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Asymptotic Solutions of a Semi-submerged Sphere in a Liquid under Oscillations

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Abstract

One of the most widely used techniques to look into transient behaviour of vibrating systems is the Krylov-Bogoliubov-Mitropolskii (KBM) method, which was developed for obtaining the periodic solutions of second order nonlinear differential systems of small nonlinearities. Later on, this method was studied and modified by numerous scholars to obtain solutions of higher order nonlinear systems. This article modified the method to study the solutions of semi-submerged sphere in a liquid which is floating owing to the gravitational force and the upward pressure of the liquid. This paper suggests that the results obtained for different sets of initial conditions by the modified KBM method correspond well with those obtained by the numerical method.

Index Terms: Asymptotic solution, perturbation solution, oscillatory system, half submerged sphere.

1. Introduction

A perturbation method has been used by Krylov and Bogoliubov [1] to discuss transients in the second order nonlinear differential system with a small nonlinearity

$$x + \omega_0^2 x = -\varepsilon f(x, x) \tag{1}$$

where over dots denote the first and second order differentiation with respect to t, $\omega_0 > 0$, ε is a sufficiently small parameter and $f(x, x)$ is the nonlinear function. Bogoliubov and Mitropolskii [2] amplified and

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justified this method and then in present the method is a well-known method as Krylov-Bogoliubov-Mitropolskii (KBM) [1, 2] method in the literature of nonlinear oscillations. Later, the method was extended by Popov [3] to the following damped oscillatory system

$$x + 2kx + \omega^2 x = -\varepsilon f(x, x) \tag{2}$$

where $\omega > k > 0$. Mendelson [4] has rediscovered the Popov's results. If $k \geq \omega$, then it is clear that the system (2) becomes non-oscillatory. Murty, Deekshatulu and Krisna [5] have used the MBK method to discuss transients in equation (2) for the over-damped case, $k > \omega$. Murty [6] has presented a unified KBM method for solving equation (2). Sattar [7] has found a solution of (2) characterized by critically damping, *i.e.* $k = \omega$. Later, Shamsul [8] has extended the unified method of Murty [6] to critically damped nonlinear systems. Here, a semi-submerged sphere in a liquid under oscillations due to the gravitational force is considered and derived the governing equation for this system. The solution of the system for both oscillatory and damped oscillatory motions is also investigated. In these cases, therefore, the eigenvalues are pure imaginary and complex conjugate for undamped and damped motions respectively. The obtained perturbation results reveal well coincidence with the numerical results obtained by using *Mathematica* for different sets of initial conditions as well as different sets of eigenvalues.

2. Related Work

During last several decades in the 20th century, some Russian scientists like Mandelstam and Papalexi [9], Krylov and Bogoliubov [10], Bogoliubov and Mitropolskii [11] jointly worked on the nonlinear dynamics. To solve nonlinear differential equations there exist some methods, among the methods, the method of perturbations, *i.e.*, an asymptotic expansion in terms of a small parameter is the most advanced. A simple analytical method was presented by Murty and Deekshatulu [12] for obtaining the time response of second order nonlinear over-damped systems with small nonlinearity based on the Krylov-Bogoliubov method of variation of parameters. Lin and Khan [38] have also used the KBM method for some biological problems, and Bojadziev *et al*. [13] have investigated periodic solutions of nonlinear systems by the KBM and Poincare method and compared the two solutions. Osiniskii [14] has also extended the KBM method to a third order nonlinear partial differential equation with initial friction and relaxation. Mulholland [15] studied nonlinear oscillations governed by a third order differential equation. Lardner and Bojadziev [16] investigated nonlinear damped oscillations governed by a third order partial differential equation. They introduced the concept of "couple amplitude" where the unknown functions A_k, B_k and C_k depend on the both amplitude *a* and *b*. Bojadziev [17], and Bojadziev and Hung [18] used at least two trial solutions to investigate time dependent differential systems one is for resonant case and the other is for the non-resonant case. But Shamsul used only one set of vibrational equations, arbitrarily for both resonant and no resonant cases. Shamsul *et al*. [19] presented a general form of the KBM method for solving nonlinear partial differential equations. Raymond and Cabak [20] examined the effects of internal resonance on impulsive forced nonlinear systems with two-degree-of-freedom. Later, Shamsul [21, 22] has extended the method to n^{th} order nonlinear systems. Shamsul [23, 24] has also extended the KBM method for certain non oscillatory nonlinear systems when the eigenvalues of the unperturbed equation are real and negative. Ali Akbar *et al*. [25] extended the KBM method which present in [26] for the fourth order damped oscillatory systems. Ali Akbar *et al*. [27] presented a unified KBM method for solving n^{th} order nonlinear systems under some special conditions including the case of internal resonance. Ali Akbar *et al*. [28] also extended the theory of perturbation for fourth order nonlinear systems with large damping. Abul Kawser *et al.* [29] present an Asymptotic Solution for the Third Order Critically Damped Nonlinear System in the Case for Small Equal Eigenvalues.

3. Formulation of the System

Let us suppose that a semi-submerged sphere of radius R is floating in a liquid and x is the instantaneous displacement of its diametric plane from the equilibrium position.

Partial volume the sphere from the bottom of height h is,

$$V_h = \frac{1}{3}\pi h^2 (3R - h) \tag{3}$$

Volume of half sphere, $V_R = \frac{2}{3}\pi R^3$

Partial volume of sphere from the bottom of height $R + x$ is,

$$V_{R+x} = \frac{\pi}{3}(R+x)^2(3R - R - x) = \frac{\pi}{3}(R+x)^2(2R - x)$$

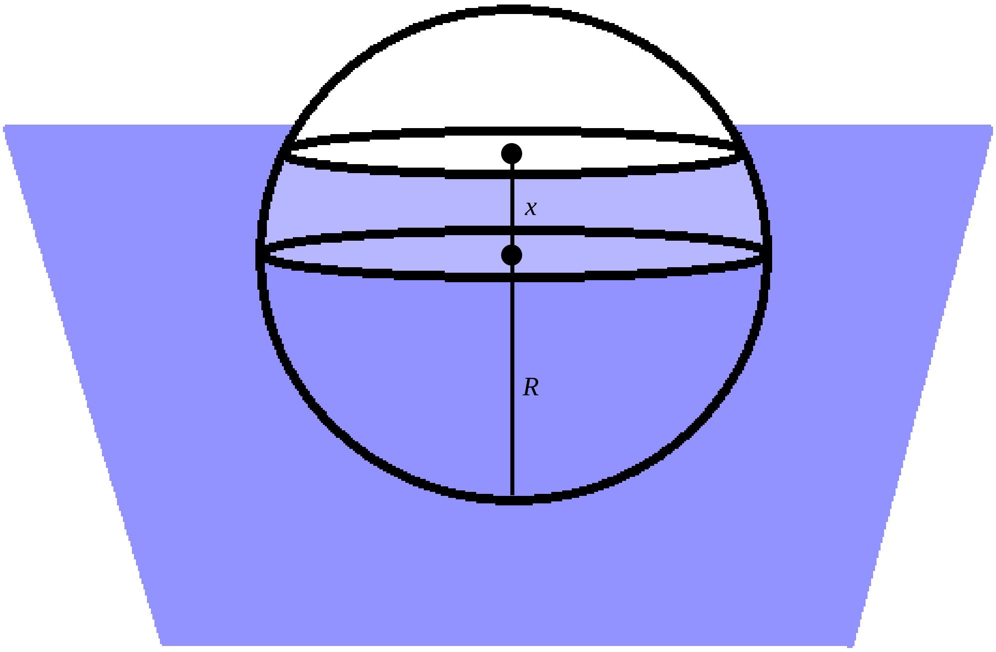

Fig.1. Submerged Sphere in A Liquid is Set into Vibration

Thus the volume of the partial part of height x from the diametric plane is of the sphere is,

$$V_x = V_{R+x} - V_R = \frac{\pi}{3}\left[3R^2 x - x^3\right]$$

Suppose m is the mass of the sphere, m_1 is the mass of the liquid occupied by the volume V_x of the sphere, ρ is the density of the liquid, $\frac{1}{2}\rho$ is the density of the sphere and g is the gravitational force. Suppose the semi-submerged sphere is oscillating in the liquid without damping, then the Newton's second law of motion gives

$$mx = -m_1 g$$

$$i.e.\ \frac{2}{3}\pi R^3 \rho x = -\frac{\pi}{3}\left[3R^2 x - x^3\right]\rho g$$

$$i.e.\ \ x + \frac{3g}{2R}x = \frac{g}{2R^3}x^3 \tag{4}$$

Also if the half submerged sphere is floating in the liquid under damping, then the equation of the system is given by

$$x + 2kx + \frac{3g}{2R}x = \frac{g}{2R^3}x^3 \tag{5}$$

where $2k$ is the damping constant.

Consider a second order weakly nonlinear ordinary differential system

$$x + 2kx + \omega^2 x = -\varepsilon\, f(x, x) \tag{6}$$

where over dots are used for the first and second derivatives of x with respect to t; k is a non-negative characteristic parameter, ε is a sufficiently small parameter and $f(x, x)$ is the nonlinear function. As the equation is second order, so, we shall get two eigenvalues, for damped oscillatory system the eigenvalues are complex conjugate *i.e.* $-k \pm i\lambda$, where $\lambda = \sqrt{\omega^2 - k^2}$ and $\omega > k$, and for the oscillatory systems *i.e.* $k = 0$, then the eigenvalues of system (6) are $\pm i\omega$.

When $\varepsilon = 0$ the solution of the corresponding linear equation (6) is

$$x(t, 0) = e^{-kt}\left(a_0 \cos \lambda t + b_0 \sin \lambda t\right) \tag{7}$$

where a_0 and b_0 are arbitrary constants.

Now we seek a solution of (6) that reduces to (7) as the limit $\varepsilon \to 0$. We look for an asymptotic solution of (6) in the form

$$x(t, \varepsilon) = e^{-kt}\left(a \cos \lambda t + b \sin \lambda t\right) + \varepsilon u_1\left(a, b, t\right) + O(\varepsilon^2) \tag{8}$$

where a and b are functions of t, defined by the first order differential equations

$$\begin{aligned}\frac{da}{dt} &= \varepsilon A_1(a,b,t) + ...\\ \frac{db}{dt} &= \varepsilon B_1(a,b,t) + ...\end{aligned} \tag{9}$$

Now differentiating (8) two times with respective to *t*, substituting for the derivatives x, x and x in (6), utilizing relations (9) and comparing the coefficients of various powers of ε, we get for the coefficients of ε:

$$\begin{aligned}& e^{-kt}\left\{\left(\frac{\partial A_1}{\partial t} + 2\lambda B_1\right)\cos\lambda t + \left(-2\lambda A_1 + \frac{\partial B_1}{\partial t}\right)\sin\lambda t\right\}\\ &+ \frac{\partial^2 u_1}{\partial t^2} + 2k\frac{\partial u_1}{\partial t} + 2k\frac{\partial u_1}{\partial t} + \omega^2 u_1 = -f^{(0)}(a,b,t)\end{aligned} \tag{10}$$

where $f^{(0)} = f(x_0, x_0)$ and $x_0 = e^{-kt}(a\cos\lambda t + b\sin\lambda t)$.

For the oscillatory system to obtain the solution, we have to put $k = 0$ and replacing λ by ω in (10).
Thus for oscillatory system, we get

$$\left(\frac{\partial A_1}{\partial t} + 2\omega B_1\right)\cos\omega t + \left(-2\omega A_1 + \frac{\partial B_1}{\partial t}\right)\sin\omega t + \frac{\partial^2 u_1}{\partial t^2} + \omega^2 u_1 = -f^{(0)}(a,b,t) \tag{11}$$

Usually, equation (10) or (11) is solved for the unknown functions A_1, B_1 and u_1 under the assumption that u_1 does not contain first harmonic terms. We shall follow this assumption (early imposed by KBM [1, 2]) partially to obtain approximate solutions of nonlinear systems with large damping. We assume that u_1 does not contain first harmonic terms of $f^{(0)}$.

4. Solution of the System

Oscillatory Motion: For the oscillatory motion from equation (4), we have

$$x + \frac{3g}{2R}x = \varepsilon x^3 \tag{12}$$

where $\varepsilon = \dfrac{g}{2R^3}$

Thus the solution of equation (12) is given by putting $k = 0$ and replacing λ by ω in equation (8), we get

$$x(t,\varepsilon) = a\cos\omega t + b\sin\omega t + \varepsilon u_1(a,b,t) \tag{13}$$

where $\omega = \sqrt{\dfrac{3g}{2R}}$.

Comparing equation (12) with the equation (6), we obtain

$$f(x,x) = x^3$$

Therefore, $f^{(0)} = [a\cos\omega t + b\sin\omega t]^3$

$$= \frac{3}{4}\left(a^3 + ab^2\right)\cos\omega t + \frac{3}{4}\left(b^3 + a^2 b\right)\sin\omega t + \left(\frac{1}{4}a^3 - \frac{3}{4}ab^2\right)\cos 3\omega t + \left(\frac{3}{4}a^2 b - \frac{1}{4}b^3\right)\sin 3\omega t \tag{14}$$

Substituting $f^{(0)}$ from equation (14) into equation (11), we obtain

$$-2A_1\omega\sin\omega t + \frac{\partial A_1}{\partial t}\cos\omega t + 2B_1\omega\cos\omega t + \frac{\partial B_1}{\partial t}\sin\omega t + \left(D^2 + \omega^2\right)u_1 = \frac{3}{4}\left(a^3 + ab^2\right)\cos\omega t + \frac{3}{4}\left(b^3 + a^2 b\right)\sin\omega t + \left(\frac{1}{4}a^3 - \frac{3}{4}ab^2\right)\cos 3\omega t + \left(\frac{3}{4}a^2 b - \frac{1}{4}b^3\right)\sin 3\omega t \tag{15}$$

According to our assumption, u_1 does not contain first harmonic terms of $f^{(0)}$, the following equations can be obtained by comparing the coefficients of $\sin\omega t$ and $\cos\omega t$ the higher argument terms of $\sin\omega t$ and $\cos\omega t$ as

$$\left(D^2 + 4\omega^2\right)A_1 = -\frac{3\omega}{2}b^3 - \frac{3\omega}{2}a^2 b \tag{16}$$

$$\left(D^2 + 4\omega^2\right)B_1 = \frac{3\omega}{2}a^3 + \frac{3\omega}{2}ab^2 \tag{17}$$

$$\left(D^2 + \omega^2\right)u_1 = \left(\frac{1}{4}a^3 - \frac{3}{4}ab^2\right)\cos 3\omega t + \left(\frac{3}{4}a^2 b - \frac{1}{4}b^3\right)\sin 3\omega t \tag{18}$$

The solutions of the equations (16) to (18) are respectively

$$A_1 = -\frac{3(b^3 + a^2 b)}{8\omega} \tag{19}$$

$$B_1 = \frac{3(a^3 + ab^2)}{8\omega} \tag{20}$$

$$u_1 = \frac{(3ab^2 - a^3)\cos 3\omega t + (b^3 - 3a^2 b)\sin 3\omega t}{32\omega^2} \tag{21}$$

Substituting the values of A_1, B_1 from equations (19) and (20) into equation (9), we obtain

$$\frac{da}{dt} = -\varepsilon \frac{3(b^3 + a^2 b)}{8\omega} \tag{22}$$

$$\frac{db}{dt} = \varepsilon \frac{3(a^3 + ab^2)}{8\omega} \tag{23}$$

Therefore, under the transformation, $a = c\cos\phi$ and $b = -c\sin\phi$ equations (21) to (23) respectively become

$$u_1 = -\frac{c^3}{32\omega^2}\cos(3\omega t + 3\phi) \tag{24}$$

And $c = 0$

$$\phi = -\frac{3\varepsilon c^2}{8\omega}$$

That is $c = c_0$ (25)

$$\phi = \phi_0 - \frac{3\varepsilon c^2 t}{8\omega} \tag{26}$$

Thus by substituting $a = c\cos\phi$ and $b = -c\sin\phi$ into equation (13) and after simplification it becomes

$$x(t,\varepsilon) = c\cos(\omega t + \phi) + \varepsilon u_1 \tag{27}$$

Therefore, equation (27) represents a first order oscillatory solution of equation (12), where c, ϕ, u_1 is given by equations (25), (26) and (24).

Damped Oscillatory Motion: For the damped oscillatory motion, we have from equation (5)

$$x + 2kx + \frac{3g}{2R}x = \varepsilon x^3 \tag{28}$$

where $\varepsilon = \frac{g}{2R^3}$

Comparing equation (28) with the equation (6), we obtain

$$f(x, x) = x^3 \tag{29}$$

Therefore, $f^{(0)} = [e^{-kt}(a\cos\lambda t + b\sin\lambda t)]^3$

$$= e^{-3kt}\left[\left(\frac{3}{4}a^3 + \frac{3}{4}ab^2\right)\cos\lambda t + \left(\frac{3}{4}b^3 + \frac{3}{4}a^2 b\right)\sin\lambda t + \left(\frac{1}{4}a^3 - \frac{3}{4}ab^2\right)\cos 3\lambda t + \left(\frac{3}{4}a^2 b - \frac{1}{4}b^3\right)\sin 3\lambda t\right] \tag{30}$$

where $\lambda = \sqrt{\frac{3g}{2R} - k^2}$.

Substituting $f^{(0)}$ from equation (30) into equation (10), we obtain

$$e^{-kt}\left(-2A_1\lambda\sin\lambda t + \frac{\partial A_1}{\partial t}\cos\lambda t + 2B_1\lambda\cos\lambda t + \frac{\partial B_1}{\partial t}\sin\lambda t\right) + \left(D^2 + 2kD + \frac{3g}{2R}\right)u_1 = e^{-3kt}\left[\frac{3}{4}\left(a^3 + ab^2\right)\cos\lambda t + \frac{3}{4}\left(b^3 + a^2 b\right)\sin\lambda t + \left(\frac{1}{4}a^3 - \frac{3}{4}ab^2\right)\cos 3\lambda t + \left(\frac{3}{4}a^2 b - \frac{1}{4}b^3\right)\sin 3\lambda t\right] \tag{31}$$

Since u_1 does not contain first harmonic terms, the following equations can be obtained by comparing the coefficients of $\sin\lambda t$ and $\cos\lambda t$ the higher argument terms of $\sin\lambda t$ and $\cos\lambda t$ as

$$\left(D^2 + 4\lambda^2\right)A_1 = -\frac{3}{2}e^{-2kt}\left\{k\left(a^3 + ab^2\right) + \lambda\left(b^3 + a^2 b\right)\right\} \tag{32}$$

$$\left(D^2 + 4\lambda^2\right)B_1 = \frac{3}{2}e^{-2kt}\left\{\lambda\left(a^3 + ab^2\right) - k\left(a^2 b + b^3\right)\right\} \tag{33}$$

$$\left(D^2 + 2kD + \frac{3g}{2R}\right)u_1 = e^{-3kt}\cos 3\lambda t\left(\frac{1}{4}a^3 - \frac{3}{4}ab^2\right) + e^{-3kt}\sin 3\lambda t\left(\frac{3}{4}a^2 b - \frac{1}{4}b^3\right) \tag{34}$$

The solutions of the equations (32) to (34) are respectively

$$A_1 = -\frac{3e^{-2kt}\left\{k\left(a^3 + ab^2\right) + \lambda\left(b^3 + a^2 b\right)\right\}}{8\left(k^2 + \lambda^2\right)} \tag{35}$$

$$B_1 = \frac{3e^{-2kt}\left\{-k\left(b^3 + a^2 b\right) + \lambda\left(a^3 + ab^2\right)\right\}}{8\left(k^2 + \lambda^2\right)} \tag{36}$$

$$u_1 = \frac{e^{-3kt}}{16(k^4+5k^2\lambda^2+4\lambda^4)}\Big[(3ab^2-a^3)\{3k\lambda\sin 3\lambda t-(k^2-2\lambda^2)\cos 3\lambda t\} \\ +(3a^2b-b^3)\{(k^2-2\lambda^2)\sin 3\lambda t+3k\lambda\cos 3\lambda t\}\Big] \quad (37)$$

Substituting the values of A_1, B_1 from equations (35) and (36) into equation (9), we obtain

$$\frac{da}{dt} = -\varepsilon\frac{3e^{-2kt}\left\{k\left(a^3+ab^2\right)+\lambda\left(b^3+a^2b\right)\right\}}{8\left(k^2+\lambda^2\right)} \quad (38)$$

$$\frac{db}{dt} = \varepsilon\frac{3e^{-2kt}\left\{-k\left(b^3+a^2b\right)+2\lambda\left(a^3+ab^2\right)\right\}}{8\left(k^2+\lambda^2\right)} \quad (39)$$

Therefore, under the transformation, $a = c\cos\phi$ and $b = -c\sin\phi$ equations (11) and (12) respectively become

$$u_1 = \frac{c^3e^{-3kt}}{16(k^4+5k^2\lambda^2+5\lambda^4)}[(k^2-2\lambda^2)\cos\left(3\lambda t+3\phi\right)-3k\lambda\sin\left(3\lambda t+3\phi\right)] \quad (40)$$

And $c = -\dfrac{3k\varepsilon c^3e^{-2kt}}{8(k^2+\lambda^2)}$

$$\phi = -\frac{3\lambda\varepsilon c^2e^{-2kt}}{8(k^2+\lambda^2)}$$

Or, $\phi = \phi_0 + \dfrac{3\lambda\varepsilon c^2}{16k\left(k^2+\lambda^2\right)}\left(e^{-2kt}-1\right)$ (41)

$$c = c_0 + \varepsilon\frac{3c_0^{\ 3}}{16\left(k^2+\lambda^2\right)}\left(e^{-2kt}-1\right) \quad (42)$$

Thus by substituting $a = c\cos\phi$ and $b = -c\sin\phi$ into equation (28) and after simplification it becomes

$$x(t,\varepsilon) = ce^{-kt}\cos\left(\lambda t+\phi\right)+\varepsilon u_1 \quad (43)$$

Therefore, equation (43) represents a first order damped oscillatory solution of equation (28), where c, ϕ, u_1 are given by equations (42), (41) and (40).

5. Results and Discussion

To make sure the efficiency of our results, we compare our results to the numerical results obtained by the *Mathematica* program for the different sets of initial conditions. First of all, when the system is oscillatory, $x(t,\,\varepsilon)$ has been computed by the approximate solution given by equation (27) in which $c,\,\phi$ and u_1 are calculated by equations (25), (26) and (24) together with three sets of initial conditions. Again for damped oscillatory motion, equation (43) is used to compute the asymptotic solution $x(t,\,\varepsilon)$, wherein $c,\,\phi$ and u_1 are obtained from equations (42), (41) and (40) together with three sets of initial conditions. The corresponding numerical solutions for both cases have been computed by the *Mathematica* program for various values of *t* and all the perturbation solutions have been developed by a code in *Mathematica* program. Finally, we get different results for both oscillatory and damped oscillatory motion for different sphere and damping constant $2k$. All the results are shown in the **Figure 2** to **Figure 4** for oscillatory motion and **Figure 5** to **Figure 7** for damped oscillatory motion respectively.

Corresponding Figures for Undamped Oscillatory Motion

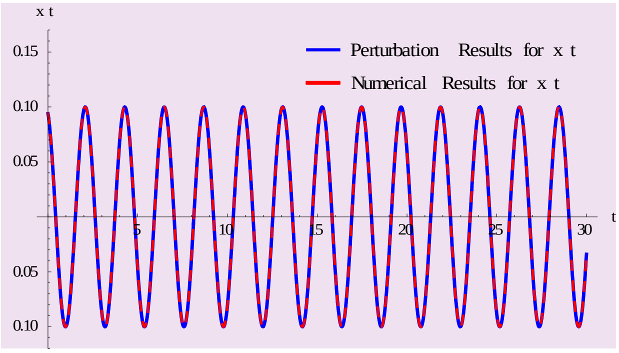

Fig.2. Comparison between perturbation and numerical results for $R = 1.8\ m,\ \ g = 9.8\ ms^{-2}$ with the initial conditions $c_0 = 0.10\ m,\ \phi_0 = 20°$.

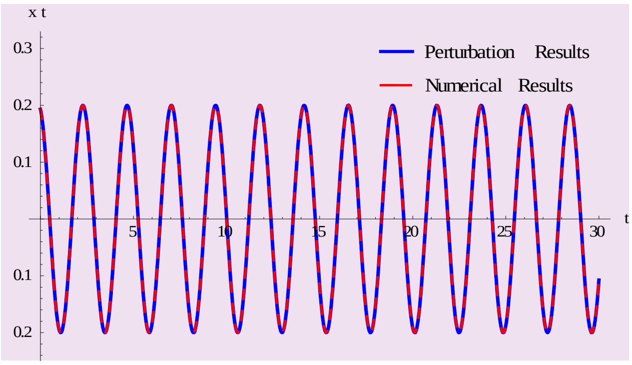

Fig.3. Comparison between perturbation and numerical results for $R = 2.1\ m,\ \ g = 9.8\ ms^{-2}$ with the initial conditions $c_0 = 0.20\ m,\ \phi_0 = 15°$.

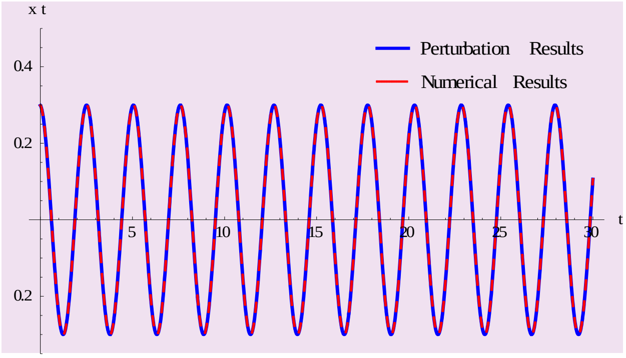

Fig.4. Comparison between perturbation and numerical results for $R = 2.4\ m$, $g = 9.8\ ms^{-2}$ with the initial conditions $c_0 = 0.30\ m$, $\phi_0 = 5°$.

Corresponding Figures for Damped Oscillatory Motion

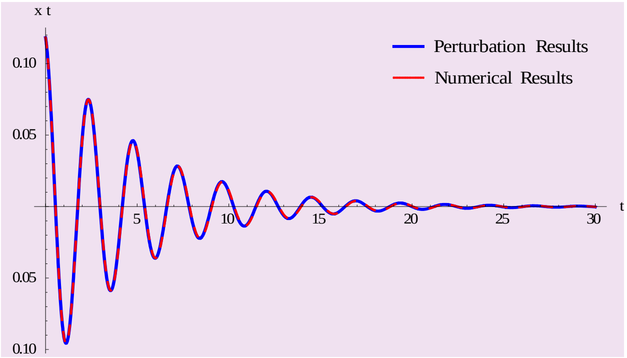

Fig.5. Comparison between perturbation and numerical results for $R = 2.2\ m$, $k = 0.2\ s^{-1}$, $g = 9.8\ ms^{-2}$ with the initial conditions $c_0 = 0.12\ m$, $\phi_0 = 10°$.

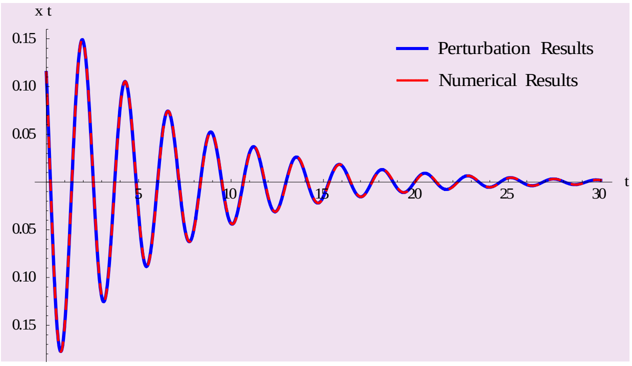

Fig.6. Comparison between perturbation and numerical results for $R = 2.0\ m$, $k = 0.15\ s^{-1}$, $g = 9.8\ ms^{-2}$ with the initial conditions $c_0 = 0.20\ m$, $\phi_0 = 55°$.

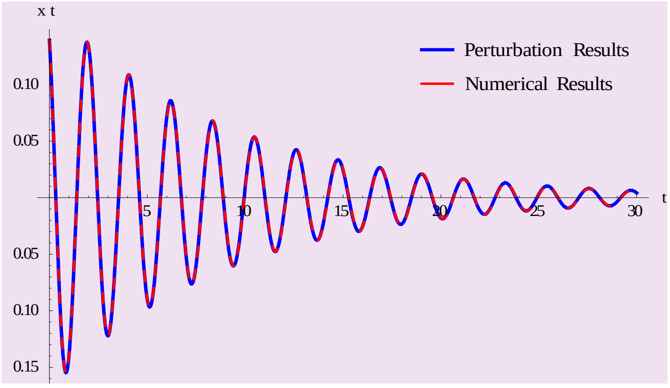

Fig.7. Comparison between perturbation and numerical results for $R = 1.7\ m,\ \ k = 0.11\ s^{-1},\ g = 9.8\ ms^{-2}$ with the initial conditions $c_0 = 0.17\ m,\ \phi_0 = 35°$.

6. Conclusions

In this paper, we have carried out the modification of the KBM method and successfully applied the modified method to the half-submerged sphere for oscillatory and damped oscillatory nonlinear systems. At first, we have derived the equations for these systems. In this article, we have been studied on semi-submerged sphere which is oscillating in a liquid due to the gravitational force and upward pressure of the liquid. Based on the modified KBM method transient responses of nonlinear differential systems have been investigated. The second order nonlinear systems for an oscillating half submerged sphere, the solutions are looked for such circumstances wherein the eigenvalues are pure imaginary and complex conjugate for oscillatory and damped oscillatory motion respectively. For different sets of initial conditions the modified KBM method provides solutions which show well agreement with the numerical solutions.

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