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Accelerated Simulation Scheme for Solving Financial Problems

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Abstract— The Monte Carlo simulation method uses random sampling to study properties of systems with components that behave in a random state. More precisely, the idea is to simulate on the computer the behavior of these systems by randomly generating the variables describing the behavior of their components. In this paper, we propose an efficient and reliable simulation scheme based on Monte Carlo algorithm and combining two variance reduction procedures. We simulate a European option price numerically using the proposed simulation scheme.

Index Terms— Monte Carlo Simulation, Option Pricing, Variance Reduction, European Option

I. Introduction

Monte Carlo simulation dominates many areas of science and engineering, largely because of their ability to treat complex problems that previously could be approximated only by simplified deterministic methods. Various Monte Carlo methods now are used routinely by scientists, engineers, mathematicians, statisticians, economists, and others who work in a wide range of quantitative disciplines. Recent years have seen a lot of improvements in Monte Carlo simulation with high potential for success in applications. Today, Monte Carlo simulation is natural and essential tools in computational finance [1-8].

The basic idea of the Monte Carlo simulation is to approximate an expected value $E[X]$ by an arithmetic average of the large number of independent realizations which all have the same distribution as X. As the Monte Carlo estimator is a random variable, each run of it produces new values. Therefore, the variance of the estimator is a measure for its accuracy and reducing this variance by efficient algorithm is the usual way of speeding up the Monte Carlo simulation.

In this paper, we introduce the naïve Monte Carlo simulation for European call or put options pricing, and then concentrate on two variance reduction procedures, antithetic variates and control variates for variance reduction. Finally, by combining two variance reduction procedures, we propose an efficient Monte Carlo simulation schema to reduce the variance of estimated option prices financial engineering. The paper is organized as follows: Section 2 describes European option pricing. Section 3 discusses the main point of Monte Carlo simulation and variance reduction techniques. In sections 4, we propose a new simulation scheme for European option pricing under Black-Scholes model. Section 5 summarizes the most important results. Finally, we conclude.

II. European Options

A call option (put option) gives the right, but not the obligation, to buy (sell) an underlying asset at a fixed price (exercise price or strike price) at or before a specified date (maturity date or expiry date). Gain or loss on the option is called payoff. The simplest option is the European option that can be exercised only on the maturity. Let $S(T)$ denote the underlying asset price at maturity and K be the exercise price. For a call option, if $S(T) > K$ then the older of the option exercise it for a profit of $S(T) - K$. On the other hand, if $S(T) \le K$ the option expires worthless. Thus the payoff function of the European call option at maturity data is as follows [6]

$$(S(T) - K)^{+} = \max\{S(T) - K, 0\}. \tag{1}$$

Based on the risk-neutral valuation method the price of the call option is equal to expectation of the discounted payoff with risk-free interest rate, i.e. as

$$C = e^{-rT} E\,[\max\{0, S(T) - K\}]. \tag{2}$$

One of the most important models for describing underlying asset price is the Black-Scholes model [1]. Based on this model, the price of the underlying asset is assumed to follow the Geometric Brownian Motion and thus satisfy in the following stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)\,dW(t), \quad \mu \in \mathbb{R}, \sigma > 0 \tag{3}$$

where $W(t)$ is a Brownian motion. The parameter μ, σ is called drift and volatility parameter, respectively. By using Ito's lemma, the unique solve of the above stochastic differential equation at maturity T is given by [6]

$$S(T) = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W(T)\right\} \tag{4}$$

with S_0 is the initial price of the underlying asset, assumed to be known. If Z be a standard normal random variable, then based on the characteristics of the Brownian motion increments the random variables $\sqrt{T}Z$ and $dW(T)$ have identical distributions. Therefore, Eq. (4) can be rewritten as follows:

$$S(T) = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,Z\right\} \tag{5}$$

According to the risk neutral valuation principle, drift parameter μ is equal to the risk-free interest rate r. Using the relationship between normal and log-normal distributions and based on the above equation, it can be shown that $S(T)$ has a log-normal distribution with mean $(\mu - \frac{1}{2}\sigma^2)T + \ln S_0$ and variance $\sigma^2 T$. Therefore, we will get

$$S(T) = S_0 e^{\mu T}. \tag{6}$$

We can easily seen that if Z be a standard normal random variable and $a, b,$ and c be positive constants, then we have

$$E[\max\{ae^{bZ} - c, 0\}] = a\,e^{\frac{b^2}{2}}\Phi\left(b + \frac{1}{b}\log\frac{a}{c}\right) - c\,\Phi\left(\frac{1}{b}\log\frac{a}{c}\right) \tag{7}$$

where the function $\Phi(.)$ is the cumulative distribution function of the standard normal distribution. Thus, by substituting $a = S_0 e^{-\frac{\sigma^2 T}{2}}$, $b = \sigma\sqrt{T}$ and $c = K$ in the above equation and by Eq. (2) we conclude that

$$\begin{aligned} C &= e^{-rT} E\,[\max\{S(T) - K, 0\}] \\ &= e^{-rT} e^{-rT}\, S_0 \Phi\left(\frac{1}{\sigma\sqrt{T}}\log\frac{S_0 e^{rT}}{K} + \frac{\sigma\sqrt{T}}{2}\right) \\ &\quad - K e^{-rT}\,\Phi\left(\frac{1}{\sigma\sqrt{T}}\log\frac{S_0 e^{rT}}{K} - \frac{\sigma\sqrt{T}}{2}\right) \\ &= S_0\Phi(d_1) - K e^{-rT}\Phi(d_2). \end{aligned}$$

The values d_1 and d_2 are given by

$$d_1 = \frac{\log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\log(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

III. Monte Carlo Simulation and Variance Reduction Procedures

Suppose that a simulation analyst wishes to compute a quantity θ that can be expressed as the expectation of a real-valued random variable X, so that $\theta = E[g(X)]$. The conventional sampling-based algorithm for computing θ involves simulating n independent, identically distributed copies of the sequence of independent, identically distributed X denoted $X_1, X_2, \ldots, X_n$. The corresponding estimator for θ is then just the following sample mean

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} g(X_i). \tag{8}$$

Since $\hat{\mu}_n$ is the sample mean of independent, identically distributed random variables

$g(X_1),..., g(X_2)$ having expectation μ, the strong law of large numbers says $\hat{\mu}_n$ converges almost surely to μ as the number of simulations goes to infinity [5]

$$\hat{\mu}_n \xrightarrow{a.s.} \mu, \quad n \to \infty. \tag{9}$$

The statistical error of the Monte Carlo error is $\frac{\sigma_n^2}{\sqrt{n}}$, which is independent of the problem dimension. Also, the confidence interval by formula (8) is as follows

$$(\hat{\mu}_n - 1.96\sqrt{\frac{\sigma_N{}^2}{N}}\ ,\ \ \hat{\mu}_n + .96\sqrt{\frac{\sigma_N{}^2}{N}}).$$

The basic idea for increasing the accuracy of the Monte Carlo methods is to use the variance reduction techniques to reduce the variance of the samples X_i directly. A number of techniques have been developed and that help to reduce the number of simulations required for a given accuracy [1-5, 8].

3.1 Antithetic Variates Procedures (AV)

Consider the parameter $\theta = E[g(X)]$. Let Y_1 and Y_2 are two Monte Carlo estimators for parameter θ. A new estimator can be defined by $\hat{\theta} = (Y_1 + Y_2)/2$. The variance of this estimator is as follows

$$\mathrm{Var}(\hat{\theta}) = \frac{1}{4}\begin{Bmatrix} \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) \\ +2\mathrm{Cov}(Y_1, Y_2) \end{Bmatrix}$$

If $\mathrm{Cov}(Y_1, Y_2) < 0$ then the new estimator $\hat{\theta}$ has smaller variance than Y_1, Y_2 [6]. It is well known that if U be a uniform random variable on $[0,1]$ then $1-\mathrm{U}$ also will be uniformly distributed on this interval. As a result $g(U)$ and $g(1-U)$ will be unbiased estimators for θ. It is shown that [3] if $g(U)$ be a non-decreasing or non-increasing function of U, then $\mathrm{Cov}\big(\, g(U),\, g(1-U)\,\big) < 0$. Thus, for random numbers $U_1,\ldots,U_N$ from $[0,1]$ two Monte Carlo estimators $\hat{\theta}_1 = \frac{\sum_{i=1}^{N} g(u_i)}{N}$ and $\hat{\theta}_2 = \frac{\sum_{i=1}^{N} g(1-u_i)}{N}$ can be combined to get antithetic Monte Carlo estimator as follows

$$\hat{\theta}_{AV} = \frac{\hat{\theta}_1 + \hat{\theta}_2}{2} = \frac{1}{2N}\sum_{i=1}^{N}\left[g(u_i) + g(1-u_i)\right]$$

where U and $1-\mathrm{U}$ is called antithetic variates.

3.2 Control Variates Procedure (CV)

Consider the parameter $\theta = E[Y]$ and suppose X be a random variable with known expectation. We know that X and Y are unbiased estimators for their expectations. With these estimators a general class of unbiased estimators for θ can be constructed as follows

$$\hat{\theta}_{CV} = Y + c(X - E[X])$$

where c is a real number. We want to choose c such that to minimize $\mathrm{Var}(\hat{\theta}_{CV})$ that has the following form

$$\begin{aligned} \mathrm{Var}(\hat{\theta}_{CV}) &= \mathrm{Var}(Y) + c^2\mathrm{Var}(X) \\ &\quad + 2c\,\mathrm{Cov}(X,Y) \end{aligned} \tag{10}$$

With differentiating from Eq. (10) and equal to zero we obtain the optimal c

$$c^* = -\frac{\mathrm{Cov(Y,Y)}}{\mathrm{Var(X)}}$$

By substituting c^* in Eq. (10), we will obtain

$$\begin{aligned} \mathrm{Var}(\hat{\theta}_{CV}) &= \mathrm{Var}(Y) - \frac{\mathrm{Cov}(Y, X)}{\mathrm{Var}(X)} \\ &= \mathrm{Var}(\hat{\theta}) - \frac{\mathrm{Cov}^2(Y, X)}{\mathrm{Var}(X)} \end{aligned}$$

Thus, to achieve variance reduction we must have $\mathrm{Cov}(Y, X) \neq 0$.

IV. The Proposed Simulation Scheme

Here, by combining AV and CV procedures, we achieve an efficient simulation scheme, namely AVCV scheme to solve an option pricing problem. This simulation scheme is as follows:

1. Input S0,K,r,T,sigma,NRepl.

2. Set nuT =(r-0.5*sigma^2)*T.

3. Set siT =sigma*sqrt(T).

4. **For** i=1 **to** NRepl **do**

***Simulate** a standard normally distributed random variable Z.*

***Set** StVals=S0*exp(nuT+siT*Z)*

***Set** OptVals=exp(-r*T)*max(0,StVals-K)*

***Set** MatCov =cov(StVals,OptVals)*

Set

*VarY=S0^2*exp(2*r*T)*(exp(T*sigma^2)-1*

***Set** c = - MatCov(1,2)/VarY.*

***Set** ExpY = S0 * exp(r*T).*

***Set** NewStVals=S0*exp(nuT+siT*Z).*

***Set** NewStVals1=S0*exp(nuT+siT*-Z).*

*NewOptionVals= exp(-r*T)*0.5* (max(0,NewStVals- K)+max(0,NewStVals1-K)).*

***Set** NewStVals=(NewStVals+NewStVals1)*0.5.*

***Set** ControlVars= NewOptVals+c*(NewStVals-ExpY).*

5. End for

6. Set Price= ControlVars/NRepl.

7. End of Algorithm

In Fig. 1 a sample trajectory of three estimators is shown. The simulation is done using Naïve estimator, antithetic variates estimator (AV), and antithetic control variates estimator (AVCV) with the following parameters:

$$S_0 = 50,\ K = 50,\ \sigma = 0.4,\ r = 0.1,\ T = 1\,(one\ year).$$

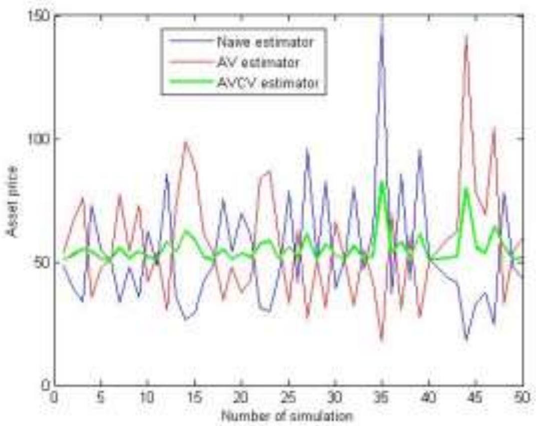

Fig. 1: Comparison of three estimators

V. Simulation Results

In this section, we present the simulation results for the accuracy and the convergence of the proposed simulation scheme (i.e., AVCV) for European option pricing. We have used rand function of Matlab software for generating random numbers. In Table 1 the simulation error by the naïve Monte Carlo simulation (NMC), the antithetic Monte Carlo simulation (AV), the control variates Monte Carlo simulation (CV), and the proposed simulation (AVCV) are outlined. The purpose of Figures 2-5 is to understand the asset volatility factor that influence and move asset prices.

Also, in Figures 6-9 we see the performance of AVCV simulation in comparison to the other methods. For all calculations we used the following parameters:

$$S_0 = 50,\ K = 50,\ \sigma = 0.4,\ r = 0.1,\ T = 1\,(one\ year).$$

Table 1: Comparison of simulation error

Number of Simulation	NMC simulation error	AV simulation error	CV simulation error	AVCV simulation error
1000	0.5879	0.3291	0.2526	0.2281
2000	0.4126	0.2301	0.1793	0.1623
4000	0.2862	0.1591	0.1346	0.1155
8000	0.1962	0.1121	0.0984	0.0814
16000	0.1379	0.0790	0.0724	0.0573
32000	0.0974	0.0555	0.0544	0.0406

Table 2: The option value by AVCV simulation, NRepl=10000

Maturity Time	Strike price (K=30)	Strike price (K=40)	Strike price (K=45)	Strike price (K=50)
3/12	20.7501	11.4485	7.5714	4.6013
6/12	21.5618	13.0515	9.5968	6.8172
9/12	22.4137	14.5013	11.2846	8.6182
12/12	23.2632	15.8197	12.7719	10.1942

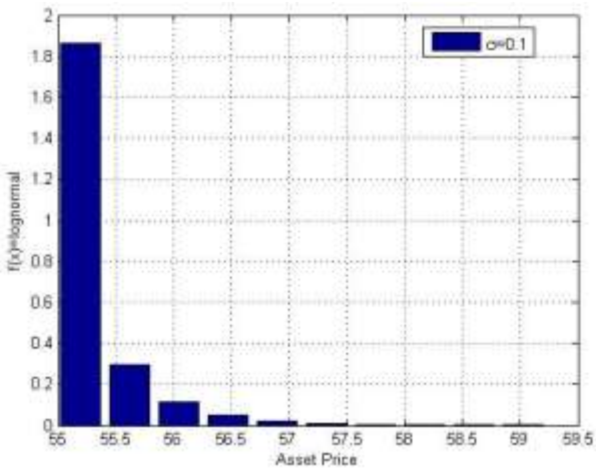

Fig. 2: 5000 discrete asset paths and final time histogram

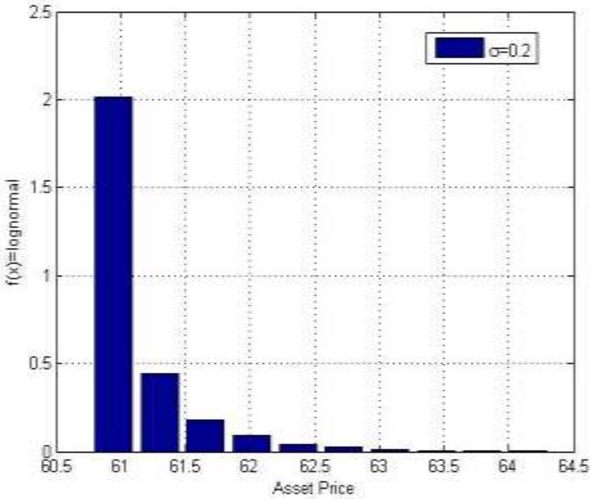

Fig. 3: 5000 discrete asset paths and final timehistogram

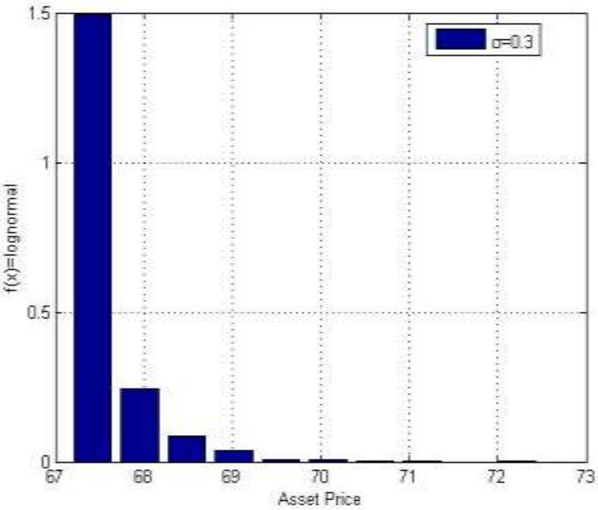

Fig. 4: 5000 discrete asset paths and final time histogram

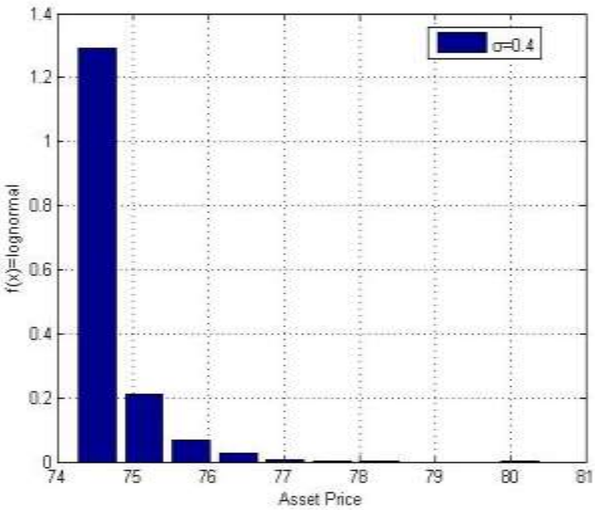

Fig. 5: 5000 discrete asset paths and final time histogram

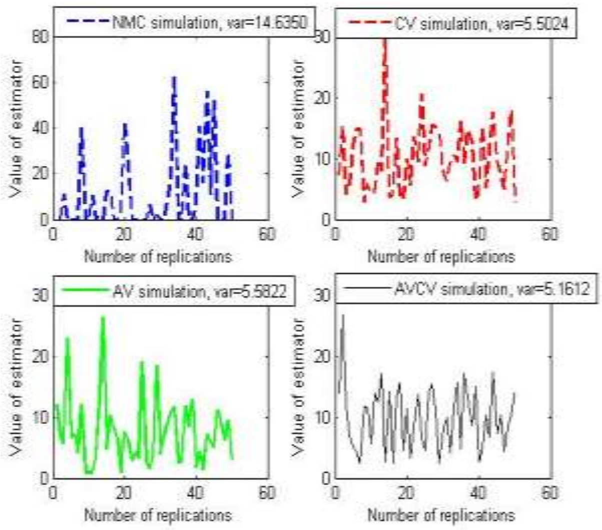

Fig. 6: Comparison of simulation schemes.

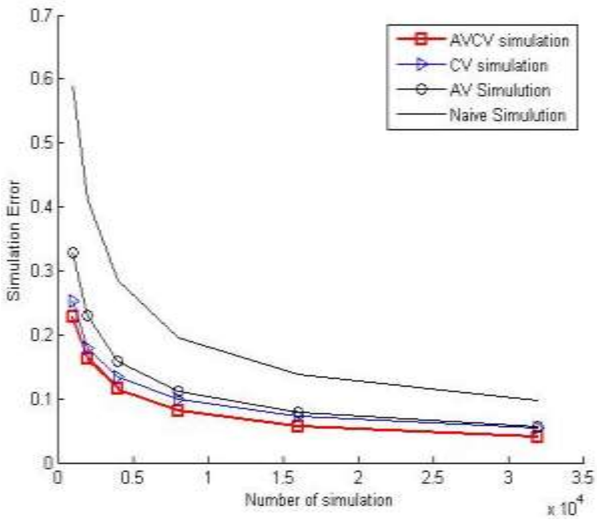

Fig. 7: Comparison of simulation error by NMC, AV, CV, AVCV.

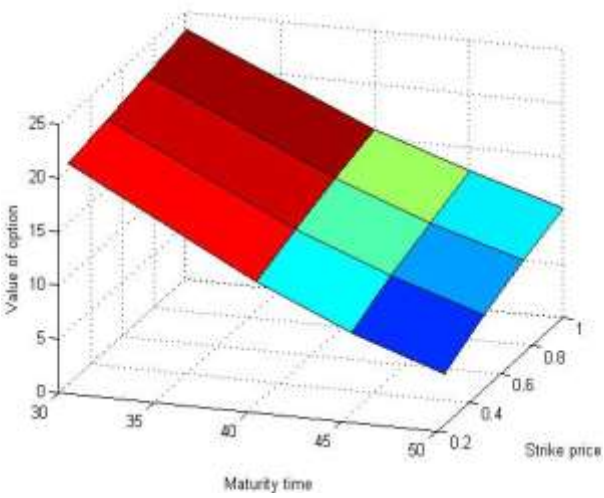

Fig.8: Price of the European call option by AVCV simulation scheme.

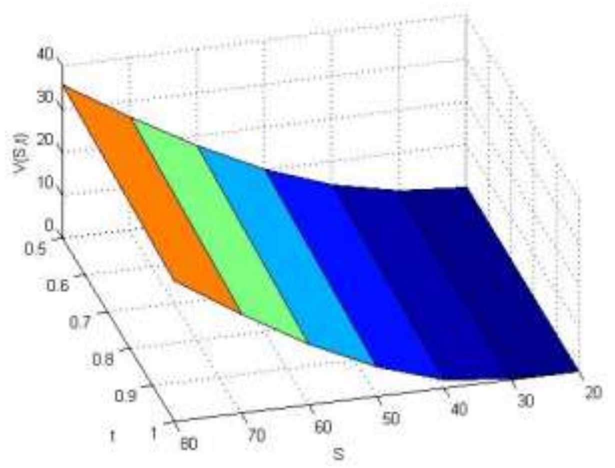

Fig. 9: Price of the European call option by AVCV simulation scheme.

VI. Conclusion

In this paper, a new and efficient simulation scheme for European call or put option pricing based on the Monte Carlo algorithm and two variance reduction procedures is proposed. The simulation results show that AVCV algorithm is always a good candidate for variance reduction in the Monte Carlo simulation for European options pricing problem.

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