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Proving Riemann Hypothesis through the Derivative of Zeta Reflection

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Abstract: A novel proof is presented in this work indicating that all non-trivial zeros of the zeta function has a real part equal to ½. As no proof has been validated yet, this work was successful in introducing a proof through the use of elementary calculus theorems. A second version of the proof was also shown where a more advanced series analysis (e.g. Fourier series) is used.

Index Terms: Riemann Zeta Function.

1. Literature Review

The Riemann hypothesis (RH) is well known in mathematics and many schemes have been investigated to prove it. More thorough analysis of the zeta function (and applications of RH) can be found in many books and papers, e.g. [1-3].

This results in many approaches to prove RH. Unfortunately, the complex approaches went too far and the supporters deny any possibility for simple proofs [4,5]. On the other hand, most simple approaches have a hidden incorrect assumption [6,7].

The current proof is considered simple and the author has made the best of his efforts to avoid these drawbacks. One of these drawbacks is that the assumption of 2 corresponding zeros [8]. Obviously, this necessitates the critical line. The approach should be, in the author opinion, is to consider 4 rather than 2 corresponding zeros in the critical strip.

Although no proof has been validated till now, numerical methods have excluded a large region from having a zero as demonstrated by [9,10]. This work is not intended to survey previous work on the function but rather concentrate on the proof itself.

2. Introduction

The description and features of the zeta function will not be re-introduced in this work, interested readers can refer to any text book, e.g. [1]. This work starts from the reflection introduced through mathematical continuation. Consider ([1] pp 16)

$$\zeta(1-s) = \pi^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \zeta(s) \tag{1}$$

Taking the magnitude square of (1) and rewriting, s=σ+*i*t

$$P(\sigma,t) = C(\sigma,t)Q(\sigma,t) \tag{2a}$$

$$\frac{P(\sigma,t)}{Q(\sigma,t)} = C(\sigma,t) \tag{2b}$$

$$C(\sigma,t) = \pi^{(1-2\sigma)} \frac{\left(\frac{1-\sigma}{2}\right)^2 + \left(\frac{t}{2}\right)^2}{\left(\frac{\sigma}{2}\right)^2 + \left(\frac{t}{2}\right)^2} e^{-\gamma(2\sigma-1)} \prod_{n=1}^{\infty} e^{(2\sigma-1)/n} \left\{ \frac{\left(1+\frac{1-\sigma}{2n}\right)^2 + \left(\frac{t}{2n}\right)^2}{\left(1+\frac{\sigma}{2n}\right)^2 + \left(\frac{t}{2n}\right)^2} \right\}$$

$$C(\sigma,t) = \pi^{(1-2\sigma)} e^{\gamma(1-2\sigma)} \prod_{k=1}^{\infty} e^{(2\sigma-1)/k} \prod_{n=0}^{\infty} \left\{\frac{(2n+1-\sigma)^2+t^2}{(2n+\sigma)^2+t^2}\right\} \tag{2c}$$

$$log\big(C(\sigma,t)\big) = (1-2\sigma)log(\pi) + (1-2\sigma)\gamma + (2\sigma-1)\sum_{k=1}^{\infty}\frac{1}{k} + \sum_{n=0}^{\infty} log\left\{\frac{(2n+1-\sigma)^2+t^2}{(2n+\sigma)^2+t^2}\right\} \tag{2d}$$

Avoiding the special cases of C (called the reflection function from now on), poles (σ = –2n, t = 0) and zeros (σ = 2n+1, t = 0) where n=0, 1, … ,, the higher order derivatives of (2a) w.r.t t (other derivatives will be addressed later) are

$$P^{(1)}(\sigma,t) = C(\sigma,t)Q^{(1)}(\sigma,t) + C^{(1)}(\sigma,t)Q(\sigma,t) \tag{3a}$$

$$P^{(2)}(\sigma,t) = C(\sigma,t)Q^{(2)}(\sigma,t) + 2C^{(1)}(\sigma,t)Q^{(1)}(\sigma,t) + C^{(2)}(\sigma,t)Q(\sigma,t) \tag{3b}$$

$$P^{(3)}(\sigma,t) = C(\sigma,t)Q^{(3)}(\sigma,t) + 3C^{(1)}(\sigma,t)Q^{(2)}(\sigma,t) + 3C^{(2)}(\sigma,t)Q^{(1)}(\sigma,t) + C^{(3)}(\sigma,t)Q(\sigma,t) \tag{3c}$$

Restricting ourselves to simple zeros (will be relaxed later), P and Q have their zeros of order 2. Evaluating (3) at any zero (σ_0, t_0),

$$\frac{P(\sigma_0,t_0)}{Q(\sigma_0,t_0)} = C(\sigma_0,t_0) \tag{4a}$$

$$P^{(1)}(\sigma_0,t_0) = Q^{(1)}(\sigma_0,t_0) = 0 \tag{4b}$$

$$P^{(2)}(\sigma_0,t_0) = C(\sigma_0,t_0)Q^{(2)}(\sigma_0,t_0) \tag{4c}$$

$$P^{(3)}(\sigma_0,t_0) = C(\sigma_0,t_0)Q^{(3)}(\sigma_0,t_0) + 3C^{(1)}(\sigma_0,t_0)Q^{(2)}(\sigma_0,t_0) \tag{4d}$$

3. Methodology

Assuming that the zeros of the zeta function are all simple, they can be one of the three following types

1. Single zeros: along t=0.
2. Double zeros: along the line σ=½ (occurring in complex conjugate pairs).
3. Quadruple zeros: having σ≠½ and t≠0 and occur symmetrically in the four "quadrants".

These three types will be investigated in the following sub-sections.

3.1. Single Zeros

These are obviously the trivial zeros (negative even integers). Consider

$$\zeta(\sigma)(1-2^{1-\sigma}) = \eta(\sigma) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma} = \sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)^\sigma} - \frac{1}{(2n)^\sigma}\right), \qquad \sigma > 0$$

As the convergent sum is >0, zeta can never be zero. Due to reflection (1), negative values of σ (not including negative even integers) will not be zeros. The other special case is σ=0 where zeta is not zero due to reflection with σ=1.

3.2. Quadruple Zeros

The formula in (2b) can be considered as a description of the third "quadrant" (σ<½ and t<0) in reference to the first "quadrant" (σ>½ and t>0). Taking the derivative of (2b) w.r.t t, we have

$$\frac{P^{(1)}(\sigma,t)Q(\sigma,t)-P(\sigma,t)Q^{(1)}(\sigma,t)}{Q(\sigma,t)Q(\sigma,t)} = \frac{P^{(1)}(\sigma,t)}{Q(\sigma,t)} - \frac{P(\sigma,t)}{Q(\sigma,t)}\frac{Q^{(1)}(\sigma,t)}{Q(\sigma,t)} = C^{(1)}(\sigma,t) \tag{5}$$

Evaluating (5) at any zero and using (4a), recall L'Hopital's rule, hence ($C(\sigma_0, t_0)$ is not zero/pole)

$$\lim_{s\to s_0}\left(\frac{P^{(2)}(\sigma,t)}{Q^{(1)}(\sigma,t)}\right) - C(\sigma_0,t_0)\lim_{s\to s_0}\left(\frac{Q^{(2)}(\sigma,t)}{Q^{(1)}(\sigma,t)}\right) = \lim_{s\to s_0}\frac{P^{(2)}(\sigma,t)-C(\sigma_0,t_0)Q^{(2)}(\sigma,t)}{Q^{(1)}(\sigma,t)} = C^{(1)}(\sigma_0,t_0)$$

$$\lim_{s\to s_0}\frac{P^{(3)}(\sigma,t)-C(\sigma_0,t_0)Q^{(3)}(\sigma,t)}{Q^{(2)}(\sigma,t)} = C^{(1)}(\sigma_0,t_0) \tag{6}$$

Evaluating (6) and using (4d), we have

$$\frac{C(\sigma_0,t_0)Q^{(3)}(\sigma_0,t_0)+3C^{(1)}(\sigma_0,t_0)Q^{(2)}(\sigma_0,t_0)-C(\sigma_0,t_0)Q^{(3)}(\sigma_0,t_0)}{Q^{(2)}(\sigma_0,t_0)} = C^{(1)}(\sigma_0,t_0)$$

$$3C^{(1)}(\sigma_0,t_0) = C^{(1)}(\sigma_0,t_0) \Rightarrow C^{(1)}(\sigma_0,t_0) = 0 \tag{7}$$

Differentiating (2d) w.r.t *t*

$$\frac{\partial}{\partial t}C(\sigma,t) = C(\sigma,t)\left(\sum_{n=0}^{\infty}\frac{2t}{(2n+1-\sigma)^2+t^2} - \sum_{n=0}^{\infty}\frac{2t}{(2n+\sigma)^2+t^2}\right) \tag{8}$$

Evaluating (8) at the zero and using (7)

$$\sum_{n=0}^{\infty}\frac{2t_0\{(2n+\sigma_0)^2-(2n+1-\sigma_0)^2\}}{\{(2n+\sigma_0)^2+t_0^2\}\{(2n+1-\sigma_0)^2+t_0^2\}} = 0 \Rightarrow t_0(2\sigma_0-1)\sum_{n=0}^{\infty}\frac{4n+1}{\{(2n+\sigma_0)^2+t_0^2\}\{(2n+1-\sigma_0)^2+t_0^2\}} = 0 \tag{9}$$

Clearly (9) restrict the zeros to the real axis (t=0) and the vertical line (σ=½) as the series is convergent. As (7) should be satisfied for all derivatives, *we can not have zeros in the third quadrant and by reflection we can not have quadruple zeros*.

Extension to zeros of higher order (if any) can be easily achieved by the extension of (3c), (4d) and (7).

3.3. Double Zeros

The above argument can be extended to the double zeros (the critical line) where (9) is satisfied. Unfortunately, as the critical line is the reflection boundary, extension to other derivatives is not valid for the double zeros. The reason is obvious by considering the first "quadrant" (σ>½ and t>0) and the second "quadrant" (σ<½ and t>0). For this case, to the left of the line the reference is (σ>½ and t>0), while, to the right of the line the reference is (σ<½ and t>0). Please note that the argument here is not regarding an undefined derivative (w.r.t σ) due to different values at each side, but the fact that we have descriptions at each side that do not represent one entity. The author would like to emphasize that the non-availability of the derivative is for the function C, see (2), and not the zeta function itself.

We can overcome that by considering C to the left and the constant 1 to the right of the critical line, and in this case the derivative w.r.t σ is not defined for most of the critical line. Hence, as σ=½ is the reflection boundary for C (extension of the zeta function) (5) the only valid derivative of C on the critical line is w.r.t t. Hence, the only possible solutions are of type

$$s = \frac{1}{2} + it \tag{10}$$

Obviously, (10) proves the Riemann Hypothesis. Unfortunately, it reduces C to 1 and making it independent of t as in (2c). Therefore, other alternatives are needed to find the t values of the non-trivial zeros.

4. Other Forms of Reflection

As indicated by [1], the reflection of zeta function can have many forms. Consider

$$\zeta(1-s) = 2(2\pi)^{-s}cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s) \tag{11}$$

Hence,

$$C(\sigma,t) = 2(2\pi)^{-2\sigma}\left\{cos^2\left(\frac{\pi\sigma}{2}\right) + sinh^2\left(\frac{\pi t}{2}\right)\right\}|\Gamma(\sigma)|^2\prod_{k=0}^{\infty}\left\{1+\frac{t^2}{(k+\sigma)^2}\right\}^{-1} \tag{12}$$

Evaluating (7) using (12) results in (differentiating w.r.t t)

$$\frac{\left(\frac{\pi}{2}\right)sinh(\pi t_0)}{\left\{cos^2\left(\frac{\pi\sigma_0}{2}\right)+sinh^2\left(\frac{\pi t_0}{2}\right)\right\}} - \sum_{k=0}^{\infty}\frac{2t_0}{\{(k+\sigma_0)^2+t_0^2\}} = 0 \tag{13}$$

The reflection should also satisfy (13), hence

$$\sum_{k=0}^{\infty}\frac{2t_0}{\{(k+1-\sigma_0)^2+t_0^2\}} - \sum_{k=0}^{\infty}\frac{2t_0}{\{(k+\sigma_0)^2+t_0^2\}} = 0 \tag{14}$$

Obviously, (14) will lead to the same result as in (9). In fact, substituting σ=½ in (13) gives

$$\frac{\left(\frac{\pi}{2}\right)sinh(\pi t)}{\left\{\frac{1}{2}+sinh^2\left(\frac{\pi t}{2}\right)\right\}} - \sum_{k=0}^{\infty}\frac{8t}{\{(2k+1)^2+4t^2\}} = 0$$

$$\frac{\left(\frac{\pi}{2}\right)sinh(\pi t)}{\left\{\frac{1}{2}+sinh^2\left(\frac{\pi t}{2}\right)\right\}} - \sum_{k=0}^{\infty}\frac{8t}{\{k^2+4t^2\}} + \sum_{k=0}^{\infty}\frac{8t}{\{(2k)^2+4t^2\}} = 0 \tag{15}$$

A closed form solution for the summation terms can be found using Fourier analysis. Consider the signal e^{ax} with a period of [–π, π], hence the complex Fourier series is given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-jnx}dx = \frac{1}{2\pi}\frac{e^{\pi(a-jn)}-e^{-\pi(a-jn)}}{a-jn} \Rightarrow |c_n|^2 = \frac{1}{4\pi^2}\frac{(e^{\pi a}-e^{-\pi a})^2(cos(\pi n))^2}{n^2+a^2} = \frac{sinh^2(\pi a)}{\pi^2(n^2+a^2)} \tag{16}$$

Applying Parseval's theorem

$$\sum_{n=-\infty}^{\infty}\frac{sinh^2(\pi a)}{\pi^2(n^2+a^2)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2ax}dx = \frac{sinh(2a\pi)}{2a\pi} \Rightarrow 2\sum_{n=1}^{\infty}\frac{1}{(n^2+a^2)} = \frac{\pi cosh(a\pi)}{asinh(a\pi)} - \frac{1}{a^2} \tag{17}$$

Substituting (17) into (15) we have

$$\frac{\left(\frac{\pi}{2}\right)sinh(\pi t_0)}{\left\{\frac{1}{2}+sinh^2\left(\frac{\pi t_0}{2}\right)\right\}} - 4t_0\left\{\frac{\pi cosh(2\pi t_0)}{2t_0 sinh(2\pi t_0)} - \frac{1}{4t_0^2}\right\} + t_0\left\{\frac{\pi cosh(\pi t_0)}{t_0 sinh(\pi t_0)} - \frac{1}{t_0^2}\right\} = 0 \Rightarrow 1 + 2sinh^2\left(\frac{\pi t_0}{2}\right) = cosh(t_0\pi) \tag{18}$$

Clearly, (18) is always satisfied. As expected, results are similar to those in (9), however, the derivations using (10) are rather involved compared to the use of (1).

5. Discussion and Conclusions

This work provides a simple proof that the zeta function can only have zeros on the lines Re{s}=½ and Im{s}=0. The result was obtained by noticing that the derivative of the reflection function should be zero for all zeros of the zeta function.

More insight is needed to investigate the availability of some distribution that can describe the positions of the zeros. Fourier analysis could be of help in this regard.

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