non-Euclidean examples, they evaluated the effectiveness of incorporating diverse geometric perspectives. [17] explored how a teacher could use methods that help students develop a deep and symbolic understanding of the fundamental ideas of non-Euclidean geometry. Their study suggested that such interventions would motivate more students to pursue advanced geometry courses at the university level. [18] highlighted the importance of using non-Euclidean geometries to challenge conventional thinking and promote openness to new methods and perspectives. They emphasized Lobachevsky's geometry, specifically focusing on the introduction of basic concepts and their comparison with Euclidean geometry, as a key approach for enhancing conceptual understanding.

[19] aimed to revisit the foundational framework of Lobachevsky's geometry. He discussed how non-Euclidean geometries arose as a result of mathematicians' efforts to prove Euclid's fifth postulate as a theorem. He referenced Lobachevsky's proposal of a non-standard model of the Euclidean plane in hyperbolic 3-space, an unusual contribution for his time. Additionally, Lobachevsky's use of the parallel postulate instead of Euclid's fifth postulate led to the emergence of unique structures in hyperbolic geometry.

[20] discuss the difficult process of the recognition of non-Euclidean geometry, noting that the history of Lobachevsky's geometry was simplified in the research. They point out that the earlier contributions of mathematicians like Aristotle, Omar Khayyam, Schweikart, Taurinus, Lambert, Saccheri, and others were largely overlooked. The study also explains why non-Euclidean geometry was not immediately accepted.

[21] review the independent work of Bolyai, Gauss, and Lobachevsky in the early 19th century on non-Euclidean geometry, following references to Euclid's fifth postulate by various mathematicians. They focus on the infinite number of parallel lines and the new "parallel angle" concept. They highlight that the work of Bolyai, Gauss, and Lobachevsky was not immediately accepted, with their ideas gaining recognition only after their deaths. The study also discusses further developments in non-Euclidean geometry by mathematicians like Riemann and Christoffel, and the application of this geometry in Einstein's physics.

[22] mention that students are able to compare the deductive structures of Lobachevsky geometry with Euclidean geometry, which enables them to integrate multiple elements into a unified whole and generalize to new structures. In addition conducted an exploratory study where they aimed to define students' understanding of triangles in Lobachevsky's geometry through the lens of local culture. Their research analyzes students' ability to comprehend the characteristics of Lobachevsky geometry, particularly focusing on triangles.

A study by [23] from Queen's Mathematics Research Institute in London emphasizes the significant research conducted on Lobachevsky's geometry. It argues that treating geometry as a separate science and using Lobachevsky geometry as a scientific stepping stone can aid in teaching other areas of geometry. The study presents a theoretical and practical analysis of Lobachevsky geometry and its key components.

[24] explored how new information interacts with entrenched knowledge through conceptual change, introducing students to hyperbolic geometry, a branch of non-Euclidean geometry. In their research, they provided training on

hyperbolic geometry through lines and closed shapes, along with experimental activities. They concluded that an integrated teaching model, which provides holistic knowledge about geometric relationships, was more effective than a model focused on specific points, particularly highlighting the challenges in integrating new knowledge.

In more recent studies on Lobachevsky geometry, we encounter updated works that, with the support of technology, facilitate the teaching process. In a study focusing on certain basic concepts of hyperbolic geometry for high school students, the use of WebSketchpad, a free online dynamic geometry application, encourages students to engage in experiments that can form the basis for their proofs [25].

Holonomy, a virtual environment based on the concept of hyperbolic geometry, allows users to physically walk and explore an infinite hyperbolic space, unlike other environments. Standing out with its unique features compared to other VRs, it offers students the ability to comfortably create complex hyperbolic environments and find the shortest paths in non-Euclidean geometry, demonstrating how technology-supported education can create wonders [26].

[27] based on the findings and discussions of their research on examining the awareness of gifted students in Turkey regarding non-Euclidean geometries through Euclid's 5^{th} postulate, recommend that the Ministry of Education include non-Euclidean geometries (hyperbolic and elliptic geometries) in mathematics curricula during updates.

[28], while investigating teachers' mathematical knowledge and mathematical connections for teaching geometry, found that although prospective middle school teachers' mathematical knowledge for teaching geometry was below average, the curriculum connections made had a statistically significant and positive effect on their mathematical knowledge for teaching geometry. In addition, [29] emphasizes that improving the effectiveness of interaction between teachers and students impacts the quality of education. Beyond determining mathematical competencies and performance in geometry, [30] examines whether high school students' performance in geometry significantly affects their mathematical competencies in terms of mathematical concepts and problem-solving skills. The study found that high school students' learning styles, class size, classroom structure, and mathematical competence level significantly influence their geometry performance.

[31] mentions that students are more interested in learning mathematics and experience less anxiety. They attribute this to the pedagogically fun educational game they designed. It is suggested that students feel more at ease when dealing with numerical arithmetic, and thus, technology-enriched educational games lead to positive individual attitudes toward mathematics and a better classroom experience, ultimately resulting in more effective mathematics teaching and learning.

However, [32] points out that the emergence of numerous e-learning platforms and classrooms has not resolved the issue of improving the quality of education. In technology-enriched learning systems, the level of adaptation to individual student characteristics and the organization of human-computer interaction become important, and it is noted that technology can increase students' cognitive comfort levels and optimize learning time.

Today, [33] states that learning through different technologies has become a global phenomenon. It emphasizes that the internet is often a value-neutral tool that enables individuals to overcome the restrictions of traditionally exclusive fields and gain unrestricted access to learning.

[34] notes that the potential of technology to enhance the learning and teaching of mathematics is widely acknowledged. As a result of a case study investigating how technology can improve mathematics teaching, it was concluded that when teachers integrated technology into their instructional practices, students made significant progress. However, [35] argues that we need to be more cautious and critical about the terminology we adopt to define this area. Similarly, [36] draws attention to the significant lack of academic research on the use of technology tools that are easily accessible to teachers.

In [37]'s research, they assessed students' current knowledge and proficiency in non-Euclidean geometries and provided insights into the potential inclusion of these concepts in the mathematics curriculum. Their findings strongly suggest that incorporating non-Euclidean geometries into the standard mathematics curriculum is essential. Such inclusion has the potential not only to broaden students' mathematical horizons and develop their critical thinking skills, but also to spark curiosity about non-Euclidean geometries. By doing so, they emphasize that students can be better equipped with a more holistic mathematics education, promoting the development of essential analytical skills.

[38] mentions that current trends in mathematics education and modern teaching methods are rapidly evolving at all levels, and that introducing non-Euclidean geometry concepts in schools can play an important role in developing students' spatial imagination and enhancing their scientific research competencies. By introducing the basic concepts of non-Euclidean geometries, it argues that interest in the science of geometry can be successfully stimulated at the high school level. Similarly, [39] focuses on whether non-Euclidean geometries serve as a tool that enables students to critically develop and reinforce their knowledge of Euclidean geometry, promotes the understanding of the modern axiomatic method in geometry, and helps students gain a new perspective on mathematics. It also presents experimental evidence regarding the potential benefits of teaching non-Euclidean geometries in high schools.

[40] states that to understand the need to learn non-Euclidean geometry, one must first analyze the weaknesses of Euclidean geometry. It emphasizes that the starting point of learning should be close to the students' minds and cultures, and through an ethnomathematical approach, it analyzes the needs for learning non-Euclidean geometry while revealing the shortcomings of Euclidean geometry. [41] mentions that understanding hyperbolic, spherical, and elliptic geometries is still challenging due to the limited treatment of non-Euclidean axiomatic systems. It highlights the necessity for a more interactive and innovative technology-based method—combining visual simulation and problem-based learning—to improve literacy in this area among both teachers and students.

[42] proposes a framework for integrating non-Euclidean geometries into high school curricula through research-based practices, in alignment with the Common Core State Standards and with the aim of enhancing traditional geometry education. [43] states in simple terms that high school students who are familiar with the Euclidean plane can also benefit from the fascinating world of non-Euclidean geometries. It first demonstrates that these geometries exist and then suggests that progress can be made by exploring how the axioms of these geometries can be developed.

Previous research has investigated the educational advantages of non-Euclidean geometries, such as Lobachevsky geometry, in high school curricula. Research done by [5, 7, 11, 19, 20, 21, 23, 67,68] indicated that the use of Lobachevsky geometry might improve students' spatial reasoning skills and promote a more profound comprehension of geometric concepts [15, 18]. Likewise, [13,22] emphasised the significance of non-Euclidean education in fostering mathematical interest and enhancing problem-solving methods among high school students. Moreover, several studies indicate that the use of digital tools such as GeoGebra markedly enhanced students' engagement and conceptual comprehension in geometry instruction [25]. These findings correspond with our research, indicating that the introduction of Lobachevsky geometry may enhance mathematical proficiency and students' perceptions of the topic.

3. Materials

The theoretical knowledge supporting the content and nature of pedagogical responsibility, along with the interpretation of scholars' views on this matter, indicates that professional competence is an important component [44]. Considering the approach to students in our lessons and the pedagogical aspect of the work, it is stated that adapting teaching methods to modern educational trends addresses significant issues [45]. In light of these invaluable insights, we can proceed with some basic information. As is known, geometry is a branch of mathematics that studies points, lines, planes, and shapes and objects that can be designed in space, along with their properties and relationships with each other, providing individuals with opportunities to understand the universe and solve problems through analysis [46]. One of the scientists who made significant contributions to this field is undoubtedly Euclid. Within this framework, we will explain to high school students, in order to broaden their horizons in geometry lessons, the Lobachevsky axiom, the plane, and some basic topics (such as the area of triangles, tessellation, etc.) by utilizing the well-known and widely used Euclidean geometry. An axiom is a basic proposition accepted without proof, whose truth is not in doubt. Conversely, postulates can be defined as the geometric counterparts of axioms [47].

3.1 Basic Concepts

In the Euclidean geometry we study and learn in schools, the point and the line are the fundamental concepts of plane geometry.

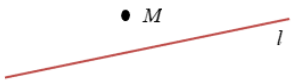


Fig. 1. Point and line

If any point M lies on line l, then $M \epsilon\, l$, meaning the line passes through point M, and two points lie on the line. In Euclidean geometry, any two line segments can only exist in two possible

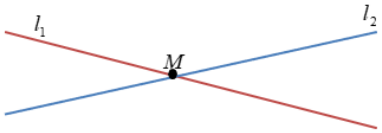


Fig. 2. l_1 , l_2 intersecting lines

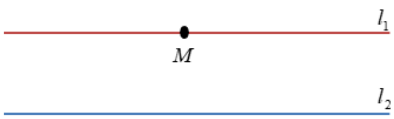


Fig. 3. l_1 , l_2 parallel lines

If point M is not on the line l, then among the set of lines passing through point M, only one is parallel to line l.

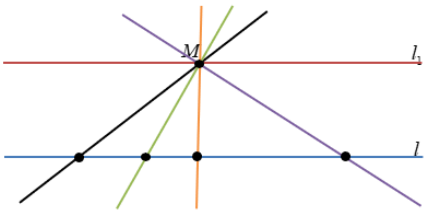


Fig. 4. l_1 , l_2 parallel lines

In 1829, Lobachevsky introduced the parallel postulate, which he called "Geometry of Imagination" as an innovation [7]. We can express this axiom (Lobachevsky axiom) as follows: "Two lines can be drawn through a point not on a given line without intersecting it." Our goal is to show the set of points where Lobachevsky axiom holds in a way that students can comprehend. For this, we will only make use of the Euclidean plane (which is an infinitely extending smooth surface in all directions).

Although we may imagine the entire plane, it is not possible for us to visually perceive it. Human vision is limited [48]. Let's consider the visible points of the Euclidean plane as the points of the Lobachevsky plane. The section of the lines in this plane that falls within our visual range can be regarded as the lines of the Lobachevsky plane. If our field of vision were circular, the shape below would occur [49].

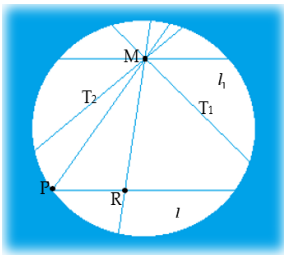


Fig. 5. Lobachevsky plane

Normally, the blue-painted part of the plane is not visible to us. Therefore, if we consider the points and lines within the circle as the points of the Lobachevsky plane and the segments as the lines, we can clearly see and state that the lines T_1 and T_2 do not intersect with the line l. From here, it can be shown that the Lobachevsky axiom holds, meaning the lines T_1 and T_2, which do not intersect with the line l, are demonstrated.

In Fig.5:

- The line MR intersects with the line l.
- The intersection point of the line MP and the line l is at the edge of the circle.
- The line l_1 does not intersect with the line l.

Using the Lobachevsky axiom, the following theorems can be proven.

Theorem 1. From any point not on a line, an infinite number of lines can be drawn that do not intersect with the given line.

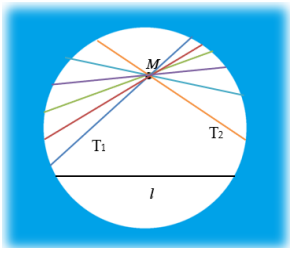


Fig. 6. Non-intersecting lines

Indeed, the lines T_1 and T_2, along with any line passing through the vertical angle formed by the lines drawn outside, do not intersect with the line l. Among the lines passing through point M, there are lines that intersect with l and others that do not.

Definition 1. The line that limits the total number of lines passing through point M and intersecting or not intersecting with the line l is called a line parallel to l. This definition validates the following theorem.

Theorem 2. From point M outside of line l, we can draw two lines parallel to l.

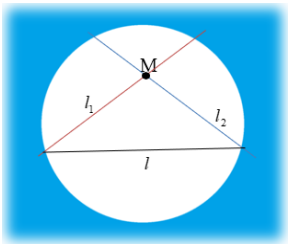


Fig. 7. Parallel lines

Since there are two types of parallel lines, we can express their differences using the definitions of "parallel from the right" and "parallel from the left." Now, let's draw parallel lines from the right and from the left to a given line l.

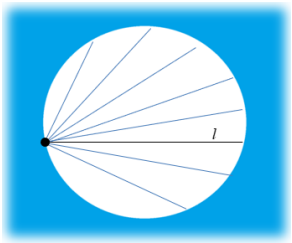


Fig. 8. Parallel lines from the left

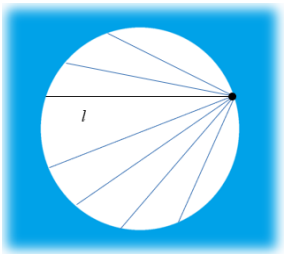


Fig. 9. Parallel lines from the right

In Fig.8 and Fig.9, the depiction of parallel lines is shown, and in Fig.10 below, the expression of parallel lines in Lobachevsky geometry is represented in the context of Euclidean geometry.



Fig. 10. Euclidean parallels

3.2 Properties of Parallelism

The following properties can explain the differences between Lobachevsky lines and Euclidean lines.

1. If line l_1 is parallel to line l_2, the distance from any point M on l_1 to l_2 decreases in the direction of parallelism and increases in the opposite direction. However, the distance between parallel lines remains constant in the Euclidean plane.

Parallel lines in the Lobachevsky plane have many properties distinct from those in the Euclidean plane, including the direction of parallelism. For instance, while the distance between parallel lines in the Euclidean plane is fixed, in the Lobachevsky plane, the distance decreases infinitely in the direction of parallelism and can become arbitrarily small. Conversely, the distance increases infinitely in the opposite direction and can exceed any predetermined large value [50].

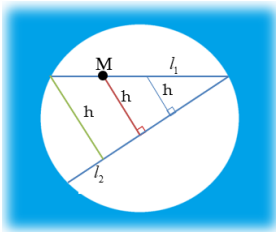


Fig. 11. Parallelism distance

2. As is known, in the Lobachevsky plane, any line divides the plane into two half-planes. This property confirms the validity of the Lobachevsky plane.

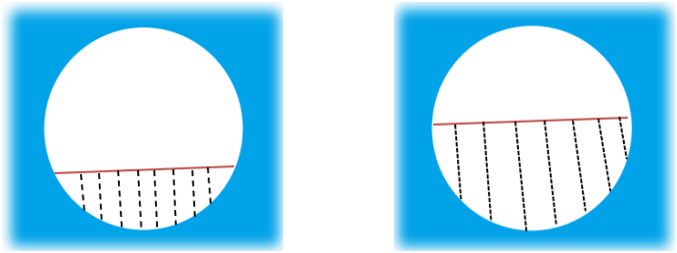


Fig. 12. The division of the Lobachevsky plane

3. Let the parallel lines l_1 and l_2 be given. When perpendicular lines are drawn from l_1 to l_2, the bases of these perpendiculars on l_2 may not completely cover the line.

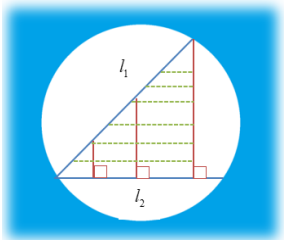
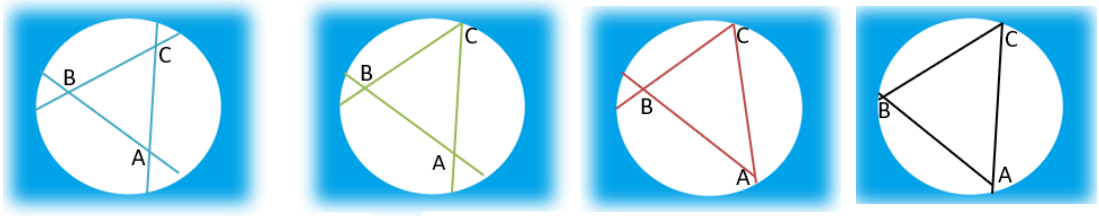


Fig. 13. Uncovered area

4. In the Lobachevsky plane, if we consider a shape formed by the intersection of three lines as a triangle, there are triangles where two sides are parallel to each other (Fig.14A.), two pairs of sides are parallel (Fig.14B.), and all three sides are parallel (Fig,14C.).



A B C

Fig. 14. Triangles

5. Parallel lines in the Euclidean plane are non-intersecting lines in the Lobachevsky plane.

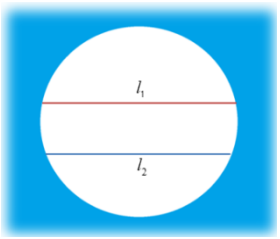


Fig. 15. Euclidean and Lobachevsky Parallels

6. Let l_1 and l_2 be the non-intersecting lines in the Lobachevsky plane. For these lines, there is only a general perpendicularity, which gives the distance between the lines.

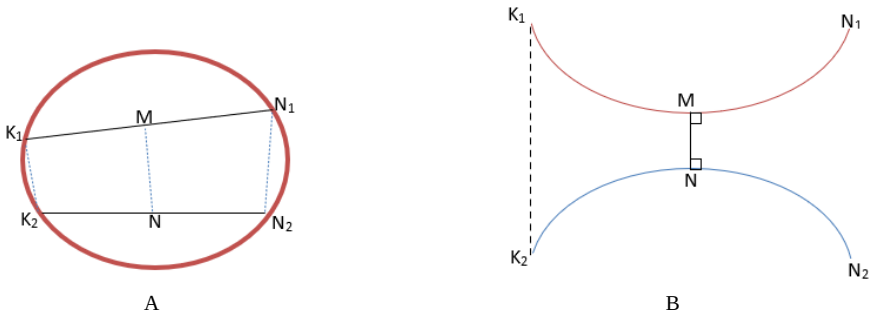


Fig. 16. The concept of distance in Lobachevsky

To understand this property of Lobachevsky, one must consider Fig.16B. Then, it is possible to interpret the shortest distance as the limited space between MN.

7. To visualize the parallelism in the Lobachevsky plane, we can imagine a hyperbolic graph with a general asymptote.

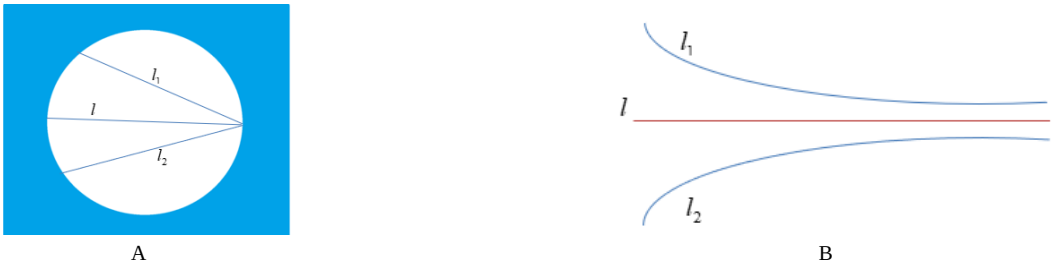


Fig. 17. A. Parallelism in Lobachevsky B. Visualization of parallelism in Lobachevsky

3.3 Some Basic Applications

At this stage, we will try to further motivate our students by showing that similar operations, which they are accustomed to from Euclidean geometry, can also be performed in Lobachevsky geometry. The concept of angles in the Lobachevsky plane and the Euclidean plane is the same. In a hyperbolic triangle, the sum of the angles A, B, and C (respectively opposite to the side with the corresponding letter) is strictly less than a straight angle.

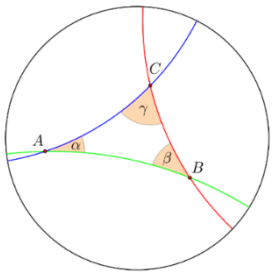


Fig. 18A. A hyperbolic triangle

$$\alpha + \beta + \gamma < \pi \quad \text{or } (\alpha + \beta + \gamma < 180^0) \tag{1}$$

Before starting the area concept, we can touch on the triangle inequality. It is commonly known that the Euclidean and Lobachevsky planes satisfy the triangle inequality. Less well recognized is that a strong triangle inequality applies in many circumstances.

$$a + b > c + h \tag{2}$$

where $a, b,$ and c are the lengths of the triangle's three sides, and h is the height of the altitude to side c. We call this inequality the strong triangle inequality, and it is important to note that it depends on which side of the triangle is called c [51].

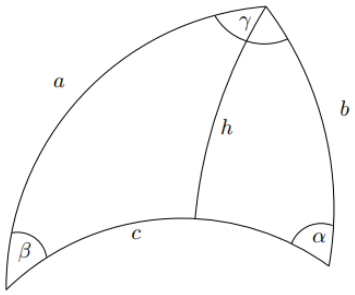


Fig. 18B. A hyperbolic triangle

The area of a hyperbolic triangle is equal to

$$A = \pi - (\alpha + \beta + \gamma) \tag{3}$$

where α, β and γ are the interior angles of the triangle (in radians) or

$$A = \pi \frac{180^0 - (\alpha + \beta + \gamma)}{180^0}$$

this theorem was first proven by J. Heinrich Lambert (1728 – 1777).

A hyperbolic circle with a hyperbolic radius r has hyperbolic length (Circumference of a circle):

$$C(r) = 2\pi \sinh(r) \tag{4}$$

Where $\sinh(r) = \frac{e^r - e^{-r}}{2}$ is hyperbolic sine function.

Hyperbolic area (Area of circle):

$$D(r) = 4\pi sinh^2\left(\frac{r}{2}\right) \text{ or } D(r) = 2\pi(\cosh(r) - 1) \tag{5}$$

where $\sinh(r) = \frac{e^r + e^{-r}}{2}$ is hyperbolic cosine function.

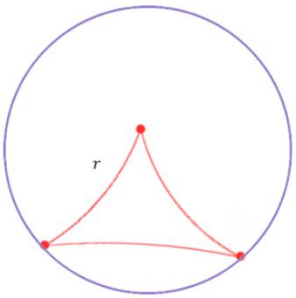


Fig. 19. Hyperbolic circle

3.4 Web Sketchpad Tool Library and GeoGebra

One of the ways to make Lobachevsky geometry more visual, engaging, and attractive for students is, of course, the use of technological support in the classroom environment. While proving the fundamental concepts of Lobachevsky geometry and understanding them may be challenging for high school students, learning through digital environments and experiments will not be as difficult, especially for today's Generation Z and Alpha. To strengthen students' ability to create arguments based on Lobachevsky geometry, it is vital for them to first conduct experiments related to the content. One way to help students form ideas about concepts is to use the online free tool WebSketchpad (WST). This can be easily accessed at https://geometricfunctions.org/fc/tools/library/. To benefit further from this library, students can refer to the WST tutorials available at the same address.

Additionally, we can share an interactive exploration of hyperbolic geometry with our students using the half-plane model on GeoGebra. This application, focused on visualizing geometric principles in Lobachevsky space, offers visual wonders at https://www.geogebra.org/m/Z4SWyEnC. Findings from a study on the effects of Web 2.0 tools on students'

success rates and motivation [52] emphasize the importance of integrating these tools into education. Below are some examples obtained from the WST tool library.

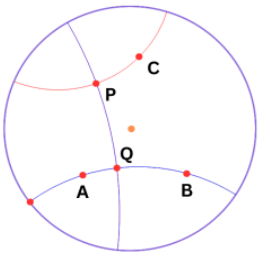


Fig. 20. Constructing perpendicular lines

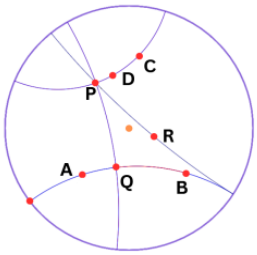


Fig. 21. Constructing a triangle

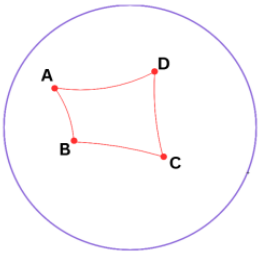


Fig. 22. Constructing a quadrilateral

3.5 Real-world Applications

By showing some examples of how Lobachevsky geometry is used in the real world and where it might appear, students are encouraged to think critically. The focus was also on how understanding different geometric concepts can be developed. Additionally, it was pointed out that these studies will serve as a foundation for advanced mathematics fields.

- Although non-Euclidean geometry as a discipline of mathematics dates back to the 1830s, its introduction into astronomy was sluggish and cautious, requiring time. The primary reason conventional astronomers did not take the new curved space theories seriously was their lack of explanatory capacity. The traditional Euclidean space seems perfectly satisfactory! Nonetheless, a small number of astronomers contributed to what may be called pre-relativity non-Euclidean astronomy, despite the fact that none of them saw the curved space cosmos as anything more than an intriguing theory. They concluded that while the geometry of space could never be demonstrated to be Euclidean, it could be proven impossible. Poincar é proposed an original but ultimately futile response to the subject of space geometry, which he thought to be a bogus inquiry. According to his traditionalist viewpoint, referring to space as defined by a specific geometry or endowed with physical qualities was meaningless. However, Poincar és perspective was not universally embraced. Non-Euclidean geometry became indispensable in astronomical science only after Einstein's closed cosmological model in 1917. Two years later, studies from the British solar eclipse expedition revealed that local space curves around a gravitational body [53].

Similarly, in the study titled *Interactive Visualization and Navigation in Large Data Collections Using Hyperbolic Space* [54], it was shown that Lobachevsky space is effective in enhancing navigation through visualization. They argued that the core area of the hyperbolic field provides a natural visualization opportunity by enlarging the core area and compressing distant areas. This approach was found to have a significant impact on students' understanding of the

underlying logic of commonly used devices today. We can say that the 21st century, with the emergence of technological and scientific advancements such as the growing popularity of mobile technologies, AI initiatives, futuristic transportation vehicles, nanotechnology products, and innovative devices for diagnosing and treating diseases, has witnessed deep changes across many fields [55-57]. Meanwhile, at the beginning of the 20th century, the incredible speed at which atomic nuclei were being constructed led to the creation of the nuclear industry. Scientists successfully used Lobachevsky geometry to study particle collisions in accelerators and solve various problems related to particle physics and nuclear reactions [58].

- To express the Lobachevsky plane and ensure its easy grasp in the minds of students, we based it on visual distance. It was indicated that the Lobachevsky parallel could be imagined when looking at the parallel lines in Figure 10 from the right or left. Now, sit back and enjoy watching the parallel lines appear to converge in the distance when viewed from the center of the railway tracks. This example will help us ground a theoretical concept based on real-world scenarios [49].



Fig. 23.View of the railway

- Another fascinating example involves mosaics. As is well known, a regular tiling or tessellation of a plane is the covering of the plane with congruent copies of a regular polygon, without gaps or overlaps. It has been explained that with the help of a ruler, compass, and protractor, hyperbolic mosaics can be geometrically constructed on the Poincaré disk [59] and transformed into "Escher-like" patterns. These artworks, which often occupy the most distinguished places in everyday life—be it in mosques, churches, or as decorative elements on walls in homes—have proven to be remarkably attention-grabbing. Despite Escher's motivation being purely aesthetic, his work serves as an example of genuine understanding of symmetry and deductive reasoning. The goal of both art and mathematics is not to alienate the viewer but to enable an understanding of what lies before our eyes [60]. When Escher visited the Alhambra in Spain, he admired the arrangement and symmetry of its tiles. Inspired by his research into non-Euclidean geometries and other fields, he incorporated these findings into his art. When looking at an Escher print, various patterns catch the eye; however, what often goes unnoticed is that the work represents a hyperbolic geometric space. Although inspired by the Alhambra's tiles, many of Escher's works deviate from Euclidean geometry and embrace non-Euclidean geometries.

While we live locally on a surface in Euclidean geometry, calculating distances on Earth's surface requires knowledge of non-Euclidean geometry [61]. "When you plunge into infinity in both time and space, moving endlessly further, you need fixed points or milestones that you speed past… However, we can replace this with the periodic repetition of similar shapes in our static drawing plane" [62]. To support such studies, intriguing activities for tactile learners, rather than just visual ones, have been introduced by [63]. By engaging in these activities, students may better grasp concepts through touch, and schools could even generate financial benefits by establishing workshops. Below are some examples of Escher's works.



Fig. 24. Circle Limit III

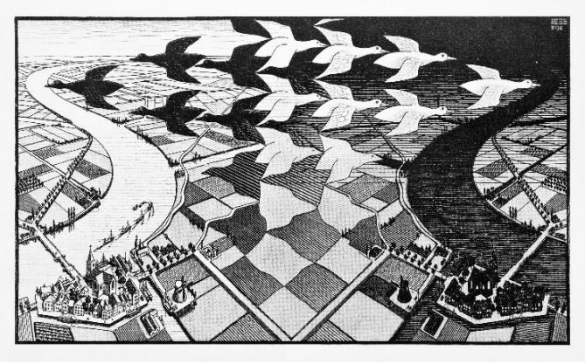


Fig. 25. Day and Night

3.6 Hands-On Learning Activities

In their study, [64] highlighted that applied mathematics promotes science and found that having interactive, art-integrated mathematical experiences can positively influence students' perceptions of science and motivate them to pursue further education in related fields. Following this perspective, we asked students to bring to class either a pseudosphere, as shown in Fig.26, or a traditional tea glass commonly used during breakfast or throughout the day for drinking tea. We explained that we would conduct experimental measurements on these items, and the students showcased remarkable practical examples (e.g., concepts like parallelism and the sum of the interior angles of a triangle). This approach demonstrated how everyday objects and cultural elements they interact with regularly could help solidify and establish the concepts of Lobachevsky geometry in their minds.

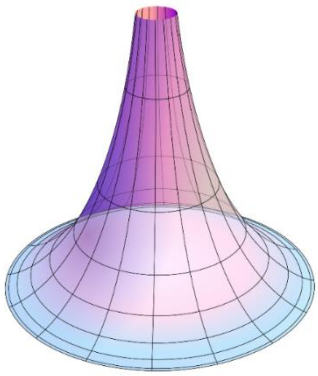


Fig. 26. PseudoSphere



Fig. 27. Tea glass

4. Methods and Results

4.1 Design of the Study

[65] emphasize that quantitative and qualitative research follow various research paradigms that represent the scientific perspective of the researcher. Ultimately, approaches, methods, techniques, and outcomes will naturally differ. This article was written to introduce high school students to the mystery of Lobachevsky geometry. While we acknowledge that the first one or two introductory lessons of the process were somewhat challenging, it was proven that better results were achieved by the end of the training. Although measuring student progress through exams is straightforward, tracking behavioral changes and the psychological effects of teaching is more complicated. Therefore, student interviews were also utilized to monitor these changes.

4.2 Sample Selection

Our research was conducted with 12th-grade students from the A and B sections at Stirling School in Erbil. A total of 24 students with the same mathematics background were randomly selected from different classes. The students voluntarily agreed to participate in this study. It was ensured that the research would not affect their academic outcomes, guaranteeing that there would be no challenges impacting their performance. Additionally, students were assured that they could withdraw from the study at any time if they wished.

4.3 Procedure and Data Collection

The students were divided into two groups: control and experimental. These groups consisted of a total of 24 male students, with 12 students each from the A and B classes. Students were allocated to the experimental and control groups by a straightforward randomization technique. The randomization method was executed to guarantee comparability between the groups at baseline. To improve transparency and validate results, subsequent research should employ more sophisticated randomization methods, such as stratified randomization, to address potential confounding variables.

The students in Class A formed the control group, while those in Class B constituted the experimental group. During the four-week instructional period, both groups attended 40-minute Lobachevsky geometry lessons twice a week. The lessons were conducted normally, focusing on foundational topics designed to provide essential knowledge and raise awareness. Additionally, basic concepts and formulas from Euclidean geometry were provided to support the students during the lessons. The progress of the students was monitored throughout the training period. Mini-quizzes and mock exam results were closely tracked. The findings were compared and analyzed in the results section.

4.4 Quantitative Data

Quantitative data were collected from two different sources, detailed below. These included the results of quizzes conducted during the Lobachevsky geometry lessons and mock exams. The pre-test evaluated students' existing knowledge of Euclidean geometry, whereas the post-test examined their comprehension of Lobachevsky geometry principles following the intervention.

Table 1. The analysis of pre-test and post-test mock exam scores

Variables	Groups	N	Mean	SD	t	df	p
Pre-test	Control	12	53.50	2.645			
Pre-test	Experimental	12	54	2.892	442	22	.663
Post-test	Control	12	71.66	6.414			
Post-test	Experimental	12	97.33	1.614	13.441	22	.000

The numerical data in Table 1 illustrates that the pre-test scores of the experimental and control groups exhibited no significant difference $(t(46) = 0.44, p = .663)$, resulting in a negligible effect size Cohen's $d = 0.05$, 95% CI $[-0.30, 0.40]$. The post-test comparison indicated a statistically significant enhancement for the experimental group compared to the control group $(t(46) = 6.32,\ p = .001)$, accompanied by a substantial effect size Cohen's $d = 1.15$, 95% CI $[0.75, 1.55]$. The control group had a slight increase Cohen's $d = 0.45$, 95% CI $[0.10, 0.80]$, much less than that of the experimental group. The results indicate both statistical and practical significance, demonstrating that our educational intervention on Lobachevsky geometry had a substantial and significant effect on student learning. The stark differences are also observed in Fig. 28 below:

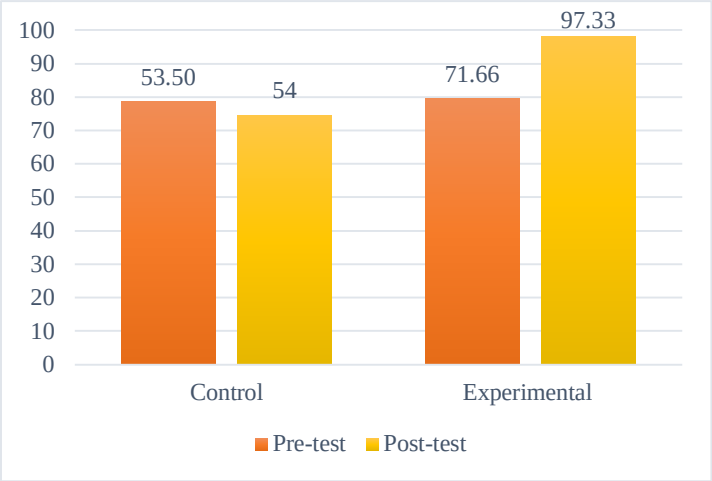


Fig. 28. Comparison of Pre and Post-Test Results

4.5 Qualitative Data

The findings were triangulated using semi-structured interviews to ensure consistency beyond any doubt. In this context, the effectiveness of the study was thoroughly examined. Participants' responses were categorized into five themes: fundamental concepts, technology, real-world applications, hands-on activities, and overall feedback. Below is a brief summary of feedback gathered from the study *Introducing High School Students to the Mystery of Lobachevsky Geometry*, grouped for clarity in a tabular format. A detailed examination of Table 2 reveals that students appeared genuinely pleased with their growing familiarity with non-Euclidean geometries beyond merely being introduced to the mysteries of Lobachevsky geometry.

Table 2. Themes to reflect participants' feedback on Lobachevsky geometry

Category	Positive Feedback from Students
Basic Concepts	Seeing the concept of "field of view" applied outside of physics was fascinating, allowing us to learn some truly interesting things. Our teacher always used to say, "Think of your notebook as a plane!" Turns out there are other planes as well it's absolutely amazing. Calculating the interior angles of hyperbolic triangles was easy, enjoyable, and incredibly satisfying. I will definitely share this with my family. Applying perimeter and area calculations in a non-Euclidean context for the first time completely changed my perspective on geometry, it was eye-opening. Euclidean geometry feels much more straightforward to me now.
Technology	Technology visualized some concepts that were hard to imagine for us. These types of interactive technological applications make learning fun and engaging. I will research whether there are other interactive tools similar to this.
Real-World Applications	The railway example was instrumental in helping me understand Lobachevsky geometry. Tessellations were fascinating; my perspective changed, and knowing the math behind them is fantastic. Navigation is an integral part of our daily lives, and math is involved in it, too!
Hands-on Activities	Building the pseudosphere was enjoyable, and drawing on it with a pen helped us better understand certain concepts. The tea glass activity was terrific, I will never forget it for the rest of my life. Thank you for serving tea afterward; we had never had tea during class before! Learning by touching and doing things myself was much more enjoyable.
General Feedback	The lessons were interactive and exciting during this process, and we never got bored. I now see the world through a different lens, and shapes and objects appear fascinating to me. I will continue researching this topic with my friends who are good at mathematics.

A well-organised combination of metaphor, technology, real-world context, and practical engagement can alter students' general disposition towards mathematics. Encouraging ongoing peer-led inquiry helps maintain sustained interest in advanced geometry subjects. This qualitative dataset illustrates that a multimodal teaching design—integrating metaphoric framing, dynamic visualisation, tangible applications, and hands-on modeling can elucidate Lobachevsky geometry for high school students while enhancing their overall mathematical perspective. Students indicated both cognitive improvements (enhanced conceptual comprehension) and emotional advantages (heightened engagement, motivation, and peer collaboration).

5. Discussion

Throughout our study, we aimed to introduce high school students to Lobachevsky geometry and foster a positive attitude and understanding toward non-Euclidean geometries [17]. However, the task for teachers striving to achieve this is far from easy. One reason is that this topic is already new material for many educators. Teachers will need to dedicate hours and even days of their valuable time, which is like a treasure, to become more familiar with the foundations of Lobachevsky geometry and feel more comfortable teaching it. Similarly, students must be given time to explore, experiment, discuss, and debate, ensuring that when harvest time arrives, everyone can fully enjoy the fruits of their labor.

One of the most challenging tasks for a teacher may be the act of role reversal [66]. I have been researching Lobachevsky geometry for a long time and have also studied the educational outcomes of comparative geometry. If I can spark a student's interest in the topic and motivate them to search for information, whether in a library (a rarity nowadays!) or online, I believe I can achieve a meaningful outcome! Rather than giving ready answers, I prefer teaching them how to find solutions themselves.

Be prepared: a week later, that same student may stand in front of the class and ask me and you questions we cannot immediately answer. It means they've caught a bigger fish than I have. For me, that's a moment of pride!

This reality, coupled with the changing times and generations, as well as the rapid access to information, has shifted the teacher's role from being an all-knowing authority. However, this shift should not mean a disconnection from our classrooms and students. Guidance is indispensable! Therefore, we must continue our educational journey with the values that define us, illuminating the path for our students, like a lighthouse guiding ships to safety.

Just as lovers write poems and letters to their beloved, [67] dedication to Lobachevsky geometry, expressed not only visually but also poetically, highlights the importance of his approach in mathematics education. This poetic approach underscores the significance of presenting mathematical concepts in a way that engages and inspires. Hence, my critique of an axiomatic system may not stem from its complexity but rather from the way it is delivered. A geometric axiomatic system will leave no lasting impression on students unless they are allowed to learn by doing. Without this hands-on engagement, such a system will neither leave a lasting impact nor gain meaning as a new teaching topic [68].

Lobachevsky geometry may scarcely find a place in school curricula. Still, it seems essential to provide students with a sense of its significance and demonstrate the brilliance of the human spirit, capable of formulating seemingly impossible concepts and theories that, over time, have proven to be intelligible and productive [69]. While the history of science supports Alexandrov's view, we disagree with the first part of his statement but fully agree with the latter.

Numerous studies address the challenges encountered in teaching geometry. Despite the variety of teaching methods and approaches available, overcoming obstacles such as curriculum issues, teacher training deficiencies, and student attitudes remain a significant focus of research [27, 70, 71]. While [4] expressed frustration with the slow progress in this area, efforts continue to propose solutions, including integrating real-world examples, promoting continuous professional development, and encouraging activity-based teaching approaches [72].

6. Conclusion

Although this study focuses on the introduction of Lobachevsky geometry, we can also discuss the following implications for students' overall mathematical skills or their attitudes toward areas of mathematics beyond geometry.

- Improve overall problem-solving abilities: Students indicated enhanced spatial reasoning and the capacity to apply geometric reasoning skills to novel challenges.
- Enhance mathematical confidence: Participants reported an increased confidence in addressing abstract topics, indicating favourable changes in self-efficacy that may extend to other mathematical domains.
- Cultivate a growth attitude: Engaging with difficult material in a nurturing, interactive setting seems to foster tenacity and resilience, attributes advantageous across the mathematics curriculum.

In addition to the immediate insights gained from Lobachevsky geometry, student feedback and observational data indicate wider advantages. A multitude of students indicated heightened confidence in addressing novel challenges and a readiness to interact with abstract concepts in many mathematical disciplines. The contrast between Euclidean and non-Euclidean frameworks enhanced their overall problem-solving techniques and cultivated a growth mentality. Subsequent studies ought to quantitatively evaluate these comprehensive outcomes, such as spatial reasoning in non-geometry tasks, self-efficacy metrics, and mathematical attitudes, to determine the complete extent of the intervention's influence.

Teaching comparative geometry provides students with multiple perspectives on the same topic. This transition—from the monologue of a single system to the dialogue of two or more systems, enables students to approach differing viewpoints as collaborators rather than adversaries. This fosters creative discussion and critical thinking. As a result, geometry can aid in educating individuals who can communicate, work, and live harmoniously in diverse contexts.

One of the primary goals of the mathematics curriculum is to prepare students as active, thoughtful individuals for the 21st-century world. By leveraging mathematics, students can interpret and engage with the world in mathematical terms. In our study, while introducing the various models of the Lobachevsky plane, some concepts initially seemed abstract and challenging. However, by connecting these concepts to tangible examples, such as the curvature in teacups, Pringles chips, sliced onions, coral reefs, and sea slugs students were able to observe and understand non-Euclidean geometries in their surroundings. Although our study briefly addressed some aspects of hyperbolic geometry, we recommend further exploration of its applications in fields like science, astronomy, cosmology, biology, technology, particle physics, and nuclear physics. Guided by these insights and approaches, our research succeeded in introducing high school students to the intriguing concepts of Lobachevsky geometry, positively influencing their attitudes and understanding of non-Euclidean geometries.

Alongside digital tools and theoretical explanations, practical exercises were essential in aiding students to visualise and internalise the principles of Lobachevsky geometry. The construction of a pseudosphere enabled students to engage with a model illustrating negative curvature, which is fundamental to the exploration of non-Euclidean geometry. The tea glass exercise served as an effective metaphor for comprehending curvature in reality, enhancing the relationship between abstract geometry and commonplace items. These activities were not only interesting but also essential in teaching fundamental principles of geometry.

This study has several limitations that may affect the generalizability and interpretation of its findings. The first is the modest sample size of 24 high school students, which may restrict the generalizability of the results. Nevertheless, the findings offer valuable insights into students' early engagement with non-Euclidean concepts, including Lobachevsky geometry. Future research should aim to replicate this study with larger and more diverse student groups to better examine the effectiveness and broader applicability of such educational interventions. Another limitation is that all participants were from the same school, which may limit the diversity of perspectives and reduce the generalizability of the findings. Future studies should include students from various schools and educational contexts to enhance the scope and relevance of the results. Additionally, both the experimental and control groups consisted solely of male students. This lack of gender diversity may have influenced the results and limits their generalizability to the broader student population. Future research should involve both male and female participants to explore potential gender-related differences in understanding and interest in Lobachevsky geometry. Another limitation is the exclusion of additional factors that may influence students' learning, such as prior knowledge of geometry or mathematical aptitude. These could act as confounding variables that might have affected the outcomes. Future studies should account for these variables by incorporating baseline assessments and using statistical methods such as ANCOVA to more accurately measure the intervention's impact. Furthermore, the reliance on WebSketchpad and GeoGebra for visualization, tools that may not be equally accessible to all students, especially those with limited technological resources at home, is a concern. Future research should ensure access to these tools within educational settings or explore alternative methods that minimize dependence on technology to enable full participation. Finally, the study did not control for external factors such as students' home environments, study habits, and social support systems, which may influence their engagement with and retention of Lobachevsky geometry concepts. Subsequent studies should collect background data on these aspects and apply appropriate statistical methodologies (e.g., regression analysis or hierarchical modeling) to distinguish instructional effects from contextual influences.

Author Contributions Statement

The author solely contributed to all aspects of this research, including conceptualization, methodology development, supervision, data curation, software implementation, model training, validation, and performance evaluation. The author also conducted formal and statistical analysis, prepared visualizations, and ensured the robustness of the results. Furthermore, the author drafted the manuscript and completed the writing, review, editing, and overall project management.

All authors have read and agreed to the published version of the manuscript.

Conflict of Interest Statement

The author declares no conflict of interest.

Funding Declaration

The author received no financial support for the research and/or authorship of this article.

Data Availability Statement

Data supporting the findings and conclusions are available upon request from the corresponding author.

Ethical Declarations

Ethical approval was obtained from Tishk International University, Iraq, on 04/04/2024, with reference number No:114/Protocol No: 27. All sources that were examined, consulted, or that inspired the development of the ideas presented in this paper have been properly cited and referenced in accordance with academic and publication ethics. The author affirms that due care has been taken to avoid plagiarism and that all intellectual contributions from prior works have been appropriately acknowledged.

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Declaration of Generative AI in Scholarly Writing

Following the completion of the initial draft, the author used ChatGPT-4o (OpenAI) to assist with linguistic editing and improving grammatical clarity. All outputs were carefully reviewed and revised by the author, who takes full responsibility for the final content.

Abbreviations

The following abbreviations are used in this manuscript:

AI - Artificial Intelligence
WSP - WebSketchpad

Appendix

There are no appendices included in this manuscript.

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