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Mathematical Based Implicit and Explicit Finite Difference Techniques for Solving the Ground Water Flow Equations Using Spreadsheets

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Abstract: In countries with arid and semi-arid climate such as Iran with water constraints, the use of groundwater resources is very important. There are various mathematical based methods and software packages for modelling groundwater resources. This paper uses groundwater flow problems to illustrate possible approaches for providing the environment of active teaching. Mathematical models supported by software applications facilitate the gaining of an insight into the physical behaviors by investigating a host of scenarios and events but they are poor in training critical thinking for encapsulating the hardcore mathematical equations describing the problems. Whilst software engineering has transformed the intellectual capitals accumulated between the 20th century and the middle of the 21th century into working tools, it has the drawback of encapsulating core mathematics away from common experience of the students and practitioners. This diminishes critical thinking in a world of increasing risks and ought to be taken a serious side effect of software engineering. This paper suggests a solution by building up a library of solvers using spreadsheets, with the effect that the encapsulated knowledge of building modelling solvers can permanently be brought to life in education with the active learning culture. Implementation was carried out in the same way for steady state flow as well as explicit 2D and 3D finite difference approximation for transient flow. This study raises concern about the encapsulated body of knowledge contributed to the emergence and the establishment of modelling software applications since 1980. This body of knowledge comprises a deeper understanding of equations of often partial differential equations describing physical problems, as well as their numerical transformation into systems of equations and their subsequent properly- and improperly posed systems of equation in terms of their assumptions and quality conditions. The outcome is the emergence of a cookbook mentality among the new breed of mathematical modelers without any critical thinking. The results revealed that spreadsheet can be used with the aid of the Solver function. This idea capitalized on the capabilities of the net-generation and opens up the possibility for the emergence of bottom-up open source modelling platforms.

Index Terms: Critical thinking; explicit; groundwater modelling; finite difference; implicit; spreadsheet

1. Introduction

Gaining an insight into reality and invoking critical thinking in the learners mind should be considered as two of important pedagogical issues, particularly in a world with increasing risks. Mathematical models supported by software applications facilitate the gaining of an insight into the physical behaviors by investigating a host of scenarios and events but they are poor in training critical thinking for encapsulating the hardcore mathematical equations describing the problems. Arguably, an insight is a comprehensive understanding within a given framework of mind. Among different definitions of critical thinking, Scriven and Paul [1] define it as skilfully conceptualizing, applying, analyzing, synthesizing, and/or evaluating information as a guide to belief and action.

This paper takes critical thinking as the ability to search and shift from one reasoned framework of mind to another. This paper views both insight and critical thinking as complementary, where one does not replace the other. Although modelling has become indispensable, the emerging modelling practices nurture the cookbook mentality among the practitioners and students. Arguably, this is not healthy and therefore measures are needed to invoke and train critical thinking among the students (Kyrpychenko et al. [2]).

One way of invoking critical thinking in teaching mathematical modelling is presented by using examples from groundwater modelling. Groundwater flow problems are governed by partial differential parabolic equations, the solutions of which are currently by numerical methods. Software engineering has been effective in transforming the solution capabilities into versatile tools but their solvers often tend to comprise of a single solution method, as their productions require considerable investments. Models built in one modelling platform are not often transferrable to other platforms. Thus, the learners cannot train his critical thinking by trying different solution approaches but this is feasible in the spreadsheet environment.

This paper seeks spreadsheet programming as a convenient environment for the training of critical thinking in the researchers and the student. Simplified groundwater problems are used to obtain solutions of the governing equations without involving any particular programming like MATLAB or FORTRAN. It shows one way to effective students' learning by simplifying the underlying mathematics through dropping out tedious components in favor of arriving at sensible solutions. In this way, it becomes feasible (i) to focus on flow patterns of practical problems caused by different boundary conditions by inserting a sink (well) or source (recharge) on flow domain, (ii) to assessing their effects on water table or sensitivity against hydraulic parameters of transitivity (T) or storivity (S), or (iii) trying out different solution schemes to invoke critical thinking.

Learning and teaching are evolving. With a focus on recent situations Jewell [3] argues that in an applied science, there exists a natural tension between the study of fundamental scientific theory and instruction in the application of analysis and design methodologies within undergraduate engineering curricula. Most engineering courses are structured to emphasize the relevant physical, chemical, and biological processes that are then reinforced by studying specific problem solving skills applied to systems of engineering interest. Over the years, a wide range of courses of thought has emerged comprising theories, concepts or philosophically-oriented doctrines.

If there can be a consensus on the most effective way of learning and teaching, it is the one that there is no such a single theory. Nonetheless, each of these methods has a positive contribution but they also tend to overlook the central themes promoted by other theories for being unable to embrace pluralism in their methodology. This is because they all focus on finding the one best technique but this is just a cultural illusion and a legacy of philosophical doctrines. Arguably, the best technique is pluralistic integrating all of these techniques on their positive contributions and wary of their inter-conflicts. Pluralism is feasible by evolutionary thinking but this is not the focus of this paper.

Whilst using modelling software applications can train the user's ability on gaining an insight into physical problems, their limitations in training alternative techniques is obvious; spreadsheets overturns this limitation by enabling the users to test different solvers and create new solvers, albeit with a limited capability, as discussed below.

Spreadsheets were originally created to help accountants to solve their accounting problems. Due to their intrinsic ease-of-use, they have been extensively applied in many different fields [4,5]. They are also widely used as a training tool in engineering. For example, Excel spreadsheet has been used in simulating engineering systems, such as logic, networks, control systems and antenna array design [6]. Spreadsheet has also been used in the field of machine design, such as design of worm gear geometry, determination of spur gear form factors, aircraft structure analysis [7]. Karahan [8] studied one-dimensional advection–diffusion equation (ADE) with finite differences method using implicit spreadsheet simulation (ADEISS). Two examples which, have the numerical and analytical solutions in the literature, are solved in order to test the ADEISS performance. Spreadsheets are typically programmed in Visual Basic to customize their use for solving complicated engineering problems.

Dehghan and Mohammadi [9] used mesh less techniques and high dimensional partial differential equations to solve the tumor growth model. Also they used a semi-closed finite difference method based on the Crank-Nicolson scheme, and the other based on open Runge-Kutta time integration. Panigrahi and Velusamy [10] developed a transient, multi-phase enthalpy based computational model to analyze the streaming and freezing characteristics of the molten fuel in a blocked fuel subassembly (SA) during total flow blockage (TFB) at the foot of the SA. The model adopts both implicit and explicit type of finite difference techniques employing a variable grid system for boundary tracking. Macías-Diaz and Morales [11] used a stochastic differential equation in material refraction modelling. The developed model considered that the spread of cracks on solids includes a deterministic and stochastic component. Sattari et al. [12] predicted groundwater level using support vector regression (SVR) and M5 tree model in Ardebil Plain in Iran. They used monthly groundwater level data from 24 piezometers for a 17 - year period (1997 to 2013). SVR and M5 models predicted groundwater level with $R = 0.996$ and $R = 0.983$, respectively.

Modelling software applications have been integrated to engineering training in various ways and the challenge of its incorporation into the teaching curricula was recognized even in the early days. Cryer [13] argued in the past in favor of a productive role for scientific calculators, equation solvers, mathematics packages, spreadsheet applications, commercial analysis software, and programming assignments, but the selected tool depends on the course context and technology. Weiss and Gulliver [14] discussed the use of spreadsheets to analyze various hydraulic design projects. They illustrate the use of spreadsheets as a tool to analyze practical problems, not only to teach valuable engineering analysis skills but also to enhance users' computer skills for preparing them to the challenges to be faced professionally.

Huddleston [15] discussed the use of spreadsheet tools to introduce students to fundamental concepts of computational fluid dynamics by using an illustration from open-channel hydraulics. Huddlesston et al. [16] uses Excel illustrations to enable users' analysis while still requiring enough manual development to reinforce the underlying engineering principles. His argument is that analysis commonly results in nonlinear differential or algebraic equations or systems of equations but the users' computational capability is a limiting factor. According to him, this impedes learning but he suggests that it can be overcome through instructors' diligence by balancing the need to emphasize the engineering system physics versus numerical complexity using worksheet capabilities.

Pandit [17] provides several examples with application of spreadsheet in the areas of fluid mechanics, hydraulics, hydrology and storm water management in a workbook. Niazkar and Afzali [18] present two engineering examples in their paper using Excel spreadsheet. These examples are (i) earthquake data bases for computing peak ground acceleration (PGA) with highly nonlinear equations and (ii) the calculation of gradually varied flow condition in channels, in this Excel spreadsheet example, a first ordinary differential equation governs the value of water depth along the channel. In order to solve this equation, a finite difference scheme, e.g. Euler's method can be applied. Divayana et al. [19] evaluated a computer-based evaluation application called the CIPP-SAW.

This study examines the use of commonly available spreadsheet package to analyze groundwater flow problems to solve nonlinear systems of equations. The application of this technology is an efficient way to enable researchers to be engaged with solving relatively complex engineering problems while minimizing the computational burden. Built-in linear and nonlinear system functions are commonly available in commercial spreadsheet programs, providing an affordable alternative to more complicated or expensive software. These facilities enable researchers to analyze realistic applications while still requiring manual development of the governing equations to reinforce their underlying engineering principles.

The remain of the article is structured as follows: hydraulics of groundwater, parabolic groundwater equations, boundary conditions, spreadsheet implementation of groundwater problem, iterative calculations, explicit 2D finite difference approximation, explicit 3D finite difference approximation, implementation of the 2D explicit solver (steady state flow), implicit 2D finite difference approximation, discussion and conclusions.

2. Hydraulics of Groundwater

The customary research practice prior to 1950 was to seek analytical solutions for practical problems (i.e. to solve the equations for any point in space and time) by imposing a set of approximations but the advent of computational capabilities in the late 1950s and 1960s turned the tide and set the pace for the research efforts on establishing the proof-of-concept for nearly all of the equations available then by sweeping away these limiting assumptions. Furthermore, the advent of software engineering since the 1980s in the consultancy industry has transformed these proof-of-concepts into successful working tools.

Consider the example of transient groundwater flows, which are described by partial differential parabolic equations describing flows of a wide range of engineering problems. Their solutions depend on the specification of boundary conditions and initial conditions, as well as the specification of a number of parameters including storitivity and permeability. The goal of the solution is the determination of the free surface but the solution strategies have gone through paradigm shifts. In the past, it was customary to simplify the equations and solve them by analytical techniques. These techniques had restrictive assumptions and therefore their solutions were not of practical significance. The advent of computers since the 1950s and their applications to groundwater problems in the 1960s-1980s removed these restrictions but the emergence of software engineering since the 1980s has given rise to commercial and the public domain software applications.

The software applications often induce the cookbook mentality among the engineers and scientists hampering direct experience of solving the equations, essential for learning. This paper reflects the past research mindsets and promotes their reactivation in pedagogy as the most convenient way to transform passive into active learning by using spreadsheets. This section presents one possible solution to support a better education of the students on groundwater problems by using spreadsheets.

2.1 Parabolic Groundwater equations

Consider the general one dimensional (1D) transient flow equation in a confined aquifer, in which the unknown variable is time dependent and referred to as unsteady, non-equilibrium, or transient problems. Their governing equation combines the continuity equation with the dynamic equation (the Darcy law) for an aquifer and expressed as [20]:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T}\frac{\partial h}{\partial t} - \frac{R(x,y,t)}{T} \tag{1}$$

where, S is storivity defined as the volume of water released from storage per unite area of aquifer per unit decline in head; $R(x, y, t)$ is sink/source used to simulate both distributed and point sources (positive values) and sinks (negative values) with units of length per unit time; T is transmissivity and K is hydraulic conductivity. Transmissivity is defined as T=KD, where D is the porous medium depth. For more details, see Bear [21], among others.

2.2 Boundary Conditions

The solution of Eq. (1) requires the specification of boundary conditions to constrain the problem and to make solutions unique. The different types of boundary conditions are (a) head is known for surfaces bounding the flow region (Dirichlet conditions), (b) flow is known across surfaces bounding the region (Neumann conditions), (c) some combination of (a) and (b) is known as the Robin Boundary Conditions (mixed conditions).

In water engineering problems, the Dirichlet boundary condition occurs when one side of the domain is maintained at a fixed water level. For example the upstream slope of an earthen dam has the Dirichlet condition, because all nods in the upstream slope is normally specified in terms of head: ($h = z + \frac{p}{\gamma} = cons.$), where z = elevation at each node from datum, p is water pressure above each node and γ is specific gravity of water [22].

One of the most common boundary conditions in water engineering problems is the no-flow boundaries (Neumann conditions). Examples are (i) dam foundations normally resting on an impervious bed rock, (ii) cut-off walls below the dams implemented for reducing seepage flow.

2.3. Spreadsheet Implementation of Groundwater Problem

Many problems of the physical sciences are solved by a set of governing equations but their solutions for practical problems are often obtained by numerical techniques. These techniques transform the governing equations into a system of equations represented in matrix formats. The automatic solutions of the matrices of often nonlinear equations are obtained by further transforming them into an algorithmic fashion, in which the solutions are obtained through iterations. The bird's eye view of the complete solutions is that the boundary conditions set the iterative solutions running through the algorithmic equations connected to one another as a chain with the following possibilities: 1D (e.g. steady solutions); 2D (e.g. 1D in time plus 1D in space); 3D (e.g. 1D in time plus 2D in space) and 4D (e.g. 1D in time plus 3D in space). Mathematically minded researchers can understand the working of such mathematical complexity and the aim here is to lay down the solutions for those of less keen on mathematics but rife for a critical understanding of the whole process, as discussed below.

2.4. Iterative Calculations

Iterative solutions of systems of equations are also feasible in spreadsheets, as they offer both grids and inbuilt iterative facilities. However, this medium of solving equations has its peculiar features, setting its strength and weaknesses, as discussed in this section. The matrix format of equations is not used in iterative spreadsheet calculations but the finite difference equations are applied directly to its each and every grid nodes, where a grid in the physical system is seen as a cell in the spreadsheet and within the domain of the solution, the adjacent cells are interconnected. This is a tedious process, as the appropriate equation has to be written for each cell. However, they are easily copied through by using the copying and pasting facilities of spreadsheets. These facilities automatically increment row and column numbers.

It is recommended to deactivate the iterative calculation feature of spreadsheets while defining the equations to avoid the temporary warning messages related to circular references, and to reactivate this iterative feature defining all the equations. The iterative spreadsheet calculations are based on the numerical concept of successive relaxation (SR), which applies to the solution of both linear and nonlinear system of equations.

2.5. Explicit 2D finite difference approximation

2.5.1. An Explicit 2D Solvers (Transient flow)

For an aquifer of infinite width of space steps of Δx, the second order finite difference scheme may be used to discretize the continuous equations for groundwater flows expressed by Eq. (1) in space and a first order finite difference scheme discretized in time, as follows (Wang and Anderson 1982):

The solution for 3D; h=f(x,y,t):

$$\frac{h_{i+1,j}^{n}-2h_{i,,j}^{n}+h_{i-1,j}^{n}}{\Delta x^2}+\frac{h_{i,j+1}^{n}-2h_{i,j}^{n}+h_{i,j-1}^{n}}{\Delta y^2}=\frac{S_{i,j}}{T_{i,j}}\left(\frac{h_{i,j}^{n+1}-h_{i,j}^{n}}{\Delta t}\right)-\frac{R_{i,j}^{n}}{T_{i,j}} \tag{2.a}$$

The solution for 2D; h=f(x,t):

$$h_i^{n+1}=h_i^n+\frac{T_i\Delta t}{S_i}\left(\frac{h_{i+1}^{n}-2h_i^n+h_{i-1}^{n}}{\Delta x^2}\right) \tag{2.b}$$

where $h_{i,,j}^{n+1}$ is head at point i,*j* and at time step: *n*+1. Explicit solutions are susceptible to instability unless they satisfy Courant-Frederick-Levy (CFR) conditions given by:

$$^{T\Delta t}/_{(S\Delta x\Delta y)} < 0.5 \tag{2.c}$$

2.5.2. Implementation of the 2D Explicit Solver (Transient flow)

Fig. 1 presents a test case comprising a simple example, and specified in terms of the following boundary condition: $h(0,t) = h_1$and $h(l,t) = h_2$ for $t > 0$, h_1=20 m, h_2=15 m for an aquifer of infinite width and length of: l=100 m. The initial condition is $h(x,0) = h_1$ for $0 < x < l$.

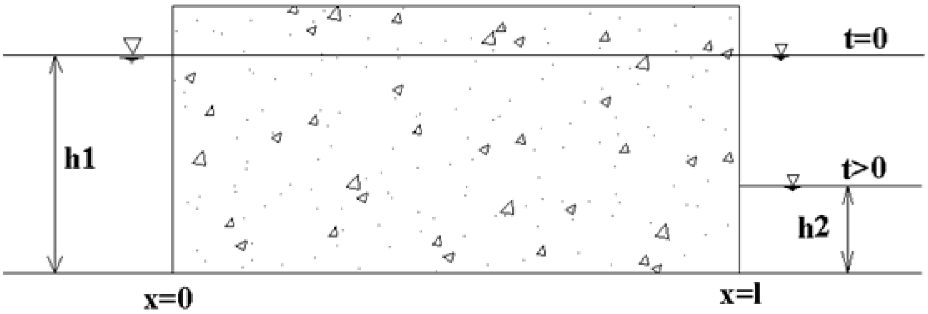


Fig. 1. Boundary and initial conditions for flow at porous media: (1D: space, 1D: time)

The simple problem is solved by (2.b) using the explicit scheme for a 2D problem (1D in space), which must satisfy the stability condition through: $T\Delta t / S(\Delta x)^2 < 0.5$. The solution of these equations engages the researcher with the direct learning of the specification of the boundary and initial conditions and with implementing the solution of (2.b).

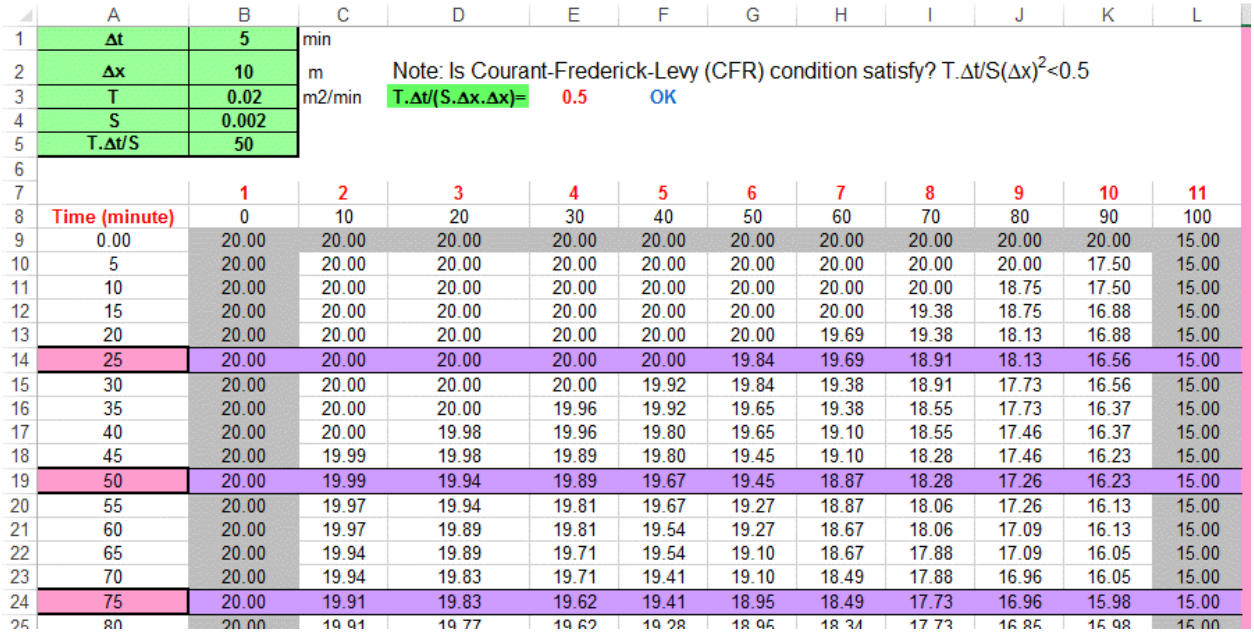
The worksheet for Fig. 1 uses (2.b) structured as follows: (i) the values of constant parameters are specified in a table, comprising: Δt, Δx, T and S, as well as the time steps. (ii) Another table presents the solver, defining the spatial layout of the test case for each time step with the grey cells specifying the Dirichlet boundary conditions. (iii) The 1D-spatial domain is set up in the cells of one row defining the computational domain by defining the equations for each cell at the computational domain. For this example, the process starts at cell C10. Its equation is displayed in the "formula bar" in Table 1: f_x =+C9+$B$5*(D9-2*C9+B9)/($B$2^2).

Evidently, cell C10 is interconnected to 3 cells of: B9, C9 and D9 at time $n - 1 = 0.0$ minute. So the value of cell C10 (at time=5 minute), depends on the value of 3 cells in the previous time, $n - 1 = 0.0$ minute. Using the Drag-and-Fill knob facilities, this equation is copied to the other cells and the computation area is filled up by the equations. Because the value of head (h) at n+1 time depends only on the values of heads in previous time (n level), equation (2.b) can be solved step by step in time. In other words, if value of head (h) at n+1 time level depends on both heads at n-1 and n+1 time levels, a system of equations at each time level can be formed for their solution.

Table 1. Different formula in cells after copying cell C10 to other cells

Cell name	Formulas
C10	=+C9+$B$5*(D9-2*C9+B9)/($B$2^2)
C11	=+C10+$B$5*(D10-2*C10+B10)/($B$2^2)
C12	=+C11+$B$5*(D11-2*C11+B11)/($B$2^2)
.	.
.	.
.	.
K28	=+K27+$B$5*(L27-2*K27+J27)/($B$2^2)
K29	=+K28+$B$5*(L28-2*K28+J28)/($B$2^2)

Notably, Cells $B$5 and $B$2 have fixed values as indicated by "$" sign in their left and right. After copying C10 formulas in all cells ranges C10 to K29, the value of cells $B$5 and $B$2 remain as per their set values at the leftmost column. So after the completion of the computational area, the option for the iteration process is activated again and this triggers the computation. The time taken depends on the defined tolerance, e.g. 0.0001. Using the drag-and-fill facilities set up the equation, in the cells in the computational area and an example is displayed in Fig. 2.



	A	B	C	D	E	F	G	H	I	J	K	L
1	Δt	5	min									
2	Δx	10	m	Note: Is Courant-Frederick-Levy (CFR) condition satisfy? T.Δt/S(Δx)^2<0.5								
3	T	0.02	m2/min	T.Δt/(S.Δx.Δx)=	0.5	OK						
4	S	0.002										
5	T.Δt/S	50										
6												
7		1	2	3	4	5	6	7	8	9	10	11
8	Time (minute)	0	10	20	30	40	50	60	70	80	90	100
9	0.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	15.00
10	5	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	17.50	15.00
11	10	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	18.75	17.50	15.00
12	15	20.00	20.00	20.00	20.00	20.00	20.00	20.00	19.38	18.75	16.88	15.00
13	20	20.00	20.00	20.00	20.00	20.00	20.00	19.69	19.38	18.13	16.88	15.00
14	25	20.00	20.00	20.00	20.00	20.00	19.84	19.69	18.91	18.13	16.56	15.00
15	30	20.00	20.00	20.00	20.00	19.92	19.84	19.38	18.91	17.73	16.56	15.00
16	35	20.00	20.00	20.00	19.96	19.92	19.65	19.38	18.55	17.73	16.37	15.00
17	40	20.00	20.00	19.98	19.96	19.80	19.65	19.10	18.55	17.46	16.37	15.00
18	45	20.00	19.99	19.98	19.89	19.80	19.45	19.10	18.28	17.46	16.23	15.00
19	50	20.00	19.99	19.94	19.89	19.67	19.45	18.87	18.28	17.26	16.23	15.00
20	55	20.00	19.97	19.94	19.81	19.67	19.27	18.87	18.06	17.26	16.13	15.00
21	60	20.00	19.97	19.89	19.81	19.54	19.27	18.67	18.06	17.09	16.13	15.00
22	65	20.00	19.94	19.89	19.71	19.54	19.10	18.67	17.88	17.09	16.05	15.00
23	70	20.00	19.94	19.83	19.71	19.41	19.10	18.49	17.88	16.96	16.05	15.00
24	75	20.00	19.91	19.83	19.62	19.41	18.95	18.49	17.73	16.96	15.98	15.00
25	80	20.00	19.91	19.77	19.62	19.28	18.95	18.34	17.73	16.85	15.98	15.00

Fig. 2. Excel spreadsheet for example of 1D explicit solver

Assuming that Δx=10 m T=0.02 m^2/s = 1728 m^2/day, S=0.002 and setting Δt= 5 minutes, it can be shown that the CFR stability condition of ${T\Delta t}/{S(\Delta x)^2} < 0.5$ is satisfied. Fig. 3 shows the results for the profile of the potentiometric surface for several time steps.

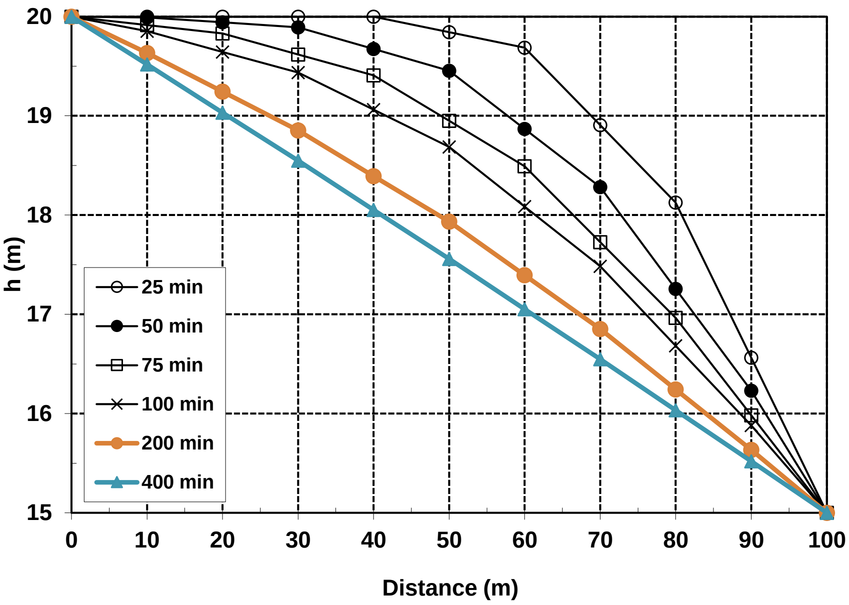


Fig. 3. Potentiometric surface in different times

2.5.3. Further notes on implementing boundary conditions

One of common boundary conditions in groundwater problems is the no-flow boundary condition (Neumann conditions), e.g. a dam resting on impervious bedrocks with the dam foundation specified in terms of a no-flow boundary. A cut-off wall below a dam (for reducing seepage flow) is also a no-flow boundary. Along the vertical boundaries specifying the no-flow boundary conditions, q_x=0 implies: $\frac{\partial h}{\partial x} = 0.0$, so $\frac{h_{i+1}-h_{i-1}}{2\Delta x} = 0.0$, hence: h_{i+1}=h_{i-1}.

For the left boundary, the point referred to by indices (i+1,j) is inside the problem domain, but the point referred to by indices (i-1,j) is outside. Therefore, we expand the finite difference problem domain by one additional column to the left by putting in a column of so-called imaginary or fictitious nodes (yellow colour nodes in Fig. 4).

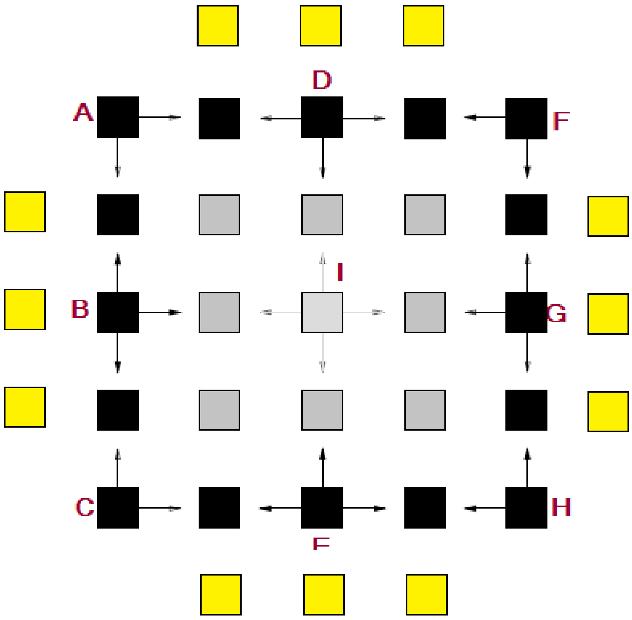


Fig. 4. Finite difference grid for no-flow boundary condition

2.6. Explicit 3D finite difference approximation

2.6.1. An Explicit 2D and 3D Solvers (Transient flow)

The finite difference method for the transient version of the well drawdown in spatial 2D is used as a test case to develop a spreadsheet solver for (1) using an explicit finite difference scheme. For a grid aquifer of sides Δx and Δy, a second order finite difference scheme discretises the continuous equation (1) in space and a first order finite difference scheme discretises it in time, as follows [23]:

$$\frac{h_{i+1,j}^{n}-2h_{i,j}^{n}+h_{i-1,j}^{n}}{\Delta x^2}+\frac{h_{i,j+1}^{n}-2h_{i,j}^{n}+h_{i,j-1}^{n}}{\Delta y^2}=\frac{S_{i,j}}{T_{i,j}}\left(\frac{h_{i,j}^{n+1}-h_{i,j}^{n}}{\Delta t}\right)-\frac{R_{i,j}^{n}}{T_{i,j}} \tag{3.a}$$

Let Δx=Δy=a, (3a) may be solved for $h_{i,j}^{n+1}$, as follows:

$$h_{i,j}^{n+1}=\left(1-\frac{4T_{i,j}\Delta t}{S_{i,j}a^2}\right)h_{i,j}^{n}+\left(\frac{4T_{i,j}\Delta t}{S_{i,j}a^2}\right)\left(\frac{h_{i+1,j}^{n}+h_{i-1,j}^{n}+h_{i,j+1}^{n}+h_{i,j-1}^{n}}{4}\right)+\frac{R_{i,j}^{n}\Delta t}{S_{i,j}} \tag{3.b}$$

The variable $h_{i,j}^{n+1}$ is unknown and evaluated in terms of the known (old) values of h (i.e. $h_{i,j}^{n}$) at the nodes surrounding (i,j) at the time step of n. Again CFL condition is given by:

$$\text{CFL Condition: } T.\Delta t/(S.a^2) < 0.5 \tag{3.c}$$

2.7. Implementation of the 2D Explicit Solver (Steady state flow)

A test case is devised in Fig. 5 in which a well is discharging at a constant rate of Q within a confined aquifer. The boundaries are treated as constant head boundaries. Fig. 5 is implemented in a worksheet with the following tables: (i) One table assigns the model and aquifer parameter values: Δx=Δy=a=200 m, T=350 m^2/day and S=0.002. The well is discharging at a constant rate of -2500 m^3/day and R=-Q/($\Delta x.\Delta$y)=-2500/(200^2)=-0.0625 m/day where R is the rate of release from the aquifer. (ii) The computational table (the solver) is developed to estimate the drawdown throughout the aquifer based on (3.e). (iii) Further tables are laid out to specify the distributed values of T, S and R. For 2D steady state ground water flow we have:

$$\frac{\partial^2 h}{\partial x^2}+=-\frac{R(x,y)}{T(x,y)} \tag{3d}$$

$$h_{i,j}=\frac{h_{i+1,j}+h_{i-1,j}+h_{i,j+1}+h_{i,j-1}+R_{i,j}/(T_{i,j}.\Delta x.\Delta y)}{4} \tag{3e}$$

The user can also specify some boundary conditions or all of them as no-flow boundary conditions. The layout of the grid is shown in Fig. 5 (left, right, top and down with no-flow boundary conditions) but the implementation of the boundary conditions is discussed further. In this test case, the grid is divided into three types of cells: (i) the computational cells are highlighted in grey; (ii) the cells representing no-flow boundary conditions are highlighted in dark and (iii) yellow cells represent fictitious nodes.

Mathematical modelling of no-flow Drichlet boundary condition in Fig. 5 for nodes B, A and H are:

$$\text{Node B: } \frac{\partial h}{\partial x} = \frac{h_{i+1,j} - h_{i-1,j}}{2\Delta x} = 0 \rightarrow h_{i+1,j} = h_{i-1,j} \tag{3.f}$$

As mentioned previously, implementation of Eq. (3.f) needs for representation of fictitious nodes/cells. Another way without need for fictitious nodes is to use Eq. (3.g) as the following. Hence with attention to (3.e), heads on boundary cells (like node B) defined as:

$$h_{i,j} = \frac{2h_{i+1,j} + h_{i,j+1} + h_{i,j-1} + R_{i,j}/(T_{i,j}.\Delta x.\Delta y)}{4} \tag{3.g}$$

At corner nodes like node A, no-flow boundary condition (BC) are both in *x* and *y* direction. In *x* direction Eq. (3f) satisfy no-flow BC but at *y* direction, Eq. (3r) must be define.

$$\frac{\partial h}{\partial y} = \frac{h_{i,j+1} - h_{i,j-1}}{2\Delta y} = 0 \rightarrow h_{i,j+1} = h_{i,j-1} \tag{3.r}$$

Again, another way without need for fictitious nodes is to use Eq. (3.n) as the following. Hence with attention to (3.e), heads on corner boundary cells (like node A) defined as:

$$\text{Node A: } h_{i,j} = \frac{2h_{i+1,j} + 2h_{i,j-1} + R_{i,j}/(T_{i,j}.\Delta x.\Delta y)}{4} \tag{3.n}$$

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	ΔX=	200											
2	ΔY=	200			Explicit 2D finite difference Solver (Steady state flow)								
3				In the following spreedsheet all boundarys (gray colour cells) are assumed constant head=10 m									
4													
5	Row	A	B	C	D	E	F	G	H	I	J	K	L
6	1	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00
7	2	10.00	9.94	9.87	9.81	9.74	9.69	9.67	9.70	9.76	9.84	9.92	10.00
8	3	10.00	9.88	9.75	9.61	9.47	9.35	9.29	9.37	9.51	9.67	9.84	10.00
9	4	10.00	9.83	9.64	9.42	9.18	8.94	8.79	8.96	9.24	9.51	9.76	10.00
10	5	10.00	9.79	9.55	9.26	8.90	8.43	7.96	8.46	8.96	9.37	9.70	10.00
11	6	10.00	9.77	9.51	9.18	8.72	7.93	6.16	7.96	8.79	9.29	9.67	10.00
12	7	10.00	9.78	9.53	9.24	8.87	8.40	7.93	8.43	8.94	9.35	9.69	10.00
13	8	10.00	9.81	9.60	9.38	9.12	8.87	8.72	8.90	9.18	9.47	9.74	10.00
14	9	10.00	9.85	9.70	9.54	9.38	9.24	9.18	9.26	9.42	9.61	9.81	10.00
15	10	10.00	9.90	9.80	9.70	9.60	9.53	9.51	9.55	9.64	9.75	9.87	10.00
16	11	10.00	9.95	9.90	9.85	9.81	9.78	9.77	9.79	9.83	9.88	9.94	10.00
17	12	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00
18													
19													
20	T=	Transmisivity		(m²/day)									
21		350	350	350	350	350	350	350	350	350	350	350	350
22		350	350	350	350	350	350	350	350	350	350	350	350
23		350	350	350	350	350	350	350	350	350	350	350	350
24		350	350	350	350	350	350	350	350	350	350	350	350
25		350	350	350	350	350	350	350	350	350	350	350	350
26		350	350	350	350	350	350	350	350	350	350	350	350

Fig. 5. Excel spreadsheet for example of 2D explicit solver

Fig. 6 presents the simulated drawdown surface. As the aim is not here to explore the accuracy of the scheme, its analytical solution is not derived. However, different aspects of this problem can be explored in group teaching.

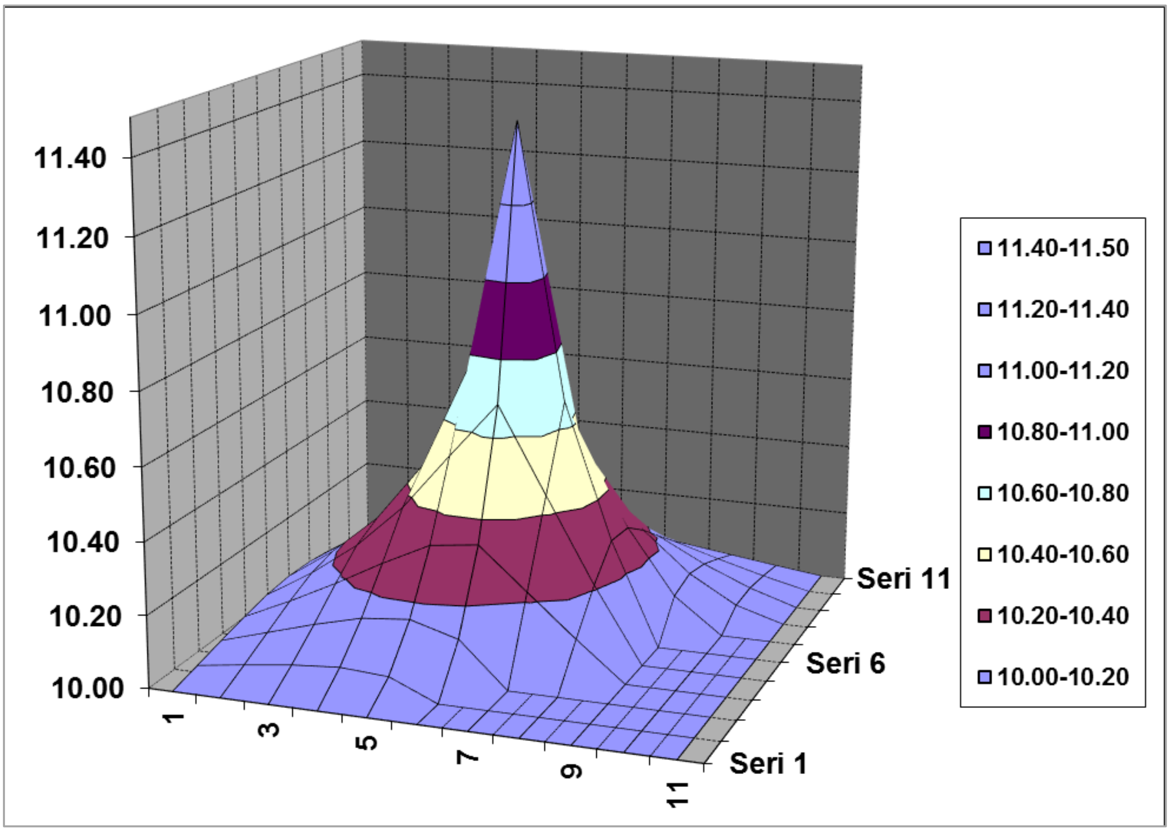


Fig. 6. Location of discharging well at cell H27 and head drawdown in other cells

2.8. Implicit 2D finite difference approximation

2.8.1. An Implicit Solver

Critical thinking in the student can be trained by learning the improvements in the solvers through using different solvers, e.g. using implicit finite difference schemes for groundwater problems. In this approach, the continuous space derivatives are discretised between $t=n\Delta t$ and $t=(n+1)\Delta t$ by weighting the average of the approximations at (n) and (n+1). The weighting parameter is represented by α, and it lies between 0.0 and 1, which normally set to 0.5 or 0.55. The following implicit finite difference scheme is often employed (Wang and Anderson 1982):

$$\frac{\partial^2 h}{\partial x^2} = a\,\frac{h^{n+1}_{i+1,j}-2h^{n+1}_{i,j}+h^{n+1}_{i-1,j}}{\Delta x^2} + (1-a)\,\frac{h^{n}_{i+1,j}-2h^{n}_{i,j}+h^{n}_{i-1,j}}{\Delta x^2} \tag{4.a}$$

$$\frac{\partial^2 h}{\partial y^2} = a\,\frac{h^{n+1}_{i,j+1}-2h^{n+1}_{i,j}+h^{n+1}_{i,j-1}}{\Delta y^2} + (1-a)\,\frac{h^{n}_{i,j+1}-2h^{n}_{i,j}+h^{n}_{i,j-1}}{\Delta y^2} \tag{4.b}$$

$$h^{*n}_{i,j} = \left(\frac{h^{n}_{i+1,j}+h^{n}_{i-1,j}+h^{n}_{i,j+1}+h^{n}_{i,j-1}}{4}\right) \tag{4.c}$$

The substitution of (4.a)-(4.c) in (1) produces the following:

$$A_{i-1,j}x^{n+1}_{i-1,j} + B_{i,j}x^{n+1}_{i,j} + C_{i+1,j}x^{t_2}_{3,1} = D_{1,1} \tag{4.d}$$

An example is presented to show how the solver described by (4.a)-(4.c) is transformed into a system of equations. Consider an aquifer transformed into a grid of 3×3 in space. Each grid is governed by (4.d) and this leads to a system of equations illustrated in Fig. 7.

$$\begin{array}{ccc}
A_{1,1}x^{t_2}_{1,1} + B_{1,1}x^{t_2}_{2,1} + C_{1,1}x^{t_2}_{3,1} = D_{1,1} & A_{2,1}x^{t_2}_{2,1} + B_{2,1}x^{t_2}_{3,1} + C_{2,1}x^{t_2}_{4,1} = D_{2,1} & A_{3,1}x^{t_2}_{1,1} + B_{3,1}x^{t_2}_{2,1} + C_{3,1}x^{t_2}_{3,1} = D_{3,1} \\
A_{1,2}x^{t_2}_{1,2} + B_{1,2}x^{t_2}_{2,2} + C_{1,2}x^{t_2}_{3,2} = D_{1,2} & A_{2,2}x^{t_2}_{2,2} + B_{2,2}x^{t_2}_{3,2} + C_{2,2}x^{t_2}_{4,2} = D_{2,2} & A_{3,2}x^{t_2}_{1,2} + B_{3,2}x^{t_2}_{2,2} + C_{3,2}x^{t_2}_{3,2} = D_{3,2} \\
A_{1,3}x^{t_2}_{1,3} + B_{1,3}x^{t_2}_{2,3} + C_{1,3}x^{t_2}_{3,3} = D_{1,3} & A_{2,3}x^{t_2}_{2,3} + B_{2,3}x^{t_2}_{3,3} + C_{2,3}x^{t_2}_{4,3} = D_{2,3} & A_{3,3}x^{t_2}_{1,3} + B_{3,3}x^{t_2}_{2,3} + C_{3,3}x^{t_2}_{3,3} = D_{3,3}
\end{array}$$

Fig. 7. Illustration of discrete equations transformed into a system of equations

These equations at each cell may be transformed into an algorithmic format, as follows:

$$h_{i,j}^{n+1} = \left(\frac{1}{[(a^2S/4T\Delta t)+a]}\right)\left[ah_{i,j}^{*n+1} + \frac{a^2S}{4T\Delta t}h_{i,j}^{n} + (1-a)\left(h_{i,j}^{*n} - h_{i,j}^{n}\right) + \frac{a^2R_{i,j}^{n}}{4T}\right] \tag{4.d}$$

2.8.2. Implementation

The implicit finite difference uses iterative spreadsheet solution technique. Thus, the formation of a system of simultaneous equations is not required using *matrix algebra*. Fig. 8 formulates a test case in which the boundary cells (the left, right and lower boundaries) are assumed to be constant head section (with h=100 m) and upper cells is assumed with no-flow BC (this made mixed BC). In addition all initial value of cells selected h=100 m. The grid size is selected at the spacing of $\Delta x = \Delta y = a = 100$ m, $\Delta t = 0.001$ days and $\alpha = 0.5$ (Crank–Nicolson scheme). The aquifer coefficients are T=300 m^2/day and S=0.002. The well discharge is at a constant rate of 5 m^3/day in cell K11. The test case also assumes that a number of cells, namely, T4, T5 and S5 cells are recharged at a constant rate of 50 m^3/day. This implicit finite difference (Eq. 4d) is solved by using the spreadsheet and corresponding graphic head changes given in Fig. 9.

	A	B	C	D	E	F	G	H	I	J	K
1	Dx	100	m								
2	Dy	100	m								
3	Dt	5	day								
4	Alpha	0.5	-								Time=30
5											
6	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000
7	100.000	99.999	99.998	99.995	99.992	99.989	99.985	99.982	99.980	99.978	99.978
8	100.000	99.998	99.993	99.988	99.981	99.973	99.966	99.959	99.953	99.950	99.949
9	100.000	99.995	99.988	99.978	99.967	99.954	99.942	99.930	99.921	99.914	99.913
10	100.000	99.992	99.981	99.966	99.950	99.932	99.913	99.896	99.881	99.872	99.868
11	100.000	99.988	99.973	99.954	99.931	99.907	99.881	99.857	99.835	99.821	99.815
12	100.000	99.985	99.965	99.940	99.912	99.880	99.847	99.813	99.783	99.761	99.753
13	100.000	99.981	99.957	99.928	99.893	99.854	99.812	99.768	99.727	99.695	99.682
14	100.000	99.978	99.951	99.917	99.877	99.831	99.780	99.725	99.670	99.625	99.604
15	100.000	99.976	99.946	99.909	99.865	99.814	99.756	99.691	99.622	99.560	99.526
16	100.000	99.975	99.943	99.905	99.859	99.806	99.744	99.674	99.598	99.523	99.467
17	100.000	99.974	99.943	99.905	99.860	99.807	99.748	99.682	99.612	99.549	99.514
18	100.000	99.975	99.945	99.908	99.866	99.818	99.764	99.707	99.650	99.603	99.580
19	100.000	99.977	99.948	99.915	99.877	99.834	99.788	99.741	99.697	99.663	99.647
20	100.000	99.979	99.953	99.924	99.890	99.854	99.816	99.779	99.745	99.720	99.708
21	100.000	99.981	99.959	99.934	99.905	99.875	99.845	99.815	99.790	99.771	99.763
22	100.000	99.984	99.965	99.944	99.921	99.896	99.872	99.850	99.830	99.817	99.810
23	100.000	99.987	99.971	99.954	99.936	99.917	99.898	99.881	99.867	99.857	99.852
24	100.000	99.990	99.978	99.965	99.951	99.937	99.923	99.911	99.901	99.894	99.891
25	100.000	99.993	99.985	99.976	99.966	99.957	99.948	99.940	99.934	99.929	99.927
26	100.000	99.996	99.992	99.987	99.983	99.978	99.973	99.970	99.966	99.964	99.963
27	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000
28											
29	300	300	300	300	300	300	300	300	300	300	300
30	300	300	300	300	300	300	300	300	300	300	300
31	300	300	300	300	300	300	300	300	300	300	300
32	300	300	300	300	300	300	300	300	300	300	300
33	300	300	300	300	300	300	300	300	300	300	300

	L	M	N	O	P	Q	R	S	T	U	V
1											
2											
3											
4											
5											
6	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000
7	99.979	99.981	99.985	99.990	99.996	100.002	100.008	100.012	100.012	100.007	100.000
8	99.951	99.956	99.964	99.974	99.986	100.000	100.012	100.022	100.024	100.014	100.000
9	99.916	99.924	99.936	99.952	99.971	99.991	100.011	100.028	100.033	100.019	100.000
10	99.873	99.884	99.902	99.923	99.948	99.974	100.000	100.023	100.030	100.017	100.000
11	99.821	99.837	99.860	99.888	99.918	99.948	99.977	100.000	100.011	100.008	100.000
12	99.761	99.782	99.813	99.847	99.883	99.918	99.949	99.974	99.990	99.997	100.000
13	99.693	99.723	99.763	99.806	99.848	99.887	99.922	99.950	99.971	99.987	100.000
14	99.622	99.663	99.715	99.767	99.816	99.860	99.898	99.930	99.957	99.979	100.000
15	99.556	99.613	99.677	99.737	99.792	99.839	99.880	99.915	99.946	99.973	100.000
16	99.517	99.587	99.657	99.721	99.778	99.827	99.869	99.906	99.939	99.969	100.000
17	99.543	99.599	99.662	99.722	99.776	99.823	99.865	99.902	99.935	99.967	100.000
18	99.596	99.636	99.686	99.737	99.784	99.828	99.867	99.902	99.935	99.967	100.000
19	99.656	99.683	99.720	99.761	99.801	99.839	99.874	99.906	99.938	99.968	100.000
20	99.713	99.731	99.759	99.790	99.822	99.854	99.885	99.914	99.942	99.971	100.000
21	99.765	99.777	99.797	99.821	99.846	99.872	99.898	99.923	99.948	99.974	100.000
22	99.811	99.820	99.834	99.851	99.871	99.892	99.913	99.934	99.956	99.977	100.000
23	99.853	99.859	99.868	99.881	99.896	99.912	99.929	99.946	99.963	99.981	100.000
24	99.891	99.895	99.902	99.911	99.922	99.933	99.946	99.959	99.972	99.986	100.000
25	99.927	99.930	99.934	99.940	99.947	99.955	99.963	99.972	99.981	99.990	100.000
26	99.963	99.964	99.967	99.969	99.973	99.977	99.981	99.986	99.990	99.995	100.000
27	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000
28											
29	300	300	300	300	300	300	300	300	300	300	300
30	300	300	300	300	300	300	300	300	300	300	300
31	300	300	300	300	300	300	300	300	300	300	300
32	300	300	300	300	300	300	300	300	300	300	300
33	300	300	300	300	300	300	300	300	300	300	300

Fig. 8. Excel spreadsheet for implicit 2D finite difference approximation

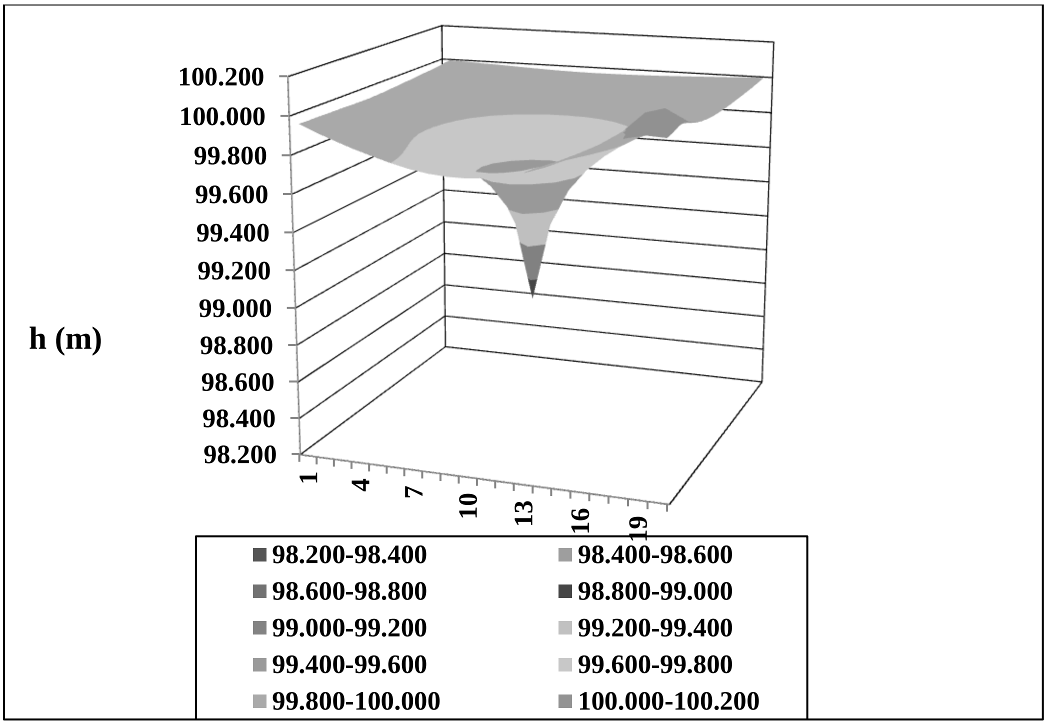


Fig. 9. 3D representation of head variation by importing discharge well in K11 cell and recharge wells in T4, T5 and S5 cells

3. Discussion

The analogy between the reproduction process of living thing and education as the reproduction of knowledge provides some food for thought. The reproduction process is an encapsulation of the primitive past at lower complexity preparing individuals for new generations. Likewise, knowledge is created and encapsulated in time but every individual has to acquire it through education and lifelong learning, during which encapsulated knowledge has to be uncovered for the users even though many parts of the learned knowledge will not be used directly during the lifetime of the user. The paper argues that a great deal of knowledge on mathematical modelling has been encapsulated and the ways of uncovering it for training and educational practices are being overlooked. Arguably, there are new opportunities as outlined below.

Self-directed lifelong learning runs simultaneously with a programmed phase of formal education at a certain stage to acquire basic skills necessary for professional practices by being taught the mainstream knowledge. In the past, education of science/engineering did not include the uptake of new knowledge produced by the on-going R&D activities as their uptakes were slow due to many barriers hampering their direct benefits for teaching. The barriers are increasingly becoming irrelevant as learning no longer relies on textbooks alone due to the emergence of (i) multimedia approaches to learning and teaching, the effectiveness of which is stressed, e.g. by Lindstrom [24], stating that simultaneously seeing and hearing and doing result in better learning; (ii) game-based learning (for more information, see Prensky [25], among others), (iii) the emergence of modelling animations of many complex scientific theories, (iii) the emergence of student-centred learning (see Brandes and Ginnis [26]).

It is often the case that the graduate student joins professional organisations with a typical cold start as practices are aligned with many drivers yet to penetrate higher educational practices. Arguably, the practising graduate engineer is often shocked by the scale of new facets of knowledge to be picked up in professional practices. In civil engineering related to water resources, these include a vast array of mathematical and spatial modelling tools and risk analysis. The task is therefore to introduce the various drivers to teaching curricula that are parts-and-parcels of professional practices. These drivers include research, policy, best practice and the various thinking sweeping industry, e.g. sustainable thinking, system thinking, uncertainty thinking, jointed-up thinking, critical thinking, etc. Curriculum designs should include appropriate levels of research-led, policy-led and best-practice-led teaching programs, as well as training the student with appropriate levels of gaining an insight into the problems and exercising critical thinking, although sustainable thinking has already penetrated into global cultures.

While researchers and students are aided in various ways in their learning as discussed above, this paper only highlights the dimension of engaging them with developing different solvers to train their critical thinking. The use of groundwater flow modelling in this paper is incidental and in particular, the state-of-the-art in this problem area has reached its classical status with no further contribution from this paper on the subject. The paper suggests the following procedure for learning and teaching practices on mathematical modelling:

1. A "living team" can be formed in universities to develop and peer reviews of new solvers by the students to build up a library of modules of programs, where each model starts from the beginning as follows.
2. The derivation of the governing equations are integrated by their quality conditions normally comprising the continuous partial differential equations satisfy the three conditions of existence, consistency and stability, together with highlighting their assumptions;
3. These equations are discretized by using numerical schemes (often using finite difference, finite elements, finite volume techniques), where their discretization satisfies the three more conditions of: convergence, consistence (compatibility) and stability. However, the transformed equations must be shown to be "properly-posed" by: (i) having the number of equations equal to the number of unknowns, (ii) the subsequent equations are consistent, (iii) they are stable; (iv) they are not ill-conditioned;
4. The formulation of their inverse problems still has to satisfy the three conditions of identifiability, uniqueness and stability.

Each of the above activities was topical in the past and now encapsulated within the expertise of a narrow section of professionals. The users are likely to learn passively these concepts, but passive learning is arguably a weakness in teaching any educational subject. The uptake of multimedia, game-based learning and model-based animations is making learning easy and releasing proportionately time resources reusable for different active learning subjects. It is suggested that the teachers can engage students with building a library of solvers based on the procedure suggested above and create the conditions for accumulating the solvers developed by the students over the years. This way of accumulating solvers is a bottom-up approach pooling together the efforts by students, which in turn can lead to the emergence of open source modelling systems. In this way, universities can change the situation on mathematical modelling and transform their capabilities into organically growing living systems and assure a better educational program.

Groundwater models may be used to predict the effects of hydrological changes (like groundwater abstraction or irrigation developments) on the aquifer by formulating their simulation problems. Nowadays the groundwater models are used in various water management plans for urban areas. Spreadsheets may be applied to these problems or any partial differential equation with specified initial and boundary conditions. These test cases can be extended to estimate uplift pressure below a dam and for accounting for recharge or discharge from aquifers, as well as that for the variations in the values of *S* and *T*. Boundaries were treated as constant values in these test runs but variable values along the boundaries can also be accommodated by spreadsheet modelling, which may be the case along earth dams. Other problems approachable by spreadsheet modelling include: simple isotropic aquifers, anisotropic aquifers, stratified aquifers, leaky aquifers, artesian aquifers, seepage from an earth dam, canal bed.

The level of sophistication on the implementation of the solvers in this paper is deliberately kept low and rather than using loops, the solver tables are repeated for each time step. However, within a reason, spreadsheet can be made relatively more versatile. Another limitation of spreadsheet calculations is that finite difference grids can only be rectangular or triangular since they are mapped onto spreadsheet cells, therefore excluding curved boundaries and inclined layer geometries. The spreadsheet solution procedure is presented in a manner that is consistent with the governing conservation statements. This, coupled with students' pre-existing familiarity with the spreadsheet package, enables students to focus on understanding of the engineering system rather than cumbersome computational procedures. Obviously, learning the skills necessary to implement and analyze the behavior of selected numerical algorithms is also an important aspect of the development and application of computational models.

4. Conclusions

This paper raises concern about the encapsulated body of knowledge contributed to the emergence and the establishment of modelling software applications since 1980. This body of knowledge comprise a deeper understanding of equations of often partial differential equations describing physical problems, as well as their numerical transformation into systems of equations and their subsequent properly- and improperly posed systems of equation in terms of their assumptions and quality conditions. The outcome is the emergence of a cookbook mentality among the new breed of mathematical modelers without any critical thinking.

A solution is offered that stems from the attributes of new technology already changing culture and now is penetrating theories of education. New technological solutions include multimedia, game-based learning and animation of mathematical modelling results, which inevitably make learning and teaching easier than the past on the assumptions that the students have been transformed into active learners. As already teenagers have demonstrated their capacity of active learning in the market-pulled and technology-pushed Internet social networking, educationists have embraced this capacity. Thus, significant time resources are releases that can be utilized by forming living teams in universities, whereby students produce a library of modelling solvers, which also peer-reviewed by them. This provides the opportunity for transforming the encapsulated knowledge on mathematical modelling into an active learning process. This paper uses spreadsheet solution of groundwater equations to illustrate the preliminary steps.

The technique used in this paper on solving groundwater equations are classic knowledge, which are transformed to solvers on commonly available spreadsheet platforms. The solvers employ explicit and implicit finite difference schemes for groundwater problems, applicable to flow and seepage problems. The main advantage of such a modelling capability is its ease of implementation and application to a wide variety of practical groundwater problems. From an educational point of view, all the assumptions, equations, and calculation steps can be captured on the spreadsheets and be clearly stated, formulated, and executed, which is rather uncommon for other nonlinear numerical techniques. The primary advantage of this suggested solution is the possibility of training the user's critical thinking.

Declarations

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Conflicts of Interest: The authors declare no conflict of interest.

Data availability: Data are available on request due to privacy or other restrictions.

References

[1] Scriven, M., Paul, R., 2003. Defining Critical Thinking: A draft statement prepared for the National Council for Excellence in Critical Thinking Instruction.

[2] Kyrpychenko O, Pushchyna I, Kichuk Y, Shevchenko N, Luchaninova O, Koval V, 2021. Communicative Competence Development in Teaching Professional Discourse in Educational Establishments, International Journal of Modern Education and Computer Science(IJMECS), Vol.13, No.4, pp. 16-27, DOI: 10.5815/ijmecs.2021.04.02

[3] Jewell, T.K., 2001. Teaching hydraulic design using equation solvers. J. Hydraul. Eng. 127(12), 1013–1021.

[4] Cheetancheri, K.G. and Cheng H.H. 2009. Spreadsheet-based interactive design and analysis of mechanisms using Excel and Ch, Adv Eng Softw 40, 274-280. doi:10.1016/j.advengsoft.2007.08.003.

[5] Yudianto MRA, Agustin T, James RM, Rahma FI, Rahim A, Utami E, 2021. Rainfall Forecasting to Recommend Crops Varieties Using Moving Average and Naive Bayes Methods, International Journal of Modern Education and Computer Science(IJMECS), 13 (3), 23-33, DOI: 10.5815/ijmecs.2021.03.03

[6] El-Hajj, A., Kabalan, K.Y., Al-Husseini, M., 2003. Antenna array design using spreadsheets. IEEE Trans Edu 46(3), 319–24. DOI: 10.1109/TE.2003.813518.

[7] Shoup, T.E., 2004. Using spreadsheet modules to augment tolerance dimensioning. In: Proceedings of the DETC. 2912, ASME, 36–44. DOI: 10.1115/DETC2004-57649.

[8] Karahan, H. 2006. Implicit finite difference techniques for the advection–diffusion equation using spreadsheets. Adv Eng Softw 37, 601-608. https://doi.org/10.1016/j.advengsoft.2006.01.003.
[9] Dehghan, M., & Mohammadi, V. (2017). Comparison between two meshless methods based on collocation technique for the numerical solution of four-species tumor growth model. Communications in Nonlinear Science and Numerical Simulation, 44, 204–219. https://doi.org/10.1016/j.cnsns.2016.07.024
[10] Panigrahi, P. K., & Velusamy, K. (2019). A robust multiphase model to investigate molten material relocation during total flow blockage in SFR fuel subassembly. Annals of Nuclear Energy, 133, 59–72. https://doi.org/10.1016/j.anucene.2019.05.007
[11] Mac ás-D áz, J. E., & Villa-Morales, J. (2017). A deterministic model for the distribution of the stopping time in a stochastic equation and its numerical solution. Journal of Computational and Applied Mathematics, 318, 93–106. https://doi.org/10.1016/j.cam.2016.11.025
[12] Sattari MT., Mirabbasi R., Sushab RS, Abraham J. 2018. Prediction of Groundwater Level in Ardebil Plain Using Support Vector Regression and M5 Tree Model. Groundwater 56(4):636-646. https://doi.org/10.1111/gwat.12620
[13] Cryer, C.W., 1970. On the approximate solution of free boundary problems using finite differences. Journal of the Association for Computing Machinery 17(3), 397–411. doi:10.1145/321592.321593.
[14] Weiss, P.T., Gulliver, J.S., 2001. What do students need in hydraulic design projects? J. Hydraul. Eng. 127(12), 984–991. DOI: 10.1061/(ASCE)0733-9429(2001)127:12(984).
[15] Huddleston, D.H., 2002. Spreadsheet tools utilised to introduce computational field simulation concepts to undergraduate engineering students. Comput. Educ. J. 12(1), 6–11.
[16] Huddlesston, D.H., Alarcon Vladimir J., Chen, W., 2004. Water distribution network analysis using Excel. Journal of hydraulic engineering 103(10), 1033-1036.
[17] Pandit A. (2016). Water engineering with the spreadsheet: A workbook for water resources calculations using excel, Publisher: Reston, Virginia: American Society of Civil Engineers, DOI: 10.1061/9780784414040.
[18] Niazkar M. and Afzali S.H. (2015). Application of Excel spreadsheet in engineering education. First International & Fourth National Conference on Engineering Education Shiraz University, 10-12 November 2015. Pp 1-7.
[19] Divayana DGH, Ariawan IPW, Giri MKW, CIPP-SAW, 2021. Application as an Evaluation Tool of E-Learning Effectiveness, International Journal of Modern Education and Computer Science (IJMECS), 13 (6), 42-59, DOI: 10.5815/ijmecs.2021.06
[20] Salmasi, F., Azamathulla, H. M., (2013). Determination of optimum relaxation coefficient using finite difference method for groundwater flow, Arabian Journal of Geo-sciences, (6): 3409-3415, https://doi.org/10.1007/s12517-012-0591-9
[21] Bear, J., 1979. Hydraulics of Groundwater, Dover Books on Engineering.
[22] Hosseinzadeh Asl, R., Salmasi, F., Arvanaghi, H., (2020). Numerical investigation on geometric configurations affecting seepage from unlined earthen channels and the comparison with field measurements. Engineering Applications of Computational Fluid Mechanics, 14:1, 236-253, https://doi.org/10.1080/19942060.2019.1706639.
[23] Wang, H.F., Anderson, M.P., 1982. Introduction to Groundwater Modeling: Finite Difference and Finite Element Methods. Academic Press, NY, ISBN-13: 978-0127345857, pp: 237.
[24] Lindstrom, R., 1994. The Business Week Guide to Multimedia Presentations: Create Dynamic Presentations That Inspire, New York: McGraw-Hill.
[25] Prensky, M., 2007. Digital Game-Based Learning (Kindle Edition), Paragon House Publishing.
[26] Brandes, D., Ginnis, G., 1994. A Guide to Student-Centred Learning.

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