I.J. Mathematical Sciences and Computing, **2017, 3, 28-38**
Published Online July 2017 in MECS (http://www.mecs-press.net)
DOI: 10.5815/ijmsc.2017.03.03

Available online at **http://www.mecs-press.net/ijmsc**

Graph Dynamic Threshold Model Resource Network: Key Features

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Abstract

In this paper, we describe a graph dynamic threshold model called *resource network*, and briefly present the main results obtained during several years of research. Resource Network is represented by a connected oriented with weighted graph with an arbitrary topology. Weights of edges denote their throughput capacities for an abstract resource. The resource is stored in vertices, which can contain its unlimited amount. Network operates in discrete time. The total amount of resource is constant, while pieces of resource are reallocating among vertices every time step, according to certain rules with threshold switching. The main objective of our research is to define for a network with an arbitrary topology all its basic characteristics: the vectors of limit state and flow for every total amount of resource W; the threshold value of total recourse T, which switches laws of operating of the network; description of these laws. It turned out that there exists several classes of networks depending on their topologies and capacities. Each class demonstrates fundamentally different behavior. All these classes and their characteristics will be reviewed below.

Index Terms: Graph dynamic models, threshold models, flows in networks, Markov chains, diffusion on graphs, resource networks.

1. Introduction

The most known and popular models of flow in networks are the classical Ford–Fulkerson flow model (Ford, Fulkerson, 1962), and its numerous dynamic modifications, described e.g. by Ahuja et al., 1993, Fleischer et al., 2003, and many other authors. However, there are many applied problems of resource distribution in networks that cannot be formalized in the terms of these models. These are, e.g., control problem for resource allocation in virtual networks (Szeto et al., 2003), modeling the distribution of substances in a water environment (Zhilyakova, 2009), and numerous other problems with conservation laws that do not imply a directed flow of resource from sources to sinks.

In addition to flow models, resource networks have two more predecessors. The first one is the very broad

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class of models based on random walks and diffusion on graphs (see, e.g. Blanchard & Volchenkov, 2011). The second is so called *chip-firing game*, an integer threshold model. Despite its apparent simplicity, the model gave rise to many non-trivial mathematical results (Björner et al., 1991, 1992, Biggs, 1999, etc.). Chip firing games proved to be extremely suitable for describing processes of self-organized criticality (Bak et al., 1988, Bak, 1996) and in particular for Abelian sandpile model (Dhar, 1999, Speer, 1993). A review of some non-classacal network models related to resource networks can be found in Zhilyakova, 2015.

The *resource network* combines different features of the above models, and thereby has new properties not inherent in either one of them. It was first proposed in Kuznetsov, 2009. This pioneer work described a simple particular version of a model – *complete uniform resource networks*, networks represented by complete graphs with identical capacities for all edges. Since then all the classes of graphs were investigated. In this paper, we will represent only key results and most general characteristics.

2. Basic Definitions

A *resource network* is a digraph $G = (V, E)$ with vertices $v_i \in V$ and edges $e_{ij} = (v_i, v_j) \in E$. All edges labeled with *nonnegative* numbers r_{ij} constant in time. These numbers denote *throughput capacities* of edges.

Resources $q_i(t)$ are nonnegative numbers assigned to vertices v_i and changing in discrete time t.

Vertices v_i can store an unlimited amount of resource.

A *state* $Q(t)$ of the network at time t is the vector $Q(t) = (q_1(t), \dots, q_n(t))$ that contains values of resources at every vertex at this time.

A state $Q(t)$ is called *stable* if $Q(t) = Q(t + 1) = \dots$.

A state $Q^* = \left(q_1^*,\dots,q_n^*\right)$ is called *asymptotically reachable* from state $Q(0)$ if for every $\varepsilon > 0$ there exists t_ε such that for all $t > t_\varepsilon$ $\left|q_i^* - q_i(t)\right| < \varepsilon$ $i = 1, \dots, n$.

A network state is called *limit* if it is either *stable* and reachable from $Q(0)$ in finite time or *asymptotically reachable* from $Q(0)$.

$R = (r_{ij})_{n\times n}$ is the throughput matrix of the network.

$r_i^{in} = \sum_{j=1}^{n} r_{ji}$ and $r_i^{out} = \sum_{j=1}^{n} r_{ij}$ are total input and output capacities of vertex v_i respectively. A loop's throughput, if exists, is included in both sums.

$r_{sum} = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}$ – the total throughput capacity of the network.

We denote the total resource of all vertices by W.

The network satisfies a *conservation law*: during its operation, resource does not come from outside and is not spent:

$$\forall t \ \sum_{i=1}^{n} q_i(t) = W.$$

Resource distribution in the network occurs per one of two rules. At every step vertices, must use one of the rules depending on the amount of its resource.

2.1. Resource distribution rules

At time step t, vertex v_i sends out to vertex v_k through the edge e_{ik}:

r_{ik} *units* of resource if $q_i(t) > r_i^{out}$ (rule 1);

$\frac{r_{ik}}{r_i^{out}} q_i(t)$ units of resource if $q_i(t) \le r_i^{out}$ (rule 2).

The meaning of these rules is as follows. Rule 1 is applied when a vertex contains more resource than it can send to adjacent vertices through output edges; it sends "everything it can," i.e. along each outgoing edge there it sends a resource equal to the throughput capacity of this edge; and totally the vertex sends $r_i^{out} = \sum_{j=1}^{n} r_{ij}$ of resource.

By rule 2, a vertex sends out its entire resource. Note, that if resource at a vertex equals to its output capacity $q_i(t) = r_i^{out}$, then rules 1 and 2 are identical.

The model is parallel. At every time step, all vertices that have a resource send it into all outgoing edges per rule 1 or 2; adjacent vertices receive the resource through incoming edges.

The set of vertices with resource $q_i(t)$ not exceeding r_i^{out} is called the *zone* $Z^-(t)$. Vertices belonging to $Z^-(t)$ operate according to rule 2. *Zone* $Z^+(t)$ is the set of vertices with resource exceeding their output capacity; they operate according to rule 1. For the limit state Q^* these zones are denoted as Z^{-*} and Z^{+*}.

2.2. Resource flow

Resource sent from vertex v_i along the edge e_{ij} at time t arrives at vertex v_j at time $t + 1$. We assume that in time interval $(t, t + 1)$ the resource is flowing through the edge e_{ij}. This resource is called the *flow* $f_{ij}(t)$. The flow in network is defined by matrix $F(t) = (f_{ij}(t))_{n\times n}$.

The total flow value at time t is the sum: $f_{sum}(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}(t)$.

Edges cannot pass through itself an amount of resource greater its capacity, which implies that $f_{ij}(t) \le r_{ij}$ and $f_{sum}(t) \le r_{sum}$ for every t.

Denote total outgoing flow for vertex v_i as $f_i^{out}(t) = \sum_{j=1}^{n} f_{ij}(t)$. It is obvious that $f_i^{out}(t) \le r_i^{out}$.

The value $f_j^{in}(t+1) = \sum_{i=1}^{n} f_{ij}(t)$ is the resource ingoing to vertex v_j at the next time step; $f_j^{in}(t+1) \le r_j^{in}$. In addition, define that $f_j^{in}(0) = 0$.

If there exists a limit state Q^* in the resource network, then the flow in this state will also be called the limit flow. We denote the limit flow matrix by $F^* = \left(f_{ij}^*\right)_{n\times n}$.

The subject of this article is the problem of studying the dynamics of different resource networks with different total resource. First, we are interested in analyzing the time behavior of the vectors $Q(t)$ and $F(t)$, finding the conditions for the existence of limit states, searching for methods for calculating vectors Q^* and F^*, and identifying the most important parameters that affect these calculations.

3. Classification of Resource Networks

We introduce a classification of resource networks depending on their topology and edge throughput capacities. We will consider the following classes with fundamentally different properties:

- *Eulerian networks* – ergodic regular networks (Roberts, 1976, Kemeny & Snell, 1960) in which every vertex enjoys the property:

$$r_i^{in} = r_i^{out} \tag{1}$$

- *Asymmetric regular networks* – ergodic regular networks, in which equality (1) is violated at least in one pair of vertices;
- *Absorbing networks* – non-ergodic networks with sink vertices;
- *Cyclic networks* – ergodic irregular networks.

All above classes of networks have been studied in Zhilyakova 2011 – 2016 with the following results.

3.1. Threshold value of total resource T

1. For all classes of networks (except absorbing one) there exists an integral characteristic, a *threshold value of total resource T* unique for every network, which depends on the matrix of capacity throughputs R.
2. The network operates differently with total resource on different sides of this threshold ($W \le T$ and $W > T$): for $W \le T$ all vertices pass to the $Z^-(t)$ zone in a finite number of time steps, while for $W > T$ at least one vertex in a finite number of time steps will be in the $Z^+(t)$ zone. The total resource values $W \le T$ and $W > T$ are called *small* and *large* respectively.
3. Obviously, the threshold value does not exceed the total throughput capacity of a network: $T \le r_{sum}$.
4. For *Eulerian* networks we have the equality $T = r_{sum}$.
5. *Asymmetric* networks satisfy the strict inequality $T < r_{sum}$.
6. *Absorbing* networks do not have a threshold value at all.

3.2. Small resource (W ≤ T)

If resource value is *small* ($W \le T$), there is no difference in operating of all regular networks (both Eulerian and asymmetric). The differences between classes begin to manifest with a large total resource.

When $W \le T$, all vertices of a network would pass to zone $Z^-(t)$ in finite time. Since then the law of network operating will be as follows:

$$Q(t+1) = Q(t) \cdot R',$$

where $R' = D^{-1}R$, and $D = \text{diag}\left\{r_1^{out}, \dots, r_n^{out}\right\}$. This process defines a homogeneous regular Markov chain and hence (Roberts, 1976, Kemeny & Snell, 1960):

1. The limit of degrees of matrix R' exists: $\lim_{k \to \infty} R'^k = R'^\infty$.
2. For any state, where all vertices belong to zone $Z^-(t)$ it is hold

$$Q(t) \cdot R'^\infty = Q^*.$$

The limit state exists and is unique for any initial state.

3. We denote the vector of limit state for $W = 1$ as Q^{1*}. This vector coincides with the vector of limit probabilities of a homogeneous Markov chain with stochastic matrix R'. Then

$$R'^{\infty} = \mathbf{1} \cdot Q^{1*},$$

where **1** is a column-vector consisting of n units (note that all vectors Q are row-vectors).

4. Vector Q^* is a left eigenvector of matrices R'^{∞} and R' corresponding to eigenvalue $\lambda = 1$:

$$Q^* \cdot R' = Q^*; \;\; Q^* = Q^* \cdot R'^{\infty}.$$

5. For any $W \le T$ the equality holds

$$Q^* = Q^{1*} \cdot W.$$

6. The limit vector of the input flow coincides with the transposed limit vector of the output flow, and both are equal to the limit state vector: $F^{in*} = F^{out*\mathrm{T}} = Q^*$.
7. Finally, we define the value of T. For any network (except absorbing) the formula for threshold value is:

$$T = \min_{i \in \{1,\ldots,n\}} \frac{r_i^{out}}{q_i^{1*}},$$

where q_i^{1*} are components of vector Q^{1*}.

3.3. Asymmetric regular networks

Class of asymmetric regular networks is the most studied one. It has many interesting non-trivial properties being manifested when the total resource in a network is large ($W > T$).

When $W > T$, it becomes important to divide the vertices of networks into three types. Since the network is asymmetric, there must be at least a pair of vertices for which $r_i^{in} - r_i^{out} \ne 0$. For vertex v_i we denote this difference by Δr_i: $\Delta r_i = r_i^{in} - r_i^{out}$. Then all vertices can be divided according the sign of Δr_i.

1) *receiver vertices* for which $\Delta r_i > 0$;
2) *source vertices* for which $\Delta r_i < 0$;
3) *neutral vertices* for which $\Delta r_i = 0$.

Asymmetric network has at least one receiver and one source vertex. It seems quite natural to assume that receivers will accumulate the resource. However, only a small part of the receivers turned out to can accumulate a resource, while some neutral vertices had this ability. Thus, this classification turned to be insufficient, and we need additional criterion to predict the behavior of network when $W > T$.

For asymmetric networks, we introduce the notion of *vertices – potential attractors*. Potential attractors are vertices that are capable, in case $W > T$, to end up in the Z^{+*} zone from a certain initial state. By modus operandi, attractors are subdivided into *active* and *passive* ones. *Active attractors* can attract the resource from other vertices; *passive attractors* can only hold their own resource excesses if they had them in the initial state. If a network has a unique attractor v_i, this vertex will accumulate all resource over T: it will have in the limit state the resource $r_i^{out} + (W - T)$.

A criterion of attractivity was found for vertices in all ergodic networks.

The vertex v_j is an attractor of a network iff

$$j = \arg \min_{i \in \{1,\ldots,n\}} \frac{r_i^{out}}{q_i^{1*}} .$$

We introduce such a numbering of vertices that the attractors have numbers from 1 to l ($l \geq 1$). For all non-attractive vertices for any total resource value $W > T$ the components of limit state vector are calculated by a simple formula

$$q_i^* = q_i^{1*} \cdot T,\ i > l.$$

The rest resource ($W - T$) is allocated in limit state among attractors.

Denote the vector of limit state with $W = T$ by $\tilde{Q} = (\tilde{q}_1, \ldots, \tilde{q}_n)$.

For attractive vertices we have the equality: $\tilde{q}_i = r_i^{out}$, $i \leq l$; for all the others, we have the strict inequality: $\tilde{q}_i < r_i^{out}$, $i > l$. Then $\tilde{Q} = T \cdot Q^{1*} = (r_1^{out}, \ldots, r_l^{out}, \tilde{q}_{l+1}, \ldots, \tilde{q}_n)$.

The amazing result is that for any arbitrarily large total resource $W > T$ vectors of limit flow does not change and are equal to $\tilde{Q}$:

$$F^{in*} = \left(F^{out*}\right)^{\mathrm{T}} = \tilde{F}^{in} = \left(\tilde{F}^{out}\right)^{\mathrm{T}} = \tilde{Q} = (r_1^{out}, \ldots, r_l^{out}, \tilde{q}_{l+1}, \ldots, \tilde{q}_n) .$$

Then for any $W > T$ the total limit flow is equal to T: $f_{sum}^* = T$.

The limit state vector Q^* can be expressed in terms of the vector $\tilde{Q}$ in this way:

$$Q^* = (r_1^{out} + \Delta q_1^*, \ldots, r_l^{out} + \Delta q_l^*, \tilde{q}_{l+1}, \ldots, \tilde{q}_n) ,$$

where $\Delta q_i^* \geq 0$ – resource surpluses distributed between potential attractors. $\sum_{i=1}^{l} \Delta q_i^* = W - T$.

3.4. *Eulerian networks*

Every vertex in a Eulerian network is a *passive potential attractor*.

For $W \leq T$ the vectors of limit state and limit flow can be found in an explicit form:

$$F^{in1*} = \left(F^{out1*}\right)^{\mathrm{T}} = Q^{1*} = \left(\frac{r_1^{out}}{r_{sum}}, \ldots, \frac{r_n^{out}}{r_{sum}}\right);\ F^{in*} = \left(F^{out*}\right)^{\mathrm{T}} = Q^* = \left(\frac{r_1^{out}}{r_{sum}} W, \ldots, \frac{r_n^{out}}{r_{sum}} W\right).$$

For $W = T$ and $W > T$

$$F^{in*} = \left(F^{out*}\right)^{\mathrm{T}} = \tilde{Q} = (r_1^{out}, \ldots, r_n^{out}) .$$

Directly from this equality it follows that $T = r_{sum}$.

When $W > T$, for some initial states the limit state can also be expressed explicitly via the initial state:

$$Q^* = \left(r_1^{out} + c_1(0) - \tilde{c}_1^m, \ldots, r_m^{out} + c_m(0) - \tilde{c}_m^m,\ r_{m+1}^{out}, \ldots, r_n^{out}\right),$$

where $c_j(0)$ – initial surpluses in vertices; $m \geq 1$ is the number of vertices containing surpluses. Let us call these states *good*. The initial state is good iff $c_j(0) \geq \tilde{c}_j^m$ for all j = 1, …, m. Here the vertices are numbered in order of decreasing resources $q_j(0)$ and, accordingly, in order of decreasing surpluses $c_j(0)$. The methods of finding limit surpluses $\tilde{c}_j^m$ are described inter alia in Zhilyakova, 2014.

For all *non-good* initial states the algorithm is proposed by which in some steps the vector $Q(0)$ turns into $Q(t)$ meeting the condition of *good* state. This vector $Q(t)$ becomes a new initial state $Q_t(0)$, good by definition. The limit states for $Q(0)$ and $Q_t(0)$ are the same.

3.5. Absorbing networks

Let absorbing network has l sinks with numbers from 1 to l. Then its matrix R can be represented in a block form:

$$R = \left(\begin{array}{c|c} D & O_1 \\ \hline R_1 & R_2 \end{array}\right),$$

where D is a diagonal matrix of size $l \times l$ with arbitrary nonnegative diagonal elements equal to the capacity of loops in sinks, O_1 – is a zero matrix of size $l \times (n-l)$, R_1, R_2 are matrices of size $(n-l) \times l$ and $(n-l) \times (n-l)$ respectively. The stochastic matrix corresponding to R has the form:

$$R' = \left(\begin{array}{c|c} E_1 & O_1 \\ \hline R_1' & R_2' \end{array}\right).$$

Then the limit of degrees of matrix R' can be calculated explicitly:

$$R'^{\infty} = \left(\begin{array}{c|c} E_1 & O_1 \\ \hline \left(E_2 - R_2'\right)^{-1} R_1' & O_2 \end{array}\right).$$

The matrix R'^{∞} remains constant at arbitrary change of diagonal elements of matrix R. It is not obvious because matrix R' can change quite strongly when the diagonal elements of the matrix R are changed.

For any total resource W the vector of limit state is calculated by formula:

$$Q^* = Q(0)R'^{\infty}.$$

From the expression for R'^{∞} it is seen, that Q^* has l non-zero components corresponding to sinks.

Here it becomes clear why the absorbing networks do not have a threshold value T. A method of finding its limit state vector is the same for total resource. Limit flow in such networks consists of flows in loops of sinks and may be absent.

3.6. Cyclic networks

Small resource. A network is called cyclic if the greatest common divisor (GCD) of all its cycles is greater than 1. Let us denote GCD by $d > 1$ and call the network d-cyclic. The graph of 2-cyclic network and its dynamics with $W < T$ are represented in fig. 1.

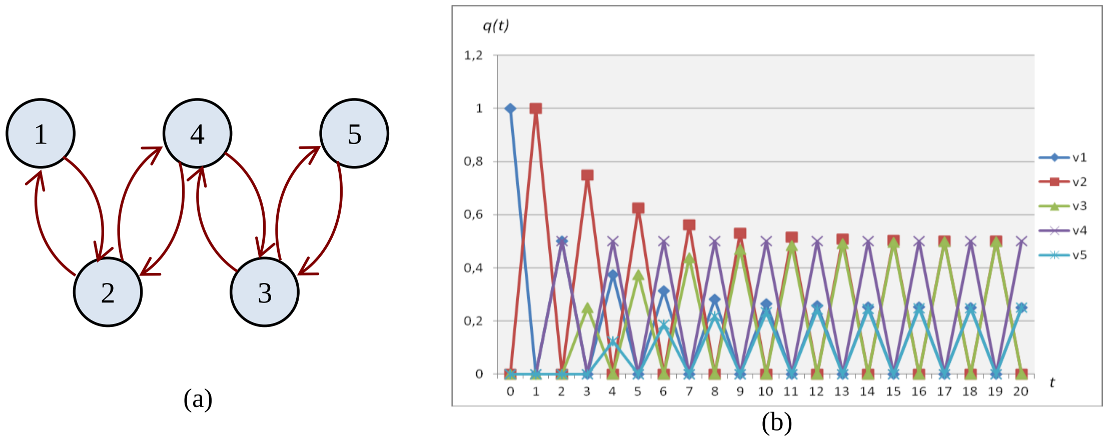

Fig.1. (a) cyclic graph, $d = 2$; (b) The oscillations on this graph with $Q(0) = (1, 0, 0, 0, 0)$.

Let R' be the stochastic matrix of the d-cyclic resource network. Consider the sequence of its powers: R', …, R'^d, R'^{d+1}, … It consists of d convergent subsequences:

1). R', $R'^d \cdot R'$, $R'^{2d} \cdot R'$, $R'^{3d} \cdot R'$,…
2). R'^2, $R'^d \cdot R'^2$, $R'^{2d} \cdot R'^2$, $R'^{3d} \cdot R'^2$,…
…
d). R'^d, $R'^d \cdot R'^d$, $R'^{2d} \cdot R'^d$, $R'^{3d} \cdot R'^d$,…

All limits of the subsequences $R_1^{'\infty}$, …, $R_d^{'\infty}$ are expressed in terms of one limit matrix $R_d^{'\infty}$:

$$R_1^{'\infty} = R_d^{'\infty} R' ;$$
$$R_2^{'\infty} = R_d^{'\infty} R'^2 ;$$
…
$$R_{d-1}^{'\infty} = R_d^{'\infty} R'^{d-1} .$$

A d-cyclic resource network doesn't have a single limit state, but has a limit cycle of length d:

$$Q_1^* = Q(t) R_d^{'\infty} R' ,\ Q_2^* = Q(t) R_d^{'\infty} R'^2 ,\ \ldots\ Q_d^* = Q(t) R_d^{'\infty} .$$

$Q(t)$ is a vector of state at any time step t, when all the vertices have already passed to the zone $Z^-(t)$.
Each vector cyclically passes to the next: $Q_{i+1}^* = Q_i^* R'$, $i = 1, \ldots, d-1$, $Q_1^* = Q_d^* R'$.
The vectors Q_1^*, …, Q_d^* are eigenvectors of matrix $R_d^{'\infty}$ corresponding to the eigenvalue $\lambda = 1$ of

multiplicity d.

The vectors $Q_1^*, \ldots, Q_d^*$ can coincide. Then a network has an equilibrium state. The equilibrium is achieved if every cyclic class has the same quantity of resource $\frac{W}{d}$ at any time step t, when all vertices are in the zone $Z^-(t)$. The equilibrium vector is vector WQ^{1*}, where Q^{1*} is any row of Cesaro limit matrix A: $A = \frac{1}{d}(E + R' + \ldots + R'^{d-1})R_d'^{\infty}$.

Vector Q^{1*} is also the only positive eigenvector of the matrix R' for $W = 1$. It can be calculated as

$$Q^{1*} = \frac{Q_1^* + \ldots + Q_d^*}{d \cdot W}$$

for any $W < T$.

Large resource. For $W \geq T$ the flows and states stabilize, and a global equilibrium is reached in the network for any initial state. The limit flow exists and is unique. The limit state exists; it does not depend on the initial state if and only if the network has one attractor.

The vertex v_i is an attractor iff

$$j = \arg\min_i \frac{r_i^{out}}{q_i^{1*}},$$

where vector Q^{1*} is Cesaro limit: $Q^{1*} = \frac{1}{d}\sum_{k=1}^{d} Q_k^{1*}$.

The threshold value T is defined similarly to other classes:

$$T = \min_i \frac{r_i^{out}}{q_i^{1*}}.$$

The only difference is that Q^{1*} is not necessarily the limit vector and in general is arithmetic mean of the vectors forming limit cycle.

Generally, the operation of cyclic network when $W \geq T$ is totally the same as operation of regular networks. Therefore, the results obtained for regular networks are completely transferred to this class.

4. Conclusion

In the article, we have represented all the main results obtained for resource networks. A brief summary of all these results for resource networks is given here for the first time.

We have defined the model of a resource network; have defined its main parameters, states, and flows. We have introduced a classification of resource networks, showing the fundamental differences between classes. For small resources, there is no need to divide the networks into classes – almost all networks operate according the same laws defined by a homogeneous Markov chain. However, when the total amount of resource exceeds the threshold value, the features of each class are manifested. The exception is made by cyclic networks; their behavior with small resources differs from other classes, but is also described by a homogeneous Markov chain, though by cyclic one.

Acknowledgements

The work was supported by the Russian Foundation for Basic Research, project no. 15-07-02488.

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How to cite this paper: L. Yu. Zhilyakova,"Graph Dynamic Threshold Model Resource Network: Key Features", International Journal of Mathematical Sciences and Computing(IJMSC), Vol.3, No.3, pp. 28-38, 2017.DOI: 10.5815/ijmsc.2017.03.03