A Study on Discrete Model of Three Species Syn-Eco-System with Limited Resources
(One Period Equilibrium States)

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Abstract—In this paper, the system comprises of a commensal (S1), two hosts S2 and S3, i.e., S2 and S3 both benefit S1, without getting themselves effected either positively or adversely. Further S2 is a commensal of S3, S3 is a host of both S1, S2 and all the three species have limited resources. The basic equations for this model constitute as three first order non-linear ordinary difference equations. All possible equilibrium points are identified based on the model equations and criteria for their stability are discussed. Further the numerical solutions are computed for specific values of the various parameters and the initial conditions.

Index Terms—Commensal, Discrete model, Equilibrium state, Integer, Host, Species, Stable.

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I. INTRODUCTION

Ecology, a branch evolutionary biology, deals with living species that coexist in a physical environment sustain themselves on common resources. It is a common observations that the species of same nature can not flourish is isolation without any interaction with species of different kinds. Syn-ecology is an ecosystem comprising of two or more distinct species. Species interact with each other in one way or other. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation Parasitism and so on. Lotka[1] and Volterra [2] pioneered theoretical ecology significantly and opened new eras in the field of life and biological sciences.

Mathematical Modeling is a vital role in providing insight io the mutual relationships (positive, negative) between the interacting species. The general concepts of modeling have been discussed by several authors, viz., [3-6]. Srinivas[7] studied competitive ecosystem of two species with three species with limited and unlimited resources. Later, Laxminarayan et al [8] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by [9-10], while Reddy [11] investigated mutualism between two species.


The present investigation is a discrete model of three species (S1, S2, S3) syn-eco system with limited resources. The system comprises of a commensal (S1), two hosts S2 and S3. Further S2 is a commensal of S3, S3 is a host of both S1 and S2. **Commensalism** is a symbiotic interaction between two populations where one population (S1) gets benefit from (S2) while the other (S2) is neither harmed nor benefited due to the interaction with (S1). The benefited species (S1) is called the commensal and the other, the helping one (S2) is called the host species. A common example is an animal using a tree for shelter-tree (host) does not get any benefit from the animal (commensal).

II. BASIC EQUATIONS OF THE MODEL

The model equations for the three species syn-eco system is given by the following system of first order non-linear difference equations employing the following notation.

A. Notation Adopted

- \( N_i(t) \) : The population strength of \( S_i \) at time \( t \), \( i = 1, 2, 3 \)
- \( t \) : Time instant
- \( a_i \) : Natural growth rate of \( S_i \), \( i = 1, 2, 3 \)
- \( a_{ii} \) : Self inhibition coefficient of \( S_i \), \( i = 1, 2, 3 \)
- \( a_{12}, a_{13} \) : Interaction coefficients of \( S_1 \) due to \( S_2 \) and \( S_1 \) due to \( S_3 \)
- \( a_{23} \) : Interaction coefficient of \( S_2 \) due to \( S_3 \)

Further the variables \( N_1, N_2, N_3 \) are non-negative and the model parameters \( a_1, a_2, a_3, a_{11}, a_{12}, a_{13}, a_{23}, a_{33} \) are assumed to be non-negative constants.
B. Basic equations

Consider the growth of the species during the time interval \((t, t + 1)\)

**Equation for the first species \((N_1)\):**

\[
N_1(t+1) = N_1(t) + a_1 N_1(t) - a_{11} N_1^2(t) + a_{12} N_2(t) N_1(t) + a_{13} N_3(t) N_1(t)
\]

\((1)\)

**Equation for the second species \((N_2)\):**

\[
N_2(t+1) = N_2(t) + a_2 N_2(t) - a_{22} N_2^2(t) + a_{23} N_2(t) N_3(t)
\]

\((2)\)

**Equation for the third species \((N_3)\):**

\[
N_3(t+1) = N_3(t) + a_3 N_3(t) - a_{33} N_3^2(t)
\]

\((3)\)

where \(a_i = a_i + 1, i = 1, 2, 3\)

III. EQUILIBRIUM STATES

For a continuous model the equilibrium states are defined by \(\frac{dN_i}{dt} = 0, i = 1, 2, 3\), the equilibrium states for a discrete model are defined in terms of the period of no growth. i.e, \(N_i(t + r) = N_i(t), r = 1, 2, 3, \ldots \), where \(r\) is the period of the equilibrium state.


**One period equilibrium states (Stage-I)**

\[N_i(t + 1) = N_i(t), i = 1, 2, 3\]

The system under investigation has fourteen equilibrium states given by

(i) Fully washed out state

\[E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0\]

(ii) States in which only one of the tree species is survives while the other two are not

\[E_2 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = \frac{a_3 - 1}{a_{33}}, \text{ when } a_3 > 1\]

\[E_3 : \bar{N}_1 = 0, \bar{N}_2 = \frac{a_2 - 1}{a_{22}}, \bar{N}_3 = 0, \text{ when } a_2 > 1\]

\[E_4 : \bar{N}_1 = \frac{a_1 - 1}{a_{11}}, \bar{N}_2 = 0, \bar{N}_3 = 0, \text{ when } a_1 > 1\]

(iii) States in which only two of the tree species are survives while the other one is not

\[E_5 : \bar{N}_1 = 0, \bar{N}_2 = \frac{1}{a_{22}} \left( (a_2 - 1) + a_{23} \left( \frac{a_3 - 1}{a_{33}} \right) \right), \bar{N}_3 = \frac{a_3 - 1}{a_{33}}\]

when \(a_2 > 1 \text{ and } a_3 > 1\)

\[E_6 : \bar{N}_1 = 0, \bar{N}_2 = a_{23} \left( \frac{a_3 - 1}{a_{33}} \right), \bar{N}_3 = \frac{a_3 - 1}{a_{33}}\]

when \(a_2 > 1 \text{ and } a_3 > 1\)

\[E_7 : \bar{N}_1 = \frac{1}{a_{11}} \left( (a_1 - 1) + a_{13} \left( \frac{a_3 - 1}{a_{33}} \right) \right), \bar{N}_2 = 0, \bar{N}_3 = \frac{a_3 - 1}{a_{33}}\]

when \(a_1 > 1 \text{ and } a_3 > 1\)

\[E_8 : \bar{N}_1 = a_{33} \left( \frac{a_3 - 1}{a_{33}} \right), \bar{N}_2 = 0, \bar{N}_3 = \frac{a_3 - 1}{a_{33}}\]

when \(a_1 = 1 \text{ and } a_3 > 1\)

\[E_9 : \bar{N}_1 = \frac{1}{a_{11}} \left( (a_1 - 1) + a_{12} \left( \frac{a_2 - 1}{a_{22}} \right) \right), \bar{N}_2 = \frac{a_2 - 1}{a_{22}}, \bar{N}_3 = 0\]

when \(a_1 > 1 \text{ and } a_2 > 1\)

\[E_{10} : \bar{N}_1 = a_{12} \left( \frac{a_2 - 1}{a_{12}} \right), \bar{N}_2 = \frac{a_2 - 1}{a_{22}}, \bar{N}_3 = 0\]

when \(a_1 = 1 \text{ and } a_2 > 1\)
(iv) The co-existent states (or) normal steady states

\[ E_{11} : R_1 = \frac{1}{a_{11}} \left( a_{11} - 1 \right) + \frac{a_{22}}{a_{22}} \left( a_{22} - 1 \right) + a_{23} \left( \frac{a_{33} - 1}{a_{33}} \right) + a_{33} \left( \frac{a_{33} - 1}{a_{33}} \right), \]

\[ R_2 = \frac{1}{a_{22}} \left( a_{22} - 1 \right) + a_{23} \left( \frac{a_{33} - 1}{a_{33}} \right), \]

when \( a_1, a_2 \) and \( a_3 > 1 \)

\[ E_{12} : R_1 = \frac{1}{a_{11}} \left( a_{11} - 1 \right) + \frac{a_{22}}{a_{22}} \left( a_{22} - 1 \right) + a_{23} \left( \frac{a_{33} - 1}{a_{33}} \right) + a_{33} \left( \frac{a_{33} - 1}{a_{33}} \right), \]

\[ R_2 = \frac{1}{a_{22}} \left( a_{22} - 1 \right) + a_{23} \left( \frac{a_{33} - 1}{a_{33}} \right), \]

when \( a_1 = 1 \) and \( a_2, a_3 > 1 \)

\[ E_{13} : R_1 = \frac{1}{a_{11}} \left( a_{11} - 1 \right) + a_{12} a_{23} \left( \frac{a_{33} - 1}{a_{33}} \right) + a_{33} \left( \frac{a_{33} - 1}{a_{33}} \right), \]

\[ R_2 = a_{23} \left( \frac{a_{33} - 1}{a_{33}} \right), \]

when \( a_2 = 1 \) and \( a_1, a_3 > 1 \)

\[ E_{14} : R_1 = \frac{1}{a_{11}} \left( a_{11} - 1 \right) + a_{12} a_{23} \left( \frac{a_{33} - 1}{a_{33}} \right) + a_{33} \left( \frac{a_{33} - 1}{a_{33}} \right), \]

\[ R_2 = a_{23} \left( \frac{a_{33} - 1}{a_{33}} \right), \]

when \( a_1, a_2 = 1 \) and \( a_3 > 1 \)

IV. STABILITY ANALYSIS

A. Stability of \( E_1 (0, 0, 0) \)

\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots = 0 \]
\[ N_2(t) = N_2(t + 1) = N_2(t + 2) = \ldots = 0 \]
\[ N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = 0 \]

i.e., \( N_i(t + r) = 0 \), where \( r \) is an integer and \( i = 1, 2, 3 \)

Hence, \( E_1 (0, 0, 0) \) is stable.

B. Stability of \( E_2 \)

\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots = 0 \]
\[ N_2(t) = N_2(t + 1) = N_2(t + 2) = \ldots = 0 \]
\[ N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = 0 \]

i.e., \( N_i(t + r) = 0 \), where \( r \) is an integer and \( i = 1, 2 \)

Hence, \( E_2 \) is stable.

C. Stability of \( E_3 \)

\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots = 0 \]
\[ N_2(t) = N_2(t + 1) = N_2(t + 2) = \ldots = 0 \]
\[ N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = 0 \]

i.e., \( N_i(t + r) = 0 \), \( N_i(t + r) = \frac{a_{33} - 1}{a_{33}} \)

Hence, \( E_3 \) is stable.

D. Stability of \( E_4 \)

\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots = 0 \]
\[ N_2(t) = N_2(t + 1) = N_2(t + 2) = \ldots = 0 \]
\[ N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = 0 \]

i.e., \( N_i(t + r) = 0 \), \( N_i(t + r) = \frac{a_{33} - 1}{a_{11}} \)

Hence, \( E_4 \) is stable.

E. Stability of \( E_5 \)

\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots = 0 \]
\[ N_2(t) = N_2(t + 1) = N_2(t + 2) = \ldots = 0 \]
\[ N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = 0 \]

i.e., \( N_i(t + r) = 0 \), \( N_i(t + r) = \frac{a_{33} - 1}{a_{11}} \)

Hence, \( E_5 \) is stable.

F. Stability of \( E_6 \)

\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots = 0 \]
\[ N_2(t) = N_2(t + 1) = N_2(t + 2) = \ldots = a_{23} \left( \frac{\alpha_3 - 1}{a_{22}a_{33}} \right) \]
\[ N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = a_{33} \left( \frac{\alpha_3 - 1}{a_{22}} \right) \]
i.e., \( N_i(t + r) = a_{23} \left( \frac{\alpha_3 - 1}{a_{22}a_{33}} \right) \), where \( r \) is an integer
Hence, \( E_3 \) is stable.

G. Stability of \( E_5 \)
\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots = \frac{1}{a_{11}} \left( \alpha_1 - 1 \right) + a_{13} \left( \frac{\alpha_3 - 1}{a_{33}} \right) \]
\[ N_2(t) = N_2(t + 1) = N_2(t + 2) = \ldots = 0 \]
\[ N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = \frac{\alpha_3 - 1}{a_{33}} \]
i.e., \( N_i(t + r) = \frac{1}{a_{11}} \left( \alpha_1 - 1 \right) + a_{13} \left( \frac{\alpha_3 - 1}{a_{33}} \right) \), where \( r \) is an integer
Hence, \( E_5 \) is stable.

H. Stability of \( E_6 \)
\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots = a_{13} \left( \frac{\alpha_3 - 1}{a_{11}a_{33}} \right) \]
\[ N_2(t) = N_2(t + 1) = N_2(t + 2) = \ldots = 0 \]
\[ N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = \frac{\alpha_3 - 1}{a_{33}} \]
i.e., \( N_i(t + r) = a_{13} \left( \frac{\alpha_3 - 1}{a_{11}a_{33}} \right) \), \( N_i(t + r) = 0 \),
where \( r \) is an integer
Hence, \( E_6 \) is stable.

I. Stability of \( E_7 \)
\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots = \frac{1}{a_{11}} \left( \alpha_1 - 1 \right) + a_{12} \left( \frac{\alpha_2 - 1}{a_{22}} \right) \]
\[ N_2(t) = N_2(t + 1) = N_2(t + 2) = \ldots = \frac{\alpha_2 - 1}{a_{22}} \]
\[ N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = 0 \]
i.e., \( N_i(t + r) = \frac{1}{a_{11}} \left( \alpha_1 - 1 \right) + a_{12} \left( \frac{\alpha_2 - 1}{a_{22}} \right) \),
\[ N_2(t + r) = \frac{\alpha_2 - 1}{a_{22}}, \quad N_3(t + r) = 0, \text{ where } r \text{ is an integer} \]
Hence, \( E_7 \) is stable.

J. Stability of \( E_{10} \)
\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots = a_{12} \left( \frac{\alpha_2 - 1}{a_{11}a_{22}} \right) \]
\[ N_2(t) = N_2(t + 1) = N_2(t + 2) = \ldots = \frac{\alpha_2 - 1}{a_{22}} \]
\[ N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = 0 \]
i.e., \( N_i(t + r) = a_{12} \left( \frac{\alpha_2 - 1}{a_{11}a_{22}} \right) \), \( N_2(t + r) = \frac{\alpha_2 - 1}{a_{22}} \),
\[ N_3(t + r) = 0, \text{ where } r \text{ is an integer} \]
Hence, \( E_{10} \) is stable.

K. Stability of \( E_{11} \)
\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots \]
\[ N_2(t) = N_2(t + 1) = N_2(t + 2) = \ldots = \frac{1}{a_{22}} \left( \alpha_2 - 1 \right) + a_{23} \left( \frac{\alpha_2 - 1}{a_{33}} \right) \]
\[ N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = \frac{\alpha_2 - 1}{a_{33}} \]
i.e., \( N_i(t + r) = \frac{1}{a_{22}} \left( \alpha_2 - 1 \right) + a_{23} \left( \frac{\alpha_2 - 1}{a_{33}} \right) \),
\[ N_2(t + r) = a_{23} \left( \frac{\alpha_2 - 1}{a_{33}} \right) \]
\[ N_3(t + r) = \frac{\alpha_2 - 1}{a_{33}}, \text{ where } r \text{ is an integer} \]
Hence, \( E_{11} \) is stable.

L. Stability of \( E_{12} \)
\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots \]
\[ N_2(t) = N_2(t + 1) = N_2(t + 2) = \ldots = \frac{1}{a_{22}} \left( \alpha_2 - 1 \right) + a_{23} \left( \frac{\alpha_2 - 1}{a_{33}} \right) \]
\[ N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = 0 \]
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\[ N_3(t) = N_3(t+1) = N_3(t+2) = \ldots = \frac{\alpha_3 - 1}{a_{33}} \]

i.e., \( N_i(t+r) = \frac{1}{a_{ii}} \left[ \left( \alpha_i - 1 \right) + a_{i2} \left( \frac{\alpha_2 - 1}{a_{22} a_{33}} \right) + a_{i3} \left( \frac{\alpha_3 - 1}{a_{33}} \right) \right] \).

\[ N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = \frac{\alpha_2 - 1}{a_{22} a_{33}} \]

\[ N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = \frac{\alpha_1 - 1}{a_{11}} \]

where \( r \) is an integer

Hence, \( E_{i3} \) is stable.

\[ N_i(t+r) = \frac{1}{a_{ii}} \left[ \left( \alpha_i - 1 \right) + a_{i2} \left( \frac{\alpha_2 - 1}{a_{22} a_{33}} \right) + a_{i3} \left( \frac{\alpha_3 - 1}{a_{33}} \right) \right], \]

\[ N_j(t+r) = \frac{\alpha_j - 1}{a_{jj}} \]

where \( r \) is an integer

Hence, \( E_{i3} \) is stable.

\[ N_i(t) = N_i(t+1) = N_i(t+2) = \ldots \]

\[ = \frac{1}{a_{ii}} \left[ a_{i2} a_{23} \left( \frac{\alpha_2 - 1}{a_{22} a_{33}} \right) + a_{i3} \left( \frac{\alpha_3 - 1}{a_{33}} \right) \right] \]

\[ N_j(t) = N_j(t+1) = N_j(t+2) = \ldots = \frac{\alpha_j - 1}{a_{jj}} \]

\[ N_i(t+r) = \frac{1}{a_{ii}} \left[ a_{i2} a_{23} \left( \frac{\alpha_2 - 1}{a_{22} a_{33}} \right) + a_{i3} \left( \frac{\alpha_3 - 1}{a_{33}} \right) \right], \]

\[ N_j(t+r) = \frac{\alpha_j - 1}{a_{jj}} \]

where \( r \) is an integer

Hence, \( E_{i4} \) is stable.

\[ N_i(t) = N_i(t+1) = N_i(t+2) = \ldots \]

\[ = \frac{1}{a_{ii}} \left[ a_{i2} a_{23} \left( \frac{\alpha_2 - 1}{a_{22} a_{33}} \right) + a_{i3} \left( \frac{\alpha_3 - 1}{a_{33}} \right) \right] \]

\[ N_j(t) = N_j(t+1) = N_j(t+2) = \ldots = \frac{\alpha_j - 1}{a_{jj}} \]

\[ N_i(t+r) = \frac{1}{a_{ii}} \left[ a_{i2} a_{23} \left( \frac{\alpha_2 - 1}{a_{22} a_{33}} \right) + a_{i3} \left( \frac{\alpha_3 - 1}{a_{33}} \right) \right], \]

\[ N_j(t+r) = \frac{\alpha_j - 1}{a_{jj}} \]

where \( r \) is an integer

Hence, \( E_{i4} \) is stable.

\[ N_i(t+2) = N_i(t), \ i = 1, 2, 3 \]

At this stage the system has 125 equilibrium states. The present paper deals only the stability of one period equilibrium states and the stability of the two period equilibrium states will be presented in the forth coming communications.

V. NUMERICAL EXAMPLES

The numerical solutions of the discrete model equations computed for specific values of the various parameters and the initial conditions. The results are illustrated in Figures 1 to 6.

\[ \alpha_1 = 3.9, \alpha_2 = 2.6, \alpha_3 = 4.4, a_{11} = 3.9, a_{12} = 0.6, a_{13} = 1.5, a_{21} = 0.8, a_{22} = 1.2, a_{23} = 1.7, a_{31} = 7.5, a_{32} = 5, a_{33} = 2 \]

\[ \alpha_2 = 1, \alpha_3 = 2.5, a_{12} = 2.3, a_{23} = 4.4, a_{13} = 0.4, N_{10} = 0, N_{20} = 7.5, N_{30} = 3.75 \]

\[ \alpha_2 = 1, \alpha_3 = 4.2, a_{11} = 3.2, a_{13} = 1.6, a_{33} = 0.5, N_{10} = 3.2, N_{20} = 0, N_{30} = 6.4 \]
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Fig. 4 Variation of population against time for $\alpha_2 =1, a_{11} =1.4, a_{12} =5.85, a_{22} =0.8, N_{10} =8.9, N_{20} =2.1, N_{30} =0$

Fig. 5. Variation of population against time for $\alpha_1 =3.8, a_{11} =0.6, N_{10} =4.7, N_{20} =0, N_{30} =0$

VI. CONCLUSION

The present paper deals with an investigation on a discrete model of three species syn eco-system with limited resources. The basic equations for this model constitute as three first order non-linear coupled ordinary difference equations. All possible equilibrium points of the model are identified at $N_i (t+1) = N_i (t)$; $i = 1, 2, 3$. It is observe that, all the fourteen equilibrium points are found to be stable. Further the numerical solutions for the discrete model equations are computed.

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