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Study of Memory Effect in an Inventory Model with Linear Demand and Shortage

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Abstract

For real market studies of any business, inclusion of memory or past experience in inventory model has great impact. Memory means it depends on the past state of the process not only current state of the process. Indeed, the inventory system is an appropriate example as a memory affected system. Presence of long past experiences or short past experiences of any company or shop has different importance on increasing or decreasing profit. The description of the memory dependent inventory model is more appropriate process compared to the memory less inventory model. Depending on demand rate, a comparison between the minimized total average costs of different numerical example has been presented. Fractional order derivative and integration have been used to establish the model. Our considered numerical example establishes that if linear type demand rate is only time proportional, profit of the business is high compared to the linear type demand rate.

Index Terms: Fractional order derivative, Fractional laplace transform method, Fractional order inventory model or memory dependent inventory model.

1. Introduction

Fractional calculus is three centuries old as the conventional calculus, but not very widespread among science or engineering community. But in the last twenty years, it was pulled to many interesting areas of financial process [1,2,3,4] biological system [5],physics[6], economic growth [**7**].Fractional calculus does not understand the calculus of fraction of any differentiation, integration. The fractional calculus is actual theory of integrations and derivatives of arbitrary order, which amalgamate and generalize the notion of integer order differentiation and *n*-fold integration to take into account memory of the system. The notion of natural number

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is a natural abstraction. The notion of a real number is a generalization of the notion of a natural number. The real number reflects real quantities, but cannot change the fact that they do not exist [8]. If one wants to compute something, then one immediately discovers that there is no place for real numbers in this world [8]. But fractional calculus is able to give a reflection of the real system or natural system.

Most of the suggestions for interpretation of fractional calculus are a little bit abstract. They do not give any physical intuition but one of the most useful interpretations is, fractional calculus has power to remember previous effects of the input in order to determine the current value of the output [7].Memory describes dependence of the variables not only current state of the process, but also on the changes of these variables in the past. Fractional calculus has been used those systems which has memory or hereditary effects. It is not explored in the inventory system still now. The theoretical breakthroughs are being continuously experienced in the field of inventory management.

The problem of inventory theory is one of the most important in organizational management. Stocks are created to carry out the normal activities of the company. There are literally millions of different type products manufactured in our society there are only two fundamental decisions that one has to make when controlling inventory (i) how large should an inventory replenishment order be?(ii)when should an inventory replenishment order be placed? The objectives of inventory management often reduce the problem if it is more profitable to do quickly but more expensive or slower but cheaper.

In practical, the demand rate is not always same; it changes with respect to time as well as other exogenous effect which depends on the position of the company, political and social conditions. On the other hand, good position or connection of the company or shop with public increase the selling. Indeed, if an object gets popularity in the market then its demand will increase or if it gets poor impression then its demand will decrease. In some sense demand of any object is not same in all shop it depends on dealing of the shopkeeper with the customer. This means the selling of any product depends on the quality as well as the shopkeeper's attitude. Hence, previous policy has effect on the present business.

For the above reasons, such systems depend on the past experiences, is called memory effect of the inventory system. Hence, inventory system is a memory dependent system. In this paper we have considered the fractional order inventory model with linear demand as well as a comparison has been considered to evaluate different effect for varying demand.

The fractional derivatives are defined using integrals, so they are non-local operators. The fractional derivative contains information about the function at earlier points, so it possesses a memory effect. Hence, fractional derivatives can be used to construct the memory dependent inventory models to take into account the memory effect of the system. Fractional order is physically treated as an index of memory [1,**5]**. When fractional order becomes less it indicates that strong memory acts on the system. System becomes memory less decreases in the sense of numerically when fraction order becomes equal to one.

In the classical calculus, ordinary derivative with respect to time provides the slope at a particular pt on a curve. It is not able to include memory or past history of the system. On the other hand, in the fractional calculus, fractional derivative signifies the memory effect of the system for its differ-integration process [7].Due to the above reasons, fractional calculus is used instead of ordinary calculus.

Our numerical example clears that in long memory effect, the minimized total average cost is much less compared to the short memory effect or without memory effect. Our analysis from the comparison of particular case of the linear demand rate provides us that when demand rate is proportional to time, the minimized total average cost and optimal ordering interval are low compared to the other cases of the considered examples. Hence, in this case, profit is high compared to the other cases. In each case, business policy initially was chosen then falls down but again good policy was chosen.

The rest part of the paper is maintained as follows, review of fractional calculus has been given in section-II, Classical model formulation is presented in section-III, Formulation of fractional order inventory model with memory kernel is given in section IV, Solution of the fractional order inventory model is presented in section IV(i),Numerical example has been furnished in section-V, Graphical presentation has been given in section-VI, Some important conclusion purpose of the paper has been presented in the section-VII.

2. Review of Fractional Calculus

The popular definition of fractional derivatives[9,10]are left and right Riemann-Liouville(R-L) derivative, Caputo fractional derivative, Modified left R-L derivative or Jumarie fractional derivative which are most times used to formulate different mathematical model. Some of them are listed below.

(i) Riemann-Liouville(R-L) Definition of Fractional Derivative

Riemann-Liouville (R-L) definition of fractional derivative of α^{th} order (where $r \le \alpha < r+1$) for any integrable function $f(x)$ on $[a,b]$ is denoted and defined as

$$ {}_{a}^{RL}D_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(r+1-\alpha)}\left(\frac{d}{dx}\right)^{r+1}\int_a^x (x-\xi)^{(r-\alpha)} f(\xi)\,d\xi \tag{1} $$

This is known as left R-L definition of fractional derivative.

The above result creates a difference between ordinary derivative and fractional derivative. To eradicate this difference M.Caputo [11] proposed a new definition of fractional order derivative for any differentiable functions.

(ii) Caputo Fractional Order Derivative

For any differentiable function $f(x)$ on $[a,b]$ Caputo fractional order derivative [11] of α^{th} order is denoted by ${}_{a}^{C}D_x^{\alpha}\left(f(x)\right)$ and defined as follows

$$ {}_{a}^{C}D_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(r+1-\alpha)}\int_a^x (x-\xi)^{(r-\alpha)} f^{(r+1)}(\xi)\,d\xi \tag{2} $$

where $r \le \alpha < r+1$.The Caputo type fractional derivative is applicable for differentiable function only.On the other hand, in terms of Caputo definition of fractional derivative of any constant function is zero. Furthermore, to solve the Caputo type fractional order differential equations, we need the initial and boundary conditions are same as of classical or integer order differential equation.

(iii) Fractional Laplace Transform Method

Laplace transformation plays an important role in integer and fractional order differential equations. The Laplace transform of the function $f(t)$ is denoted by $F(s)$ and defined as

$$ F(s)=L(f(t))=\int_0^{\infty} e^{-st} f(t)\,dt \tag{3} $$

where s>0 and s is called the transform parameter. The Laplace transformation of integer order derivative is defined as

$$L\left\{f^{r}(t)=s^{r}F(s)-\sum_{k=0}^{r-1}s^{r-k-1}f^{k}(0)\right\} \tag{4}$$

where $f^{r}(t)$ denotes *r-th* order ordinary derivative of *f* with respect to *t*.
and for non - integer α , it is defined in generalized form[**9**] as,

$$L\left\{f^{\alpha}(t)=s^{\alpha}F(s)-\sum_{k=0}^{r-1}s^{k}f^{\alpha-k-1}(0)\right\} \text{ where } (r-1)<\alpha\leq r. \tag{5}$$

(iv) Fractional Sine and Cosine Function

$\sin x$ and $\cos x$ can be represented by the exponential function as follows $\sin x=\frac{e^{ix}-e^{-ix}}{2i}, \cos x=\frac{e^{ix}+e^{-ix}}{2}$ .Fractional sin and cosine function of $\alpha-th$ th order can be represented by the Mittag-Leffler function as follows $\frac{E_{\alpha}(ix^{\alpha})-E_{\alpha}(-ix^{\alpha})}{2i}=\sin_{\alpha}(x^{\alpha}), \frac{E_{\alpha}(ix^{\alpha})+E_{\alpha}(-ix^{\alpha})}{2}=\cos_{\alpha}(x^{\alpha})$. Hence, $\sin_{\alpha}(x^{\alpha})$, $\cos_{\alpha}(x^{\alpha})$ nothing but generalization form of $\sin x$ and $\cos x$ in terms of fractional order .Taking Jumarie fractional order derivative on the $\sin_{\alpha}(x^{\alpha})$ and $\cos_{\alpha}(x^{\alpha})$ can be obtained as ${}_{0}^{J}D_{x}^{\alpha}(\sin_{\alpha}(x^{\alpha}))=\cos_{\alpha}(x^{\alpha}), {}_{0}^{J}D_{x}^{\alpha}(\cos_{\alpha}(x^{\alpha}))=-\sin_{\alpha}(x^{\alpha})$.

3. Classical Inventory Model

The classical or memory less inventory model has been developed on the basis of the bellow assumptions.

(i) Assumptions

The following assumptions are made in developing the model.

(1) The inventory consists of only one type of items.
(2) The lead time is zero or negligible.
(3) Demand rate is $\begin{cases}(a+bt) & \text{for} \quad 0\leq t\leq t_1\\ D_0 & \text{for} \quad t_1\leq t\leq T\end{cases}$
(4) The planning horizon is infinite.
(5) During the stock out period, there is complete backlogging.

(ii) Notations

Notations are listed below, which are used to develop the mathematical model.

Table 1. Used Symbols and Items

Symbols	Items	Symbols	Items
$(i) I_1(t)$	Positive inventory level at time *t*.	$(ii) I_2(t)$	Negative inventory level at time *t*.
$(iii) C_1$	Holding cost per unit per unit time.	$(iv) C_2$	Shortage cost per unit per unit time.
$(v)(\Gamma,.)$	Gamma function.	$(vi) P$	Purchasing cost per unit.
$(vii) HOC_{\alpha,\beta}$	Inventory holding cost with fractional effect.	$(viii)(B,.)$	Beta function.
$(ix) POC_{\alpha,\beta}$	Purchasing cost with memory effect.	$(x) SOC_{\alpha,\beta}$	Shortage cost with fractional effect.
$(xi) TOC_{\alpha,\beta}$	Total average cost with memory effect.	$(xii) TOC^*_{\alpha,\beta}$	Minimized total average cost with memory effect.
$(xiii) T_{\alpha,\beta}$	Ordering interval with memory effect.	$(xiv) T^*_{\alpha,\beta}$	Optimal ordering interval with memory effect.
$(xv) Q$	Total order quantity.	$(xvi) C_3$	Ordering cost per order.

(iii) Classical Model Formulation

The rate of change of inventory level during positive stock period $[0,t_1]$with linear type demand rate and constant demand rate during shortage period$[t_1,T]$ is governed by the two first order differential equations as follows

$$\frac{d\left(I_1(t)\right)}{dt}=-(a+bt) \text{ with } I_1(t_1)=0 \tag{6}$$

$$\frac{d\left(I_2(t)\right)}{dt}=-D_0 \text{ with } I_2(t_1)=0 \tag{7}$$

The classical economic order quantity model has not illustrated because memory dependent inventory model is the key area of our paper.

4. Formulation of Fractional order Inventory Model with Memory Kernel

In order to study the memory effect on the inventory management, we consider the fractional generalization of the classical inventory model. Due to observe the influence of memory effects, first the differential equation(6-7) can be written using the kernel function as follows [5].

$$\frac{d\left(I_1(t)\right)}{dt}=-\int_0^t k(t-t')(a+bt')dt' \tag{8}$$

$$\frac{d\left(I_2(t)\right)}{dt}=-\int_0^t k(t-t')D_0 dt' \tag{9}$$

in which $k(t-t')$ is the kernel function. An appropriate choice, in order to include long-term memory effects, can be a power-law function which displays a slow decay such that the state of the system at quite early times also contributes to the evolution of the system [5].

This type of kernel guarantees the existence of scaling features as it is often intrinsic in most natural phenomena. Thus, to generate the fractional order model we consider $k(t-t')=\frac{1}{\Gamma(\alpha-1)}(t-t')^{\alpha-2}$, where $0<\alpha\leq 1$ and $\Gamma(\alpha)$ denotes the gamma function. Using the definition of fractional derivative [8,9] we can re-write the Equation (8-9) to the form of fractional differential equations with the Caputo-type derivative in the following form

$$\frac{d\left(I_1(t)\right)}{dt}=-{}_0D_t^{-(\alpha-1)}\left(a+bt\right) \tag{10}$$

$$\frac{d\left(I_2(t)\right)}{dt}=-{}_0D_t^{-(\alpha-1)}\left(D_0\right) \tag{11}$$

Now, applying fractional Caputo derivative of order $(\alpha-1)$ on both sides of the Eq. (10-11), and using the fact the Caputo fractional derivative and fractional integral are inverse operators, the following fractional differential equations can be obtained for the model

$${}_0^C D_t^{(\alpha)}\left(I_1(t)\right)=-\left(a+bt\right) \tag{12}$$

$${}_0^C D_t^{(\alpha)}\left(I_2(t)\right)=-D_0 \tag{13}$$

Here, the strength of the memory is determined by the fractional order α. When, $\alpha\to 1$, it indicates that the system losses its memory and when α becomes equal to one system becomes memory less system.

Long Memory effect: The strength of memory is controlled by the order of fractional derivative or fractional integration.If order of fractional derivative or fractional integrationis in (0,0.5).Then this system may be called that it has long memory effect.

Short Memory effect: If order of the fractional derivative or the fractional integration is in [0.5,1).The system is called that it has short memory effect.

(i) Fractional order Inventory model Analysis

The fractional order model is governed by two above formulated fractional order differential equations

$$\frac{d^\alpha\left(I_1(t)\right)}{dt^\alpha}=-\left(a+bt\right) \text{ with } I_1\left(t_1\right)=0 \tag{14}$$

$$\frac{d^\alpha\left(I_2(t)\right)}{dt^\alpha}=-D_0 \text{ with } \quad I_2\left(t_1\right)=0 \tag{15}$$

Using Laplace transform and Laplace inversion method [24] on (14, 15) and we get,

$$I_1(t)=\frac{a}{\Gamma(1+\alpha)}\left(t_1^\alpha-t^\alpha\right)+\frac{b}{\Gamma(2+\alpha)}\left(t_1^{\alpha+1}-t^{\alpha+1}\right) \tag{16}$$

$$I_2(t)=\frac{D_0}{\Gamma(1+\alpha)}\left(t_1^{\alpha}-t^{\alpha}\right) \tag{17}$$

The maximum positive inventory level is

$$M=I_1(0)=\frac{at_1^{\alpha}}{\Gamma(1+\alpha)}+\frac{bt_1^{\alpha+1}}{\Gamma(2+\alpha)} \tag{18}$$

The maximum backordered units are

$$S=-I_2(T)=\frac{D_0}{\Gamma(1+\alpha)}\left(T^{\alpha}-t_1^{\alpha}\right) \tag{19}$$

Hence, the order size during *[0,T]* becomes as
Q = (Maximum positive inventory level) + (Maximum negative inventory level)

$$=M+S=\left(\frac{a}{\Gamma(1+\alpha)}\left(t_1^{\alpha}\right)+\frac{b}{\Gamma(2+\alpha)}\left(t_1^{\alpha+1}\right)\right)+\frac{D_0}{\Gamma(1+\alpha)}\left(T^{\alpha}-t_1^{\alpha}\right) \tag{20}$$

Inventory holding cost per cycle for the fractional model [1] is denoted by $HOC_{\alpha,\beta}(T)$ and defined as follows

$$\begin{aligned} HOC_{\alpha,\beta}(T)&=C_1\ {}_0D_{t_1}^{-\beta}\left(I_1(t)\right)=\frac{C_1}{\Gamma(\beta)}\int_0^{t_1}(t_1-t)^{(\beta-1)}\left(\frac{a}{\Gamma(1+\alpha)}\left(t_1^{\alpha}-t^{\alpha}\right)+\frac{b}{\Gamma(2+\alpha)}\left(t_1^{\alpha+1}-t^{\alpha+1}\right)\right)dt \\ &=\frac{aC_1t_1^{\alpha+\beta}}{\Gamma(\beta)\Gamma(1+\alpha)}\left(\frac{1}{\beta}-B(\alpha+1,\beta)\right)+\frac{bC_1t_1^{\alpha+\beta+1}}{\Gamma(\beta)\Gamma(2+\alpha)}\left(\frac{1}{\beta}-B(\alpha+2,\beta)\right) \end{aligned} \tag{21}$$

(here, β is considered as the integral memory index to include memory on different type associated cost[1])

Shortage cost per cycle with fractional effect is denoted by $SOC_{\alpha,\beta}$ and defined as follows

$$\begin{aligned} SOC_{\alpha,\beta}(T)&=-C_2\ {}_{t_1}D_T^{-\beta}\left(I_2(t)\right)=\frac{C_2}{\Gamma(\beta)}\int_{t_1}^{T}(T-t)^{(\beta-1)}\left(\frac{D_0}{\Gamma(1+\alpha)}\left(t^{\alpha}-t_1^{\alpha}\right)\right)dt \\ &=\left(\begin{array}{l} -\frac{C_2D_0t_1^{\alpha}}{\Gamma(\beta+1)\Gamma(1+\alpha)}T^{\beta}+\frac{C_2D_0\alpha t_1^{\alpha+1}}{(1+\alpha)\Gamma(\beta)\Gamma(1+\alpha)}T^{\beta-1} \\ +\frac{C_2D_0}{\Gamma(\beta)\Gamma(1+\alpha)}\left(\frac{1}{(1+\alpha)}-\frac{(\beta-1)}{(2+\alpha)}\right)T^{\beta+\alpha}+\frac{C_2D_0(\beta-1)t_1^{\alpha+2}}{(2+\alpha)\Gamma(\beta)\Gamma(1+\alpha)}T^{\beta-2} \end{array}\right) \end{aligned} \tag{22}$$

$$\left(\text{Expanding}\ \ (T-t)^{(\beta-1)}\cong T^{\beta-1}\left(1-(\beta-1)\frac{t}{T}\right),\ \ \text{neglecting higher term as}\left(\frac{t}{T}\right)\right)$$

Purchase cost per cycle becomes as

$$POC_{\alpha,\beta} = P \times Q = P\left(\left(\frac{a}{\Gamma(1+\alpha)}\left(t_1^{\alpha}\right)+\frac{b}{\Gamma(2+\alpha)}\left(t_1^{\alpha+1}\right)\right)+\frac{D_0}{\Gamma(1+\alpha)}\left(T^{\alpha}-t_1^{\alpha}\right)\right) \tag{23}$$

(where, P is the purchasing cost per unit time)

Hence, total average cost per unit time is as,

$$TOC_{\alpha,\beta}(T)=\left(\frac{HOC_{\alpha,\beta}+SOC_{\alpha,\beta}+POC_{\alpha,\beta}+C_3}{T}\right)$$

$$=\frac{1}{T}\left(\begin{array}{l}\left(HOC_{\alpha,\beta}+C_3+P\left(\frac{at_1^{\alpha}}{\Gamma(1+\alpha)}+\frac{bt_1^{\alpha}}{\Gamma(2+\alpha)}\right)-\frac{PD_0}{\Gamma(1+\alpha)}\right)\\ +\frac{aC_1}{\Gamma(\beta)\Gamma(1+\alpha)}\left(\frac{t_1^{\alpha+\beta}}{\beta}-t_1^{\alpha+\beta}B(\alpha+1,\beta)\right)+\frac{bC_1}{\Gamma(\beta)\Gamma(2+\alpha)}\left(\frac{t_1^{\alpha+\beta+1}}{\beta}-t_1^{\alpha+\beta+1}B(\alpha+2,\beta)\right)\\ -\frac{C_2D_0t_1^{\alpha}}{\Gamma(\beta+1)\Gamma(1+\alpha)}T^{\beta}+\frac{C_2D_0\alpha t_1^{\alpha+1}}{(1+\alpha)\Gamma(\beta)\Gamma(1+\alpha)}T^{\beta-1}\\ +\frac{C_2D_0}{\Gamma(\beta)\Gamma(1+\alpha)}\left(\frac{1}{(1+\alpha)}-\frac{(\beta-1)}{(2+\alpha)}\right)T^{\beta+\alpha}+\frac{C_2D_0(\beta-1)t_1^{\alpha+2}}{(2+\alpha)\Gamma(\beta)\Gamma(1+\alpha)}T^{\beta-2}+\frac{PD_0T^{\alpha}}{\Gamma(1+\alpha)}\end{array}\right)$$

$$=AT^{-1}+B_1T^{\beta-1}+CT^{(\beta-2)}+DT^{(\beta-3)}+ET^{(\alpha-1)}+FT^{(\alpha+\beta-1)}$$

$$A=P\left(\left(\frac{at_1^{\alpha}}{\Gamma(1+\alpha)}+\frac{bt_1^{\alpha}}{\Gamma(2+\alpha)}\right)-\frac{D_0}{\Gamma(1+\alpha)}\left(t_1^{\alpha}\right)\right)+HOC_{\alpha,\beta}+C_3$$

$$,B_1=-\frac{C_2D_0t_1^{\alpha}}{\Gamma(\beta+1)\Gamma(1+\alpha)},C=\frac{C_2D_0\alpha t_1^{\alpha+1}}{(1+\alpha)\Gamma(\beta)\Gamma(1+\alpha)},D=\frac{C_2D_0(\beta-1)t_1^{\alpha+2}}{(2+\alpha)\Gamma(\beta)\Gamma(1+\alpha)},$$

$$E=\frac{PD_0}{\Gamma(1+\alpha)},F=\frac{C_2D_0}{\Gamma(\beta)\Gamma(1+\alpha)}\left(\frac{1}{(1+\alpha)}-\frac{(\beta-1)}{(2+\alpha)}\right).$$

Now, we consider three cases for the memory dependent or fractional order inventory model (i) $0<\alpha\le 1.0, 0<\beta\le 1.0$, (ii) $0<\alpha\le 1, \beta=1.0$, (iii) $0<\beta\le 1.0, \alpha=1.0$.

(a). Case-1: $0<\alpha\le 1.0, 0<\beta\le 1.0$.

Therefore, generalized inventory model can be written as follows

$$\begin{cases}\text{Min}\,TOC_{\alpha,\beta}(T)=AT^{-1}+B_1T^{\beta-1}+CT^{\beta-2}+DT^{\beta-3}+ET^{\alpha-1}+FT^{\alpha+\beta-1}\\ \text{Subject to } T\ge 0\end{cases} \tag{24}$$

The corresponding dual form of (24) can be written bellow,

$$Max\,d(w)=\left(\frac{A}{w_1}\right)^{w_1}\left(\frac{B_1}{w_2}\right)^{w_2}\left(\frac{C}{w_3}\right)^{w_3}\left(\frac{D}{w_4}\right)^{w_4}\left(\frac{E}{w_5}\right)^{w_5}\left(\frac{F}{w_6}\right)^{w_6} \tag{25}$$

With the orthogonal and normal conditions respectively are

$$w_1+w_2+w_3+w_4+w_5+w_6=1 \tag{26}$$

and

$$-w_1+(\beta-1)w_2+(\beta-2)w_3+(\beta-3)w_4+(\alpha-1)w_5+(\alpha+\beta-1)w_6=0 \tag{27}$$

The primal-dual relations can be written as

$$\left.\begin{array}{l} AT^{(-1)}=w_1d(w),B_1T^{\beta-1}=w_2d(w),CT^{(\beta-2)}=w_3d(w),DT^{(\beta-3)}=w_4d(w) \\ ET^{(\alpha-1)}=w_5d(w),FT^{(\alpha+\beta-1)}=w_6d(w) \end{array}\right\} \tag{28}$$

Using the relations (28), the followings can be obtained as,

$$\left.\left(\frac{Aw_2}{B_1w_1}\right)=\left(\frac{Cw_2}{B_1w_3}\right)^{(\beta)},\left(\frac{Cw_2}{B_1w_3}\right)=\left(\frac{Dw_3}{Cw_4}\right),\left(\frac{Cw_2}{B_1w_3}\right)^{\beta-\alpha-2}=\left(\frac{Ew_4}{Dw_5}\right),\left(\frac{Ew_6}{Fw_5}\right)=\left(\frac{Cw_2}{B_1w_3}\right)^{\beta}\right\} \tag{29}$$

along with

$$T=\left(\frac{Cw_2}{B_1w_3}\right) \tag{30}$$

There are six non-linear equations (26, 27 and 29) with six unknown $w_1, w_2, w_3, w_4, w_5, w_6$. Optimal values $w_1^*, w_2^*, w_3^*, w_4^*, w_5^*, w_6^*$ are obtained solving six nonlinear equations. Optimal ordering interval $T_{\alpha,\beta}{}^*$ and the minimized total average cost $TOC_{\alpha,\beta}{}^*$ can be obtained from (30) and (24) respectively.

(b). Case-2: $0<\alpha\le 1, \beta=1.0.$

In this case, the inventory model becomes as

$$\begin{cases} \text{Min}TOC_{\alpha,1}(T)=AT^{-1}+B_1T^0+CT^{-1}+DT^{-2}+ET^{\alpha-1}+FT^{\alpha} \\ \text{Subject to } T\ge 0 \end{cases} \tag{31}$$

In similar way as case-1, the minimized total average cost and optimal ordering interval have been evaluated from (31).

(c). Case-3: $0<\beta\le 1.0, \alpha=1.0.$

In this case, generalized inventory model can be written as

$$\begin{cases}\text{Min}TOC_{1,\beta}\left(T\right)=AT^{-1}+B_1T^{\beta-1}+CT^{\beta-2}+DT^{\beta-3}+ET^{0}+FT^{\beta}\\ \text{Subject to } \quad T\ge 0\end{cases}$$
(32)

In similar approach as case-1, the minimized total average cost and optimal ordering interval have been evaluated from (32).

5. Numerical Example

(i) We consider a fractional order inventory model with the parameter values in proper units $a=20, b=30, P=50, D_0=23, C_1=15, C_2=20, C_3=300, t_1=0.6789.$ The numerical problem is solved by Matlab minimization process from the case-1.

We will also consider different numerical example depending on different values of ***a*** and ***b***.

Table 2. Optimal Ordering Interval and Minimized Total Average cost for $\beta=1.0\, and\ \ 0<\alpha\le 1.0.$ $\left(\uparrow\text{indicates increasing,}\downarrow\text{indicates decreasing}\right)$

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	1.0	43.9461	244.7051
0.2	1.0	19.8155	469.7240
0.3 (growing memory effect)	1.0	12.1259	694.2453
0.4	1.0	8.3749	913.9054
0.5	1.0	6.1480	$1.1210\text{x}10^3$
0.6	1.0	4.6682	$1.3065\text{x}10^3$
0.7	1.0	3.6156	$1.4612\text{x}10^3$
0.8	1.0	2.8383	$1.5773\text{x}10^3$
0.9↑(above)	1.0	2.2569↑9increasing above)	$1.6496\text{x}10^3$↓(decreasing above)
1.0	1.0	1.8248	$1.6771\text{x}10^3$

From table 2, it is clear that in long memory effect, minimized total average cost is low compared to the short memory effect or memory less system. But, in long memory effect, optimal ordering interval is high compared to the short memory effect or memory less system. Hence, in long memory effect, business stay long time but minimized total average cost is low compared to the short memory effect or memory less system.

(ii) We consider a fractional order inventory model with the parameter values in proper units $a=20, b=0, P=50, D_0=23, C_1=15, C_2=20, C_3=300, t_1=0.6789$

Table 3. Optimal ordering interval and minimized total average cost for $\beta = 1.0\, and \;\; 0 < \alpha \le 1.0.$

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	1.0	27.9419	215.4751
0.2	1.0	13.6716	412.5336
0.3(growing memory effect)	1.0	8.7703	610.9492
0.4	1.0	6.2170	805.9642
0.5	1.0	4.6124	989.1742
0.6	1.0	3.4945	$1.1507\text{x}10^3$↑(decreasing above)
0.7	1.0	2.6713	$1.2808\text{x}10^3$
0.8↑	1.0	2.0536↑(increasing above)	$1.3714\text{x}10^3$
0.9	1.0	1.5972	$1.4184\text{x}10^3$
1.0	1.0	1.2740	$1.4237\text{x}10^3$

In this case, for long memory effect and short memory effect, nature of working of memory effect on the minimized total average cost and optimal ordering interval is same as table-2 but the numerical value is less compared to the table 2, table-4.

(iii) We consider a fractional order inventory model with the parameter values in proper units $a = 0, b = 30, P = 50, D_0 = 23, C_1 = 15, C_2 = 20, C_3 = 300, t_1 = 0.6789$

Table 4. Optimal Ordering Interval and Minimized Total Average cost for $\beta = 1.0 \text{ and } 0 < \alpha \le 1.0.$ $\left(\uparrow \text{indicates increasing and} \downarrow \text{indicates decreasing}\right)$

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	1.0	28.0401	215.6927
0.2	1.0	13.0300	405.6592
0.3 (growing memory effect)	1.0	8.0234	590.4483
0.4	1.0	5.4609	765.54833↓(decreasing) (above)
0.5	1.0	3.8557	921.9613
0.6	1.0	2.7209	$1.0480\text{x}10^3$
0.7	1.0	1.8526	$1.1298\text{x}10^3$
0.8	1.0	1.1554↑(increasing above)	$1.1517\text{x}10^3$
0.9↑	1.0	0.6009	$1.0973\text{x}10^3$
1.0	1.0	0.2789	965.9921

From table 4, it is clear that, in long memory effect, minimized total average cost is low compared to the short memory effect. In long memory effect, business policy good. In this case, nature of working of memory effect is not same as table 2 or table 3.

In this case, for long and short memory effect, the numerical values of the minimized total average cost, optimal ordering interval are less compared to the table 2.

(a) Sensitivity Analysis

The sensitivity analysis has been performed by changing each of the parameter by +50%,+10%,-10%,-50%

to show effects on the optimal ordering interval $T^*_{\alpha,1}$ and minimized total average cost $TC^*_{\alpha,1}(T)$ taking one parameter $a, b, P, D_0, C_1, C_2, C_3, t_1$ at a time and keeping the remaining parameters unchanged for the differential memory index $\alpha = 0.1$, and 0.9 *i.e* in long memory effect or short memory effect.

Table 5. The Minimized Total Average Cost and Optimal Ordering Interval in Long Memory Effect.

Parameter	Parameter Change (%)	$T_{\alpha,1}^{\ *}$	$TOC^*_{\ \alpha,1}(T)$	Parameter	Parameter Change (%)	$T_{\alpha,1}^{\ *}$	$TOC^*_{\ \alpha,1}(T)$
a	+50%	51.5409	255.5106	C_1	+50%	44.8302	246.0410
	+10%↑	45.4799	247.0087		+10%↑	44.1232	244.9744
	-10%	42.4040	242.3194↑		-10%	43.7690↑	244.4347↑
	-50%	36.1414↑	231.8107		-50%	43.0593	243.3420
b	+50%	51.5853	255.5697	C_2	+50%	29.9525	326.8845
	+10%↑	45.4890	247.0221		+10%↑	40.1504	262.1321
	-10%	42.3984↑	242.3051↑		-10%	48.5676↓	226.6745↑
	-50%	36.0945	231.7264		-50%	85.0215	146.6384
P	+50%	61.5936	269.3703	C_3	+50%	46.1774	248.0345
	+10%↑	47.4861	250.2987		+10%↑	44.3937	245.3843
	-10%	40.4004↑	238.6749↑		-10%	43.4978↑	244.0190↑
	-50%	26.1 553	208.1995		-50%	41.6973	241.2015
D_0	+50%	31.8809	335.6325	t_1	+50%	53.9493	239.4053
	+10%↑	40.7156	263.5842↑		+10%↑	45.8719↑	243.1322↓
	-10%	47.8457↓	225.3923		-10%	42.0603	246.6529
	-50%	77.7108	142.2489		-50%	34.9645	260.8179

(i) The minimized total average cost decreases when t_1 increases and hence, profit decreases with increasing value of t_1 and the minimized total average cost increases when a, b, P, C_1, C_2, C_3 increase**.**

(ii) The optimal ordering interval gradually decreases with gradually increasing value of C_2 .The optimal ordering interval gradually increases with gradually increasing value of a, b, P, C_1, C_3, t_1. The parameter C_1 is not sensitive but other parameters are sensitive for the market studies.

Table 6. The Minimized Total Average Cost and Optimal Ordering Interval in Short Memory Effect.

Parameter	Parameter Change (%)	$T_{\alpha,1}^{\ *}$	$TOC^*_{\ \alpha,1}(T)$	parameter	Parameter Change (%)	$T_{\alpha,1}^{\ *}$	$TOC^*_{\ \alpha,1}(T)$
a	+50%	2.7148	$1.8114\text{x}10^3$	C_1	+50%	2.3330	$1.6764\text{x}10^3$
	+10%↑	2.3557	$1.6844\text{x}10^3$		+10%↑	2.2723↑	$1.6550\text{x}10^3$
	-10%	2.1535↑	$1.6131\text{x}10^3$↑		-10%	2.2414	$1.6441\text{x}10^3$↑
	-50%	1.6738	$1.4449\text{x}10^3$		-50%	2.1781	$1.6218\text{x}10^3$
b	+50%	2.5210	$1.7428\text{x}10^3$	C_2	+50%	1.8339	$1.7448\text{x}10^3$
	+10%↑	2.3122	$1.6691\text{x}10^3$		+10%↑	2.1463	$1.6719\text{x}10^3$
	-10%	2.2002↑	$1.6296\text{x}10^3$↑		-10%	2.3881↓	$1.6249\text{x}10^3$↑
	-50%	1.9564	$1.5438\text{x}10^3$		-50%	3.3301	$1.4915\text{x}10^3$
P	+50%	2.5868	$2.2554\text{x}10^3$	C_3	+50%	2.4379	$1.7135\text{x}10^3$
	+10%↑	2.3235↑	$1.7720\text{x}10^3$		+10%↑	2.2943	$1.6627\text{x}10^3$
	-10%	2.1899	$1.5264\text{x}10^3$↑		-10%	2.2189↑	$1.6361\text{x}10^3$↑
	-50%	1.9179	$1.0261\text{x}10^3$		-50%	2.0597	$1.5801\text{x}10^3$
D_0	+50%	1.4627	$2.0579\text{x}10^3$	t_1	+50%	3.0405	$1.7779\text{x}10^3$
	+10%↑	2.0722	$1.7429\text{x}10^3$		+10%↑	2.4034↑	$1.6710\text{x}10^3$
	-10%	2.4636↓	$1.5503\text{x}10^3$↑		-10%	2.1172	$1.6308\text{x}10^3$↑
	-50%	3.7247	$1.0840\text{x}10^3$		-50%	1.6543	$1.5948\text{x}10^3$

(iii) In long memory effect, the minimized total average cost gradually decreases with gradually increasing value of t_1 but in short memory effect, the minimized total average cost gradually increases with gradually increasing value of t_1 .

6. Graphical Presentation

(a) Pie diagram of the minimized total average cost with respect to differential memory index.

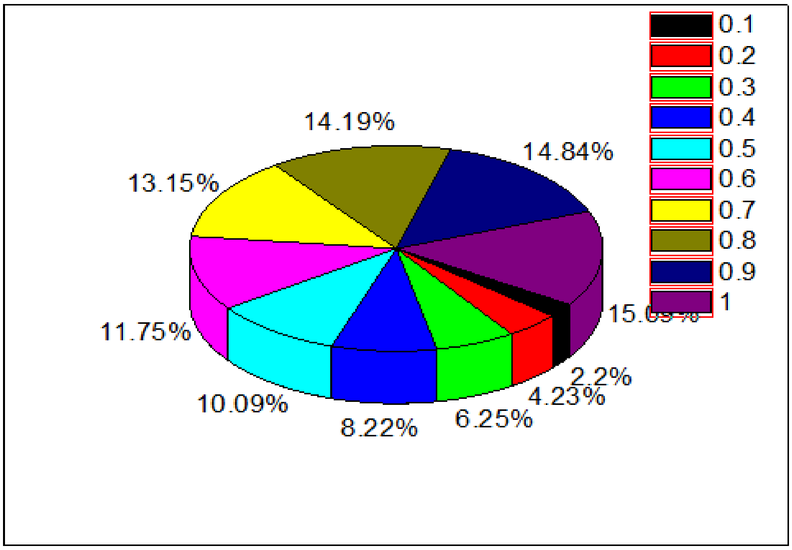

Fig.1. Pie Diagram of Minimized Total Average Cost with Respect to Differential Memory Index for Table 2.

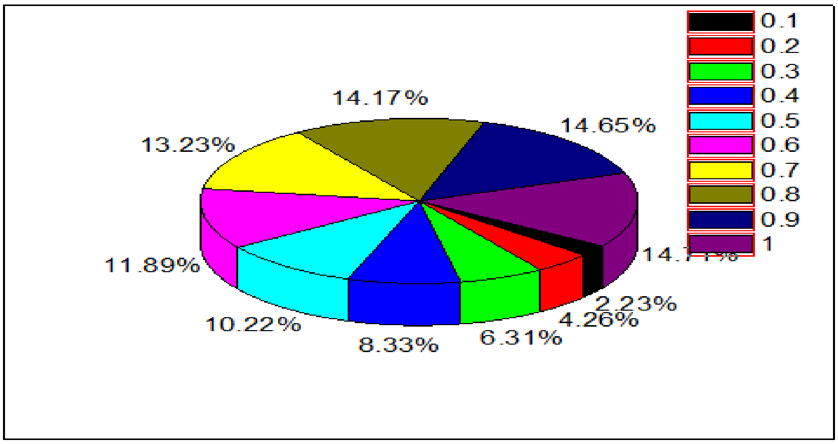

Fig.2. Pie Diagram of Minimized Total Average Cost with Respect to Differential Memory Index for Table 3.

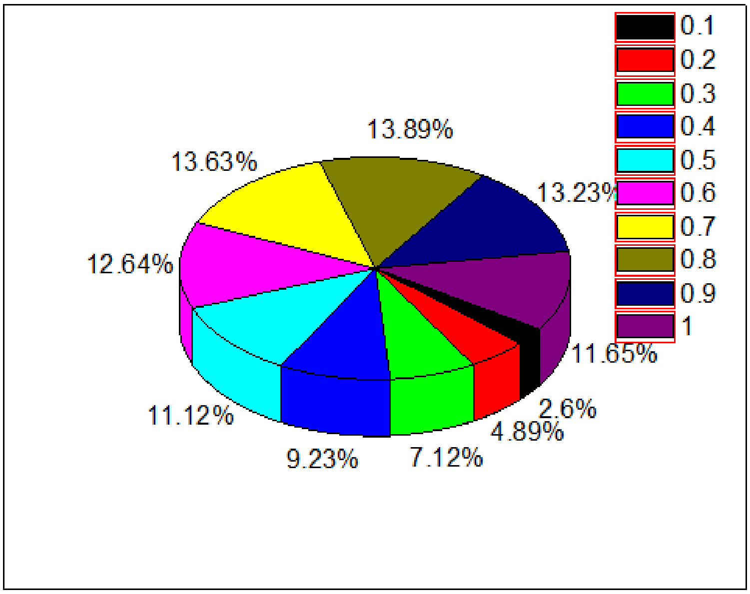

Fig.3. Pie Diagram of Minimized Total Average Cost with Respect to Differential Memory Index for Table 4.

(b) Graphical Comparison of minimized total average cost versus memory index (α) for the above different numerical example.

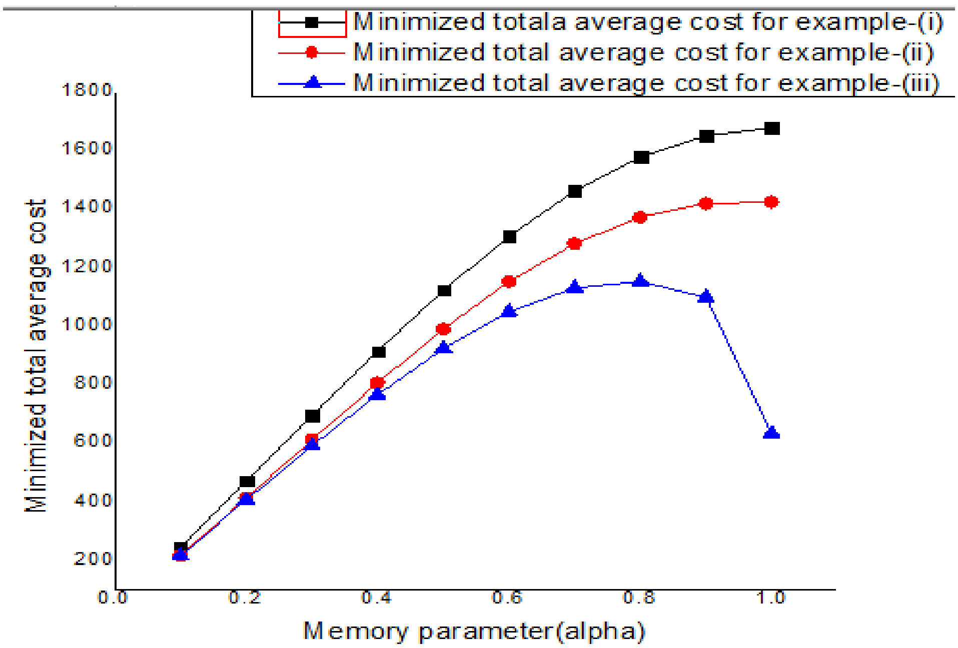

Fig.4. Memory Effect Quantity (Alpha) Versus Minimized Total Average Cost.

From the fig.4., it is found that for numerical example-(iii), nature of changes of the minimized total average cost is totally different from the numerical example-(i), (ii).

7. Conclusion

In this paper, the memory dependency has been established on the economic order quantity model with linear type demand rate and shortage. From the pie diagram, it is clear that in this system, business policy is chosen well but gradually falls down. It also shows that for time proportional demand rate, profit is maximum. Hence, in this case, selling of the product may be more compared to the other cases. To establish real market studies, memory effect should be included in the inventory model. In future study, to establish memory effect, the proposed model may be extended with including stock dependent demand rate, power pattern demand model

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