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# Bayesian Approach to Generalized Normal Distribution under Non-Informative and Informative Priors

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## Abstract

The generalized Normal distribution is obtained from normal distribution by adding a shape parameter to it. This paper is based on the estimation of the shape and scale parameter of generalized Normal distribution by using the maximum likelihood estimation and Bayesian estimation method via Lindley approximation method under Jeffreys prior and informative priors. The objective of this paper is to see which is the suitable prior for the shape and scale parameter of generalized Normal distribution. Simulation study with varying sample sizes, based on MSE, is conducted in *R*-software for data analysis.

## 1. Introduction

The generalized normal (GN) distribution, also known as power exponential, exponential error or generalized Gaussian distribution extends the normal distribution by adding a shape parameter. Being a generalization of the normal and Laplace distributions, this distribution is perhaps the most widely used in areas such as signal processing, quantitative finance, medicine, telecommunication, information systems, physics, analytical chemistry, cybernetics, energy and fuels, biology, nanotechnology, neurosciences, operations research and management science, reliability and risk. Additionally to the mean and variance, the GN distribution has the shape parameter $p$, which is a measure of the peakedness of the distribution, however, it seems that there is not a closed-form expression for estimating $p$.

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Various generalizations of the normal distribution are in use. The most well-known among them appears to be that proposed by Nadarajah (2005); his version alters the kurtosis, adjusting the sharpness of the peak, but maintains a zero-skew symmetry. Varanasi and Aazhang (1989) discuss parameter estimation for the *GN* distribution by using the methods of maximum likelihood and moments. Box (1973) first discussed its characters as well as the Bayes inference. GN distribution has been widely adopted in signal processing. Gharavi and Tabatabai (1988) adopted it to approximate the marginal distribution in image processing.

A random variable *X* is said to follow a (GN) distribution if its PDF is given by

$$f(x; \mu, \sigma, p) = \frac{p}{2\sigma\Gamma(1/p)} e^{-\left|\frac{x-\mu}{\sigma}\right|^p} \quad ; x \in R \tag{1}$$

where the parameters $\mu, \ p > 0$ denote the mean and shape parameters of GN distribution respectively, $\sigma$ is the standard deviation of the GN distribution. The shape parameter *p* denotes the rate of decay: the smaller *p*, the more peaked for the PDF, and the larger *p*, the flatter for the PDF, so it is also called as the decay rate. In most of the applications the mean can be considered as zero, i.e., $\mu = 0$, then we will be focused on estimating the parameter of the GN distribution with two parameter.

For $p = 1$, GN distribution reduces to Laplace or double Exponential distribution, for $p = 2$, GN distribution reduces to Normal distribution and for the limiting cases of *p*, we can obtain a distribution close to the uniform distribution as *p*→0. Like the normal distribution, equation (1) is bell-shaped and unimodal with mode at *x*=μ. Furthermore, $f(\mu) = \frac{p}{2\sigma\Gamma(1/p)}$.

The *GN* distribution is symmetric w.r.t. $\mu$, hence the odd order central moments are zero; i.e.,

$$E(X - \mu)^r = 0 \quad ; r = 1, 3, 5... \tag{2}$$

Let $\mu = 0$ and let $Y = |X - \mu| = |X|$, then the PDF of *Y* is given by

$$f(y; \sigma, p) = \frac{p}{\sigma\Gamma(1/p)} e^{-\left(\frac{y}{\sigma}\right)^p} \quad ; y > 0. \tag{3}$$

The factor 2 in the denominator vanishes to make *Y* > 0. Thus the absolute moments of *X* are given by

$$E|X|^r = E(y)^r = \frac{p}{\sigma\Gamma(1/p)} \int_0^\infty y^r e^{-\left(\frac{y}{\sigma}\right)^p} dy \quad ; r > 0.$$

Put $\left(\frac{y}{\sigma}\right)^p = z \Rightarrow y = \sigma z^{\frac{1}{p}} \Rightarrow dy = \frac{\sigma}{p} z^{\frac{1}{p}-1} dz.$

Also, as $y > 0$ so is $z > 0$.

Therefore, we get

$$E|X|^r = E(y)^r = \frac{\sigma^r}{\Gamma(1/p)} \Gamma\left(\frac{r+1}{p}\right). \tag{4}$$

The variance of $X$ is given by putting $r = 2$ in (4) and is given by

$$V_{ar}(X) = E[X - E(X)]^2 = E(X - \mu)^2 = E(Y)^2 = \sigma^2 \frac{\Gamma(3/p)}{\Gamma(1/p)}.$$

The expectation is still $\mu$ but variance depends on the shape parameter $p$ and decreases with respect to $p$. The CDF of GN distribution is given by

$$F(y; \sigma, p) = \frac{p}{\alpha \Gamma(1/p)} \int_0^y e^{-\left(\frac{y}{\sigma}\right)^p} dy$$

$$= \frac{1}{\Gamma(1/p)} \int_0^{\frac{y^p}{\sigma^p}} e^{-z} z^{\frac{1}{p}-1} dz = \frac{1}{\Gamma(1/p)} \gamma\left(\frac{1}{p}, \frac{y^p}{\sigma^p}\right). \tag{5}$$

In addition to the complementary incomplete gamma function mentioned above, the calculations in this paper use the Euler's psi function defined by $\psi(x) = \frac{d}{dx} \log \Gamma(x)$. This special function can be found in Prudnikov et al. (1990) and Gradshteyn & Ryzhik (2000).

## 2. Maximum Likelihood Estimation

Maximum Likelihood Estimation (MLE) method is the general method of estimating the parameters of a statistical model given observations, by finding the parameter values that maximize the likelihood of making the observations given the parameters. To obtain the MLE of $\sigma$ and $p$, we assume that $X = (x_1, x_2, ..., x_n)$ is a random sample from $\mathrm{GN}(\sigma, p)$ defined in (3), then the likelihood function of $\sigma$ and $p$ is given by

$$L = L(\sigma, p \mid x) = \frac{p^n}{[\sigma \Gamma(1/p)]^n} e^{-\sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^p} \tag{6}$$

and the log-likelihood is given by

$$l = \ln L = n \ln p - n \ln \Gamma(1/p) - n \ln \sigma - \sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^p. \tag{7}$$

Then the MLE of $\sigma$ and $p$ is obtained by the solution of the equation

$$-\frac{n}{\sigma}+\frac{p}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^p=0$$

$$\Rightarrow \hat{\sigma}=\left[\frac{p}{n}\sum_{i=1}^{n}x_i^{\ p}\right]^{\frac{1}{p}} \tag{8}$$

which depends on the shape parameter $p$.

And $1+\dfrac{1}{p}\psi(1/p)-\dfrac{p}{n}\sum_{i=1}^{n}\left(\dfrac{x_i}{\hat{\sigma}}\right)^p\ln\left(\dfrac{x_i}{\hat{\sigma}}\right)=0.$

Substituting the value of $\alpha$ from (8), we get

$$1+\frac{1}{p}\psi(1/p)-\frac{\sum_{i=1}^{n}x_i^{\ p}\ln x_i}{\sum_{i=1}^{n}x_i^{\ p}}+\frac{1}{p}\ln\left(\frac{p}{n}\sum_{i=1}^{n}x_i^{\ p}\right)=0. \tag{9}$$

Equation (9) can be solved by the iteration method of Newton-Raphson using

$$p_{k+1}=p_k-\frac{f(p_k)}{f'(p_k)} \tag{10}$$

where $f(p_k)$ is denoted by (9) and

$$f'(p_k)=-\frac{1}{p^2}\psi(1/p)-\frac{1}{p^3}\psi'(1/p)-\frac{\sum_{i=1}^{n}x_i^{\ p}(\ln x_i)^2}{\left(\sum_{i=1}^{n}x_i^{\ p}\right)^2}-\left(\frac{\sum_{i=1}^{n}x_i^{\ p}\ln x_i}{\sum_{i=1}^{n}x_i^{\ p}}\right)^2 \tag{11}$$

$$+\frac{\sum_{i=1}^{n}x_i^{\ p}\ln x_i}{p\sum_{i=1}^{n}x_i^{\ p}}-\frac{1}{p^2}\ln\left(\frac{p}{n}\sum_{i=1}^{n}x_i^{\ p}\right)+\frac{1}{p^2}$$

where $\psi(z)=\dfrac{\Gamma'(z)}{\Gamma(z)}$ denotes the digamma function $\psi'(z)$ denotes the trigamma function. Thus $\hat{p}$ is obtained

by choosing the initial value of $p_k$ and iterating the procedure till it converges. When $\hat{p}$ is obtained, $\hat{\sigma}$ can be obtained from (8).

## 3. Fisher Information of GN Distribution

For estimation procedures, the Fisher information matrix for $n$ observations is required. It is used in the asymptotic theory of MLE and Jeffreys prior The second-order derivatives of equation (7) are:

$$\frac{\partial^2 l}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{p(p+1)}{\sigma^2}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^p,$$

$$\frac{\partial^2 l}{\partial p\,\partial \sigma} = \frac{p}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^p\ln\left(\frac{x_i}{\sigma}\right) + \frac{1}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^p,$$

and $\dfrac{\partial^2 l}{\partial p^2} = -\dfrac{n}{p} - \dfrac{2n}{p^3}\psi(1/p) - \dfrac{n}{p^4}\psi'(1/p) - \sum_{i=1}^{n}\left(\dfrac{x_i}{\sigma}\right)^p\ln^2\left(\dfrac{x_i}{\sigma}\right).$

Now, since $X \sim GN(\sigma, p)$, given in (3), we have the following results:

$$E\left(\frac{x}{\sigma}\right)^r = \frac{p}{\sigma\Gamma(1/p)}\int_{0}^{\infty}\left(\frac{x_i}{\sigma}\right)^r e^{-\left(\frac{x}{\sigma}\right)^p}dx$$

$$= \frac{1}{\Gamma(1/p)}\int_{0}^{\infty}z^{\frac{(r+1)}{p}-1}e^{-z^p}dz = \frac{\Gamma\left(\overline{r+1/p}\right)}{\Gamma(1/p)},$$

$$E\left(\frac{x}{\sigma}\right)^r\ln\left(\frac{x}{\sigma}\right) = \frac{1}{p\Gamma(1/p)}\int_{0}^{\infty}z^{\frac{(r+1)}{p}-1}e^{-z}\ln z\,dz$$

$$= \frac{\Gamma'\left(\overline{r+1/p}\right)}{p\Gamma(1/p)} = \frac{\Gamma\left(\overline{r+1/p}\right)}{p\Gamma(1/p)}\psi\left(\overline{r+1/p}\right),$$

and $E\left(\dfrac{x}{\sigma}\right)^r\ln^2\left(\dfrac{x}{\sigma}\right) = \dfrac{1}{p^2\Gamma(1/p)}\int_{0}^{\infty}z^{\frac{(r+1)}{p}-1}e^{-z}\ln^2 z\,dz$

$$= \frac{\Gamma'\left(\overline{r+1/p}\right)}{p^2\Gamma(1/p)} = \frac{\Gamma\left(\overline{r+1/p}\right)}{p^2\Gamma(1/p)}\left[\psi^2\left(\overline{r+1/p}\right)+\psi'\left(\overline{r+1/p}\right)\right].$$

The elements of the Fisher information matrix are, therefore, given by:

$$E\left(-\frac{\partial^2 l}{\partial \sigma^2}\right) = \frac{n\,p}{\sigma^2},$$

$$E\left(-\frac{\partial^2 l}{\partial p\,\partial\sigma}\right)=-\frac{n}{p\,\sigma}\left[1+\psi\left(1+1/p\right)\right],$$

and $E\left(-\frac{\partial^2 l}{\partial p^2}\right)=\frac{n}{p^2}\left[1+\frac{2}{p}\psi\left(1/p\right)+\frac{1}{p^2}\psi'\left(1/p\right)+\frac{1}{p}\left\{\psi^2\left(1+1/p\right)+\psi'\left(1+1/p\right)\right\}\right].$

Thus, the Fisher information matrix is given by:

$$I(\sigma,p)=\frac{n^2}{\sigma^2 p^2}\left[\begin{array}{l} p+2\psi\left(1/p\right)+\dfrac{1}{p}\psi'\left(1/p\right)+\psi^2\left(1+1/p\right)\\[2mm] \qquad +\psi'\left(1+1/p\right)-\left\{1+\psi\left(1+1/p\right)\right\}^2 \end{array}\right]$$

$$=\frac{n^2}{\sigma^2 p^2}\left[p+2\psi\left(1/p\right)+\frac{1}{p}\psi'\left(1/p\right)+\psi'\left(1+1/p\right)-1-2\psi\left(1+1/p\right)\right] \tag{12}$$

Using the following results: $\psi\left(1+x\right)=\psi\left(x\right)+\frac{1}{x}$ and $\psi'\left(1+x\right)=\psi'\left(x\right)-\frac{1}{x^2}$.

We have

$$I(\sigma,p)=\frac{n^2}{\sigma^2 p^2}\left[\left(1+\frac{1}{p}\right)\psi'\left(1/p\right)-p\left(p+1\right)\right]. \tag{13}$$

## 4. Lindley Approximation of α and p under Jeffreys Invariant Prior

Bayesian approximation has found good sources provided by the works by Ahmad et.al (2007, 2011) discussed Bayesian analysis of exponential distribution and gamma distribution using normal and Laplace approximations. Sultan and Ahmad (2015) obtained Bayesian estimation of generalized gamma distribution using Lindleys approximation technique for two parameters. Fatima and Ahmad (2018) studied the Bayes estimates of Inverse Exponential distribution using various Bayesian approximation techniques like normal approximation, Tierney and Kadane (T-K) approximation methods.

The GN distribution has not been discussed in detail under the Bayesian approach. Our present study aims to obtain the Bayesian estimators for the shape and scale parameter of the GN distribution based on Lindley approximation technique. A simulation study is also discussed with concluding remarks.

For a Bayesian analysis of the GN distribution, we can use different prior distributions for the model parameters $\sigma$ and $p$. The Jeffreys invariant prior (Box and Tiao (1973)) and Gelman et al. (1995) for $\sigma$ and $p$ is given as:

$$g(\sigma,p)=\left[\det I(\sigma\,p)\right]^{1/2} \tag{14}$$

where $I(\sigma,p)$ is the joint Fisher information matrix defined in (13). Thus, the Jeffreys invariant prior for $\sigma$ and $p$ is given by:

$$g\left(\sigma, p\right) \propto \frac{\sqrt{A(p)}}{\sigma \, p}, \text{ where } A(p) = \sqrt{\left(1 + \frac{1}{p}\right)\psi'(1/p) - p(p+1)}.$$

Then the joint posterior distribution of $\sigma$ and $p$ is given by

$$\pi\left(\sigma, p \mid x\right) \propto L\left(x \mid \sigma, p\right) \times g\left(\sigma, p\right)$$

$$= K \frac{p^{n-1}\sqrt{A(p)}}{\sigma^{n+1}\left[\Gamma(1/p)\right]^{n}} e^{-\sum\limits_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{p}}$$

where $K$ is a normalizing constant defined as

$$K^{-1} = \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \frac{p^{n-1}\sqrt{A(p)}}{\sigma^{n+1}\left[\Gamma(1/p)\right]^{n}} e^{-\sum\limits_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{p}} d\sigma \, dp.$$

These integrals cannot be simplified in to a closed form. However, using the approach developed by Lindley (1980), one can approximate these Bayes estimators in to a form containing no integrals. This method provides a simplified form of Bayes estimator which is easy to use in practice. The basic idea in Lindley approach is to obtain Talyor Series expansion of the function involved in posterior moment. Lindley proposed the ratio for obtaining Bayes estimates as

$$E\left[h(\theta) \mid x\right] = \frac{\int\limits_{\Theta} h(\theta) e^{L(\theta)} g(\theta) d\theta}{\int\limits_{\Theta} e^{L(\theta)} g(\theta) d\theta} \tag{15}$$

where $\theta = \left(\theta_1, \theta_2 \dots \theta_l\right)$, $L(\theta)$ is the logarithmic of likelihood function, $h(\theta)$ and $g(\theta)$ is arbitrary function of $\theta$; $\Theta$ represents the space range of $\theta$. About the maximum likelihood estimator $\hat{\theta}$, Lindley approximated that posterior moment by

$$E\left[h(\theta) \mid x\right] = \left\{h + \frac{1}{2}\sum_{i}\sum_{j}\left(h_{ij} + 2h_i\phi_j\right)\phi_{ij} + \frac{1}{2}\sum_{i}\sum_{j}\sum_{k}\sum_{r} L_{ijk}\phi_{ij}\phi_{kr}\right\} + O\left(\frac{1}{n^2}\right) \tag{16}$$

where $i, j, k, r = 1, 2, \dots m$; $\theta = \theta_1, \theta_2, \dots \theta_m$

$$h_i = \frac{\partial h}{\partial \theta_i}, \; h_{ij} = \frac{\partial^2 L}{\partial \theta_i \partial \theta_j}, \; L_{ijk} = \frac{\partial^3 L}{\partial \theta_i \partial \theta_j \partial \theta_k}, \; \rho = \rho(\theta) = \log[g(\theta)], \rho_i = \frac{\partial \rho}{\partial \theta_i}$$

and $\phi_{ij}$ is the $(i,j)^{\text{th}}$ element in $\left[L_{ij}\right]^{-1}$

For two parameters case m = 2, we have

$$L_{20} = \frac{\partial^2 l}{\partial \sigma^2} = \frac{1}{\sigma^2}\left[n - p(p+1)\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^p\right],$$

$$L_{30} = \frac{\partial^3 l}{\partial \sigma^3} = \frac{1}{\sigma^3}\left[-2n + p(p+1)(p+2)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^p\right],$$

$$L_{21} = \frac{\partial^3 l}{\partial p\,\partial \sigma^2} = -\frac{1}{\sigma^2}\left[(2p+1)\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^p - p(p+1)\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^p \ln\left(\frac{x_i}{\sigma}\right)\right],$$

$$L_{02} = \frac{\partial^2 l}{\partial p^2} = -\left[\frac{n}{p^2} + \frac{2n}{p^3}\psi(1/p) + \frac{n}{p^4}\psi'(1/p) - \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^p \ln^2\left(\frac{x_i}{\sigma}\right)\right],$$

$$L_{03} = \frac{\partial^3 l}{\partial p^3} = \frac{2n}{p^3} + \frac{6n}{p^4}\psi(1/p) + \frac{6n}{p^5}\psi'(1/p) + \frac{n}{p^6}\psi''(1/p) - \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^p \ln^3\left(\frac{x_i}{\sigma}\right)$$

and

$$L_{21} = \frac{\partial^3 l}{\partial \sigma\,\partial p^2} = \frac{1}{\sigma}\left[2\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^p \ln\left(\frac{x_i}{\sigma}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^p \ln^2\left(\frac{x_i}{\sigma}\right)\right]$$

Hence, $\phi_{11} = -L_{20}^{-1} = -\dfrac{\hat{\sigma}}{n - p(p+1)\sum_{i=1}^{n}\left(\dfrac{x_i}{\sigma}\right)^p}$

and

$$\phi_{22} = -L_{02}^{-1} = \left[\frac{n}{p^2} + \frac{2n}{p^3}\psi(1/p) + \frac{n}{p^4}\psi'(1/p) - \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^p \ln^2\left(\frac{x_i}{\sigma}\right)\right]^{-1}.$$

and,

$$\rho = \ln g(\sigma, p) = -\ln \sigma - \ln p + \frac{1}{2}\ln[A(p)] \tag{17}$$

$$\therefore \rho_1 = -\frac{1}{\sigma} \text{ and } \rho_2 = -\frac{1}{p} + \frac{A'(p)}{2A(p)}$$

where $A'(p) = -\frac{1}{p^2}\left[\psi'(1/p) + \left(1 + \frac{1}{p}\right)\psi''(1/p)\right] - (2p+1)$

When $h(\sigma, p) = \sigma \Rightarrow h_1 = 1, h_{11} = h_2 = h_{22} = 0$, the Lindley approximation for $\sigma$ under Jeffreys prior is obtained as

$$E(\sigma \mid x) = \hat{\sigma}\left[1 + \frac{1}{2\left[n - \hat{p}(\hat{p}+1)\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^p\right]}\left(2 + \frac{-2n + \hat{p}(\hat{p}+1)(\hat{p}+2)\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^p}{\left[n - \hat{p}(\hat{p}+1)\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^p\right]} - \frac{2\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^p \ln\left(\frac{x_i}{\hat{\sigma}}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^p \ln^2\left(\frac{x_i}{\hat{\sigma}}\right)}{\left[\frac{n}{\hat{p}^2} + \frac{2n}{\hat{p}^3}\psi(1/\hat{p}) + \frac{n}{\hat{p}^4}\psi'(1/\hat{p}) - \sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^p \ln^2\left(\frac{x_i}{\hat{\sigma}}\right)\right]}\right)\right].$$

When $h(\sigma, p) = p \Rightarrow h_2 = 1, h_{22} = h_{21} = h_{11} = 0$, the Lindley approximation for $p$ under the non-informative prior is obtained as

$$E(p \mid x) = \hat{p} + \frac{\left[\begin{array}{l} -\dfrac{\left[\dfrac{1}{\hat{p}} + \dfrac{1}{\hat{p}^2}\psi'(1/\hat{p}) + \psi''(1/\hat{p}) + (2\hat{p}+1)\right]}{\left(1 + \dfrac{1}{\hat{p}}\right)\psi'(1/\hat{p}) - \hat{p}(\hat{p}+1)} \\[2em] +\dfrac{\dfrac{2n}{\hat{p}^3} + \dfrac{6n}{\hat{p}^4}\psi(1/\hat{p}) + \dfrac{6n}{\hat{p}^5}\psi'(1/\hat{p}) + \dfrac{n}{\hat{p}^6}\psi''(1/\hat{p}) - \sum_{i=1}^{n}\left(\dfrac{x_i}{\hat{\sigma}}\right)^{\hat{p}}\ln^3\left(\dfrac{x_i}{\hat{\sigma}}\right)}{2\left[\dfrac{n}{\hat{p}^2} + \dfrac{2n}{\hat{p}^3}\psi(1/\hat{p}) + \dfrac{n}{\hat{p}^4}\psi'(1/\hat{p}) - \sum_{i=1}^{n}\left(\dfrac{x_i}{\hat{\sigma}}\right)^{\hat{p}}\ln^2\left(\dfrac{x_i}{\hat{\sigma}}\right)\right]} \\[2em] +\dfrac{(2\hat{p}+1)\sum_{i=1}^{n}\left(\dfrac{x_i}{\hat{\sigma}}\right)^{\hat{p}} - \hat{p}(\hat{p}+1)\sum_{i=1}^{n}\left(\dfrac{x_i}{\hat{\sigma}}\right)^{\hat{p}}\ln\left(\dfrac{x_i}{\hat{\sigma}}\right)}{2\left[n - \hat{p}(\hat{p}+1)\sum_{i=1}^{n}\left(\dfrac{x_i}{\hat{\alpha}}\right)^{\hat{p}}\right]} \end{array}\right]}{\left[\dfrac{n}{\hat{p}^2} + \dfrac{2n}{\hat{p}^3}\psi(1/\hat{p}) + \dfrac{n}{\hat{p}^4}\psi'(1/\hat{p}) - \sum_{i=1}^{n}\left(\dfrac{x_i}{\hat{\sigma}}\right)^{\hat{p}}\ln^2\left(\dfrac{x_i}{\hat{\sigma}}\right)\right]}.$$

## 5. Lindley Approximation of α and p under Uniform and Gamma Prior

Assuming that $\sigma$ has a Uniform prior density given by

$$g(\sigma) = 1, \sigma > 0$$

and *p* has gamma prior with known hyper-parameters *a* and *b* given as

$$g(p) = \frac{b^{-a}}{\Gamma(a)} p^{a-1} e^{-\frac{p}{b}}, p > 0, a, b > 0.$$

Then the joint prior density is defined by

$$g(\sigma, p) \propto p^{a-1} e^{-\frac{p}{b}}. \tag{18}$$

The joint posterior distribution of $\sigma, p$ is given by

$$\pi_2(\sigma, p \mid x) \propto L(x \mid \sigma, p) \times g(\sigma, p)$$

$$= K \frac{p^{n+a-1}}{\sigma^n [\Gamma(1/p)]^n} e^{-\left[\sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^p + \frac{p}{b}\right]}$$

where *K* is a normalizing constant defined as

$$K^{-1} = \int_0^\infty \int_0^\infty \frac{p^{n+a-1}}{\sigma^n [\Gamma(1/p)]^n} e^{-\left[\sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^p + \frac{p}{b}\right]} d\sigma \, dp.$$

Again, $g(\sigma, p) \propto e^{-\frac{p}{b}} p^{a-1}$

$$\therefore \rho = \ln g(\sigma, p) = \ln \sigma + (a-1)\ln p - \frac{p}{b} \tag{19}$$

$$\Rightarrow \rho_1 = 0 \text{ and } \rho_2 = \frac{a-1}{p} - \frac{1}{b}$$

When $h(\sigma, p) = \sigma \Rightarrow h_1 = 1, h_{11} = h_2 = h_{22} = 0$, the Lindley approximation for $\sigma$ under uniform prior is

obtained as

$$
E(\sigma \mid x) = \hat{\sigma}\left[1 + \frac{1}{2\left[n - \hat{p}(\hat{p}+1)\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^{p}\right]}\left(\frac{-2n + \hat{p}(\hat{p}+1)(\hat{p}+2)\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^{p}}{n - \hat{p}(\hat{p}+1)\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^{p}} - \frac{2\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^{p}\ln\left(\frac{x_i}{\hat{\sigma}}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^{p}\ln^{2}\left(\frac{x_i}{\hat{\sigma}}\right)}{\frac{n}{\hat{p}^{2}} + \frac{2n}{\hat{p}^{3}}\psi(1/\hat{p}) + \frac{n}{\hat{p}^{4}}\psi'(1/\hat{p}) - \sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^{p}\ln^{2}\left(\frac{x_i}{\hat{\sigma}}\right)}\right)\right].
$$

When $h(\sigma, p) = p \Rightarrow h_2 = 1, h_{22} = h_1 = h_{11} = 0$, the Lindley approximation for $p$ under the gamma prior is obtained as

$$
E(p \mid x) = \hat{p} + \frac{\left[\begin{array}{l} -\dfrac{1}{b} + \dfrac{a-1}{\hat{p}} + \dfrac{\dfrac{2n}{\hat{p}^{3}} + \dfrac{6n}{\hat{p}^{4}}\psi(1/\hat{p}) + \dfrac{6n}{\hat{p}^{5}}\psi'(1/\hat{p}) + \dfrac{n}{\hat{p}^{6}}\psi''(1/\hat{p}) - \sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{p}}\ln^{3}\left(\frac{x_i}{\hat{\sigma}}\right)}{2\left[\dfrac{n}{\hat{p}^{2}} + \dfrac{2n}{\hat{p}^{3}}\psi(1/\hat{p}) + \dfrac{n}{\hat{p}^{4}}\psi'(1/\hat{p}) - \sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{p}}\ln^{2}\left(\frac{x_i}{\hat{\sigma}}\right)\right]} \\[1.5em] + \dfrac{(2\hat{p}+1)\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{p}} - \hat{p}(\hat{p}+1)\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{p}}\ln\left(\frac{x_i}{\hat{\sigma}}\right)}{2\left[n - \hat{p}(\hat{p}+1)\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{p}}\right]} \end{array}\right]}{\left[\dfrac{n}{\hat{p}^{2}} + \dfrac{2n}{\hat{p}^{3}}\psi(1/\hat{p}) + \dfrac{n}{\hat{p}^{4}}\psi'(1/\hat{p}) - \sum_{i=1}^{n}\left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{p}}\ln^{2}\left(\frac{x_i}{\hat{\sigma}}\right)\right]}.
$$

## 6. Simulation Study

The simulation study was conducted in R-software using the *pgnorm* package to examine the performance of Bayes estimates for the shape ($p$) and scale ($\sigma$) parameter of the generalized normal distribution under Jeffreys invariant prior and informative (Uniform and Gamma) prior using Lindley approximation technique. We choose n (= 20, 60, 100, 150) to represent different sample sizes, in each sequence of the ML estimates and Bayes estimates for the given values of $p = 1, 2, 4$ and $\sigma = 2, 4, 6$ the desired ML and Bayes estimates are presented in Tables 1, 2 and 3. The hyper-parameter values are chosen $a = (1, 1.7, 2.5)$ and $b = (2, 3, 3.5)$. The estimators are obtained along with their respective MSE. The results are replicated 5000 times and the average of the results has been presented in the tables below.

Table 1. ML Estimates and MSE of p and σ

| n | $p$ | $\sigma$ | $\hat{p}$ | $MSE(\hat{p})$ | $\hat{\sigma}$ | $MSE(\hat{\sigma})$ |
|---|---|---|---|---|---|---|
|  | 1.0 | 2.0 | 1.70180 | 1.035623 | 4.26663 | 7.13999 |
| 20 | 2.0 | 4.0 | 3.00955 | 6.12246 | 5.50290 | 4.17824 |
|  | 4.0 | 6.0 | 1.74799 | 5.88624 | 5.36854 | 4.27863 |
|  | 1.0 | 2.0 | 1.43941 | 0.33298 | 2.97165 | 1.46411 |
| 60 | 2.0 | 4.0 | 2.21905 | 0.55855 | 5.63489 | 3.56623 |
|  | 4.0 | 6.0 | 4.60557 | 2.67606 | 8.23932 | 5.44743 |
|  | 1.0 | 2.0 | 1.03299 | 0.04012 | 1.82053 | 0.28776 |
| 100 | 2.0 | 4.0 | 3.33154 | 2.73334 | 7.19524 | 10.6439 |
|  | 4.0 | 6.0 | 3.81543 | 0.92509 | 8.86350 | 8.61033 |
|  | 1.0 | 2.0 | 1.04799 | 0.02344 | 2.01891 | 0.16495 |
| 150 | 2.0 | 4.0 | 2.33422 | 3.52167 | 5.86343 | 3.75382 |
|  | 4.0 | 6.0 | 3.98795 | 0.92848 | 8.41659 | 6.11403 |

Table 2. Estimates of p and $\sigma$ under Jeffreys Prior

| n | $p$ | $\sigma$ | $\hat{p}$ | $MSE(\hat{p})$ | $\hat{\sigma}$ | $MSE(\hat{\sigma})$ |
|---|---|---|---|---|---|---|
|  | 1.0 | 2.0 | 1.84972 | 1.26512 | 2.07642 | 2.00822 |
| 20 | 2.0 | 4.0 | 3.77079 | 8.23897 | 4.05533 | 1.92259 |
|  | 4.0 | 6.0 | 3.54917 | 1.01794 | 6.02304 | 3.88042 |
|  | 1.0 | 2.0 | 1.09269 | 0.14849 | 2.03379 | 0.52115 |
| 60 | 2.0 | 4.0 | 2.00275 | 0.51058 | 4.00861 | 0.89343 |
|  | 4.0 | 6.0 | 3.85749 | 2.32965 | 6.00642 | 0.43292 |
|  | 1.0 | 2.0 | 1.05165 | 1.05432 | 2.02979 | 0.25644 |
| 100 | 2.0 | 4.0 | 1.98551 | 0.96055 | 4.00205 | 0.43434 |
|  | 4.0 | 6.0 | 3.95455 | 0.89309 | 6.00232 | 0.41071 |
|  | 1.0 | 2.0 | 1.04945 | 0.02359 | 2.01487 | 0.16481 |
| 150 | 2.0 | 4.0 | 1.99519 | 0.16898 | 4.00366 | 0.28146 |
|  | 4.0 | 6.0 | 3.95352 | 0.93049 | 6.00168 | 0.27412 |

Table 3. Estimates of *p* and *σ* under Gamma and Uniform prior

| n | p | σ | $\hat{p}$ | | | MSE ($\hat{p}$) | | | $\hat{\sigma}$ | MSE ($\hat{\sigma}$) |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | a=1 b=2 | a=1.7 b=3 | a=2.5 b=3.5 | a=1 b=2 | a=1.7 b=3 | a=2.5 b=3.5 | | |
| | 1.0 | 2.0 | 1.62803 | 1.71924 | 1.80844 | 0.93752 | 1.06041 | 1.19668 | 2.04116 | 2.00407 |
| 20 | 2.0 | 4.0 | 4.93438 | 4.56135 | 4.23818 | 13.7139 | 11.6638 | 10.1127 | 4.02277 | 1.92005 |
| | 4.0 | 6.0 | 4.25978 | 4.10167 | 3.98708 | 0.88218 | 0.82503 | 0.81486 | 6.00434 | 3.87991 |
| | 1.0 | 2.0 | 1.05226 | 0.14263 | 1.08517 | 0.14263 | 0.87498 | 0.14715 | 2.01697 | 0.52030 |
| 60 | 2.0 | 4.0 | 2.13753 | 2.09432 | 2.05689 | 0.52948 | 0.51947 | 0.51381 | 4.00344 | 0.89337 |
| | 4.0 | 6.0 | 4.10005 | 4.04608 | 4.00696 | 2.31935 | 2.31146 | 2.30939 | 6.00135 | 0.43288 |
| | 1.0 | 2.0 | 1.02783 | 1.03763 | 1.04721 | 0.03980 | 0.04045 | 0.04126 | 2.01371 | 0.25574 |
| 100 | 2.0 | 4.0 | 2.04208 | 2.02394 | 2.00823 | 0.96211 | 0.96091 | 0.96041 | 4.00074 | 0.43434 |
| | 4.0 | 6.0 | 4.01342 | 4.00032 | 3.99083 | 0.89120 | 0.89102 | 0.89110 | 6.00047 | 0.41070 |
| | 1.0 | 2.0 | 1.03025 | 1.03815 | 1.04588 | 0.02206 | 0.02260 | 0.02324 | 2.00721 | 0.16464 |
| 150 | 2.0 | 4.0 | 2.04271 | 2.02747 | 2.01428 | 0.17078 | 0.16971 | 0.16916 | 4.00152 | 0.28145 |
| | 4.0 | 6.0 | 4.02171 | 4.00654 | 3.99554 | 0.92880 | 0.92837 | 0.92835 | 6.00024 | 0.27412 |

## 7. Discussion

The results obtained using above programme are presented in tables 1, 2 and 3 for different values of *n* and mean (= 0) and hyper-parameters.It is observed that the values of the MSE of *σ* under both Jeffreys and Uniform prior are almost same. For the shape parameter, *p*, the Gamma prior provides values having less MSE as compared to Jeffreys prior. Also, it can beobserved that as the hyper-parameter values increase the MSE decreases simultaneously.

## 8. Conclusion

This paper deals with the approximation methods in case of the generalized normal (GN) distribution. The aim is to study the Bayes estimates of the shape and scale parameters of the generalized normal distribution. We observe that under informative as well as non- informative priors, the posterior variances under Jeffreys or Uniform prior distribution for σ are close to each other. But *p* has less posterior variances under Gamma prior. Thus, we conclude that we can prefer Jeffreys or Uniform prior distribution equally as a prior for the standard deviation and Gamma prior for *p* of GN distribution. The MSE is also seen decreasing for increasing sample size. Further, we conclude that the posterior variance based on different priors tends to decrease with the increase in sample size. It implies that the estimators obtained are consistent.

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