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Non-polynomial Spline Difference Schemes for Solving Second-order Hyperbolic Equations

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***Abstract*--In this paper, a class of improved methods based on non-polynomial cubic splines in space and finite difference in time direction are constructed for the second-order hyperbolic equations with initial boundary value problems. Truncation error and stability analysis of the methods have been carried out. It is shown that by suitably choosing the parameters, many known methods can be derived from ours. We also obtain a new high accuracy scheme of $o(k^4+h^4)$, which is conditionally stable for $r \le 1$.Finally, a numerical experiment is tested and results are compared with other published numerical solutions.**

***Index Terms*-----second-order hyperbolic equation; non-polynomial cubic spline; conditionally stable; finite difference scheme.**

I. Introduction

We consider the following second-order hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t),\ 0 < x < 1, t \ge 0, \quad (1)$$

subject to initial conditions

$$u(x,0) = g_1(x),\ 0 < x < 1, \qquad (2)$$

$$\frac{\partial u(x,0)}{\partial t} = g_2(x),\ 0 < x < 1, \qquad (3)$$

and boundary conditions

$$u(0,t) = f_1(1),\ u(1,t) = f_2(x),\ t > 0. \qquad (4)$$

The above equations occur frequently in many fields of applied science and engineering; see [1,2,4,6,7] for example Recently, the finite difference schemes for Eq,(1)were proposed in [3]with the truncation error being $o(k^2+h^p)$,where $p = 1,2,$ and k and h denote the mesh parameters for x and t , respectively. Very recently, Rashidinia *et al*. [5] developed a class of methods based on non-polynomial cubic spline and the truncation error was improved to $o(k^2+h^2)$ and $o(k^2+h^4)$. They claimed that their schemes are unconditionally stable.

In this paper, we shall first point out that Rashidinia *et al*.'s schemes are in fact conditionally stable only. Then to improve both the accuracy and the stability of their methods, a new class of spline methods is proposed for Eq. (1) by using a non-polynomial cubic spline function approximation in space direction and finite difference approximation in time direction.

II THE CONDITIONALLY STABILITY OF SPLINE METHODS IN [5]

Let Δ be a partition of the interval $0 \le x \le 1$,which divides[0,1] into N subinterval with the uniform step length $h = \frac{1}{N}$.Let $K > 0$ be the time direction. The grid points (i,j) are given by $x_i = ih, i = 0(1)N$,and $t_i = jk,\ j = 0,1,2,\cdots$. Let u_i^j be the approximate value.

In [5], an approximation for Eq. (1)was developed in which the derivative with respect to time is replaced by a finite difference approximation and the derivative with respect to space is replaced by the non-polynomial cubic spline function approximation. Denote

$$u_x(x_i,t_j) = \frac{\partial u(x_i,t_j)}{\partial x}, \qquad (5)$$

$$u_{xx}\left(x_i,t_j\right)=\frac{\partial^2 u(x_i,t_j)}{\partial x^2}, \tag{6}$$

$$u_t\left(x_i,t_j\right)=\frac{\partial u(x_i,t_j)}{\partial t}, \tag{7}$$

$$u_{tt}\left(x_i,t_j\right)=\frac{\partial^2 u(x_i,t_j)}{\partial t^2}, \tag{8}$$

$$u_{tt}(x_i,t_j)=\frac{u\left(x_i,t_{j+1}\right)-2u\left(x_i,t_j\right)+u\left(x_i,t_{j-1}\right)}{k^2}+o\left(k^2\right), \tag{9}$$

$$s''_\Delta\left(x_i,t_j\right)=M_i^j+o\left(h^2\right), \tag{10}$$

where $s_\Delta\left(x,t\right)$ and M_i^j are defined in[5].

At the gird point $\left(x_i,t_j\right)$,the given differential equation (1) was discretized as

$$u_{tt}\left(x_i,t_j\right)=u_{xx}\left(x_i,t_j\right)+f\left(x_i,t_j\right). \tag{11}$$

By putting Eqs.(5)-(11) into Eq.(11),Eq.(11) because

$$\frac{u\left(x_i,t_{j+1}\right)-2u\left(x_i,t_j\right)+u\left(x_i,t_{j-1}\right)}{k^2}+O\left(k^2\right)=M_i^j+O(h^2)+f(x_i,t_j), \tag{12}$$

and by neglecting the truncation error, the above equation leads to

$$\frac{u_i^{j+i}-2u_i^j+u_i^{j-1}}{k^2}=M_i^j+f_i^j. \tag{13}$$

Similarly, the following equations hold

$$M_{i+1}^j=\frac{u_{i+1}^{j+1}-2u_{i+1}^j+u_{i+1}^{j-1}}{k^2}-f_{i+1}^j,$$

$$M_{i-1}^j=\frac{u_{i-1}^{j+1}-2u_{i-1}^j+u_{i-1}^{j-1}}{k^2}-f_{i-1}^j. \tag{14}$$

In [5],the following non-polynomial cubic spline relation was given

$$\begin{aligned}&u_{i+2}-2u_j+u_{j-1}\\&=h^2(\alpha M_{i+1}+2\beta M_j+\alpha M_{j-1}),\end{aligned} \tag{15}$$

where $j=1(1)N-1$.

Substituting Eqs.(13)-(14) into (15),they finally obtained the following schemes

$$\begin{aligned}&\alpha u_{i-1}^{j+1}+2\beta u_i^{j+1}+\alpha u_{i+1}^{j+1}\\&-\left(2\alpha+r\right)u_{i-1}^j-\left(4\beta-2r\right)u_i^j-\\&\left(2\alpha+r\right)u_{i+1}^j+\alpha u_{i-1}^{j-1}+2\beta u_i^{j-1}\\&+\alpha u_{i+1}^{j-1}=k^2\left(\alpha f_{i-1}^j+2\beta f_i^j+\alpha f_{i+1}^j\right),\end{aligned} \tag{16}$$

where $j=0,1,2,\dots,i=1(1)N-1,$

$$r=k^2/h^2,\ \alpha=\left(\frac{w}{\sin(w)}-1\right)\Big/w^2$$

and

$$\beta=\left(1-\frac{w\cos(w)}{\sin(w)}\right)\Big/w^2$$

as defined in[5].To achieve the accuracy order $O\left(k^2+h^2\right)$ of their schemes,the parameters α and β were chosen satisfy $\alpha+\beta=\frac{1}{2}$.

The authors in [5] claimed that the stability for the difference schemes (16) are unconditionally stable. However, that is not true. In fact, by applying the root condition [9] to the Eq.(21) in [5], a necessary and sufficient condition for $|\xi|<1$ is that

$$Q>0,\quad Q-\phi+\psi>0,\quad Q+\phi+\psi>0, \tag{17}$$

where Q,ϕ and ψ are defined in [5].Eq.(17) is equivalent to

$$Q=\alpha e^{i\theta}+2\beta+\alpha e^{-i\theta}>0,\ \ \forall\theta\in[0,2\pi], \tag{18}$$

$$\begin{aligned}&Q-\phi+\psi=\left(4\alpha+r\right)e^{i\theta}\\&+\left(8\beta-2r\right)+\left(4\alpha+r\right)e^{-i\theta}>0,\end{aligned}\quad\forall\theta\in[0,2\pi], \tag{19}$$

$$Q-\phi+\psi=2r\left(1-\cos\theta\right)>0,\ \forall\theta\in[0,2\pi], \tag{20}$$

Where

$$\alpha+\beta=\frac{1}{2} \qquad (21)$$

From Eqs.(18-21),we can get

$$2\alpha<\frac{1}{1-\cos\theta},\ \forall\theta\in[0,2\pi], \qquad (22)$$

$$(8\alpha+2r)<\frac{4}{1-\cos\theta},\ \ \forall\theta\in[0,2\pi.] \qquad (23)$$

That is,

$$\alpha<\frac{1}{2}\cdot\min_{\theta\in[0,2\pi]}\frac{1}{1-\cos\theta}=\frac{1}{4} \qquad (24)$$

$$\mathrm{r}<\min_{\theta\in[0,2\pi]}\frac{2}{1-\cos\theta}-4\alpha=1-4\alpha \qquad (25)$$

Therefore Rashidinia et al.'methods[5] are conditionally stable only.

III NEW NON-POLYNOMIAL CUBIC SPLINE METHODS

To obtain some difference schemes with a better stability and accuracy, we can modify scheme (13) into

$$\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{k^2} = \alpha_1\mathrm{M}_i^{j+1}+2\beta_1\mathrm{M}_i^j+\alpha_1\mathrm{M}_i^{j-1}+f_i^{\,j}, \qquad (26)$$

where α_1, β_1 are parameters,satisfying $\alpha_1+\beta_1=\frac{1}{2}$.

Then similar to Eqs.(13-14),we obtain

$$\begin{aligned}&(\alpha_2-\alpha_1 r)(u_{i+1}^{j+1}+u_{i-1}^{j+1})+(2\beta_2+2\alpha_1 r)u_i^{j+1}\\&=(2\beta_1 r+2\alpha_2)(u_i^{j+1}+u_{i-1}^{j})+(4\beta_2-4\beta_1 r)u_i^j\\&(\alpha_1 r-\alpha_2)(u_{i+1}^{j-1}+u_{i-1}^{j-1})-(2\beta_2+2\alpha_1 r)u_i^{j-1}\\&+k^2[\alpha_1\alpha_2(f(x_{i-1},t_{j+1})+f(x_{i-1},t_{j-1}))\\&+2\alpha_2\beta_1 f(x_{i-1},t_j)+2\alpha_1\beta_2(f(x_i,t_{j+1})\\&+f(x_i,t_{j-1}))+4\beta_1\beta_2 f(x_i,t_j)\\&+\alpha_1\alpha_2(f(x_{i+1},t_{j+1})+f(x_{i+1},t_{j-1}))\\&+2\alpha_2\beta_1 f(x_{i+1},t_j)],\end{aligned} \qquad (27)$$

where $r=k^2/h^2, i=1(1)(N-1),\ j=1,2,\cdots,$ and α_2 and β_2 are parameters defined in [5].

By choosing suitable values of parameters α_1, β_1, α_2 and β_2 ,we ontain various numerical Methods to solve the hyperbolic equation (1).The truncation error and stability analysis of these methods are given in Section 4.

IV TRUNCATION ERROR AND STABILITY ANALYSIS OF THE PRESENT METHODS

Expanding Eq.(27) in Taylor series in terms of $u(x_i,t_j)$ and its derivatives, we obtain the truncation error

$$\begin{aligned}\mathrm{T}_i^j&=(2\alpha_1+2\beta_1)\frac{\partial^2 u}{\partial x^2}-(2\alpha_2+2\beta_2)\frac{\partial^2 u}{\partial t^2}\\&+\frac{1}{6}(\alpha_1+\beta_1)h^2\frac{\partial^4 u}{\partial x^4}-\alpha_2 h^2\frac{\partial^4 u}{\partial x^2 t^2}+\\&\partial_1 k^2\frac{\partial^4 u}{\partial x^2 t^2}-\frac{1}{6}(\alpha_2+\beta_2)k^2\frac{\partial^4 u}{\partial t^4}\\&-\frac{1}{180}(\alpha_1+\beta_1)h^4\frac{\partial^6 u}{\partial x^6}+\\&\frac{1}{12}(\alpha_2-\alpha_1 r)h^4\frac{\partial^6 u}{\partial x^4\partial t^2}\\&+\frac{1}{180}(\alpha_2-\alpha_1 r)k^4\frac{\partial^6 u}{\partial t^6}+\cdots.\end{aligned} \qquad (28)$$

1) If we choose $\alpha_1+\beta_1=\frac{1}{2}$ and $\alpha_2+\beta_2=\frac{1}{2}$ in (27), we get a scheme of $O(k^2+h^2)$.

2) If we choose $\alpha_1+\beta_1=\frac{1}{2}$, $\alpha_2+\beta_2=\frac{1}{2}$ and $\alpha_2=\frac{1}{12}$ in (27),we get a scheme of $O(k^2+h^4)$.

3) If we choose $\alpha_1+\beta_1=\frac{1}{2}$ $\alpha_2+\beta_2=\frac{1}{2}$ and $\alpha_2=\frac{1}{12}$ in (27), we get a scheme of $O(k^4+h^2)$.

4) If we choose $\alpha_1+\beta_1=\frac{1}{2}$ $\alpha_2+\beta_2=\frac{1}{2}$, $\alpha_1=\frac{1}{12}$ and $\alpha_2=\frac{1}{12}$ in (27),we get a scheme of O(k^4+h^4).

For various values of parameter $\alpha_1,\alpha_2,\beta_1,\beta_2$ and

γ, the truncation errors may now be obtained. Now we analyze the stability of the scheme (27).We assume that solution of (27) at the gird point (χ_i, t_j) is of the form

$$\mu_i^j = \xi^j e^{li\theta}, \qquad (29)$$

Where I= $\sqrt{-1}$, θ is real and ς is, in general, complex. Substituting (29) into (27), we obtain a characteristic equation

$$A\xi^2 + B\xi + C = 0, \qquad (30)$$

Where

$$A = C = (\alpha_2 - \alpha_1\gamma)\cos\theta + \beta_2 + \alpha_1\gamma, \qquad (31)$$

$$B = -2\beta_1\gamma\cos\theta - 2\alpha_2\cos\theta - 2\beta_2 + 2\beta_1\gamma. \qquad (32)$$

By applying the root condition [9] to Eq.(30), a necessary and sufficient condition for $|\xi| < 1$ is that $A > 0, A + B - C > 0,$ and $A - B + C > 0.$ From Eqs.(31-32), $\alpha_1 + \beta_1 = \frac{1}{2}$ and $\alpha_2 + \beta_2 = \frac{1}{2}$, we have

$$A + B + C = 2\gamma(1 - \cos\theta), \qquad (33)$$

$$A - C = 0, \qquad (34)$$

$$A - B + C = (2\alpha_2 - \alpha_1\gamma + \beta\gamma)\cos\theta + \alpha\gamma - \beta_1\gamma. \qquad (35)$$

Noticing that $A + B + C \geq 0, if \gamma > 0, \forall\theta \in [0,2\pi]$, we obtain from Eqs. (31) and (35) that

$$(\alpha_2 - \alpha_1\gamma)\cos\theta + \beta_2 + \alpha_1\gamma > 0, \forall\theta \in [0,2\pi], \qquad (36)$$

$$\begin{aligned}&(2\alpha_2 - \alpha_1\gamma + \beta_1\gamma)\cos\theta \\ &+ 2\beta_2 + \alpha_1\gamma - \beta_1\gamma > 0, \forall\theta \in [0,2\pi],\end{aligned} \qquad (37)$$

where $\alpha_1 + \beta_1 = \frac{1}{2}$ and $\alpha_2 + \beta_2 = \frac{1}{2}$ .Hence, we deduce that the scheme (27) is unconditionally stable if $\frac{1}{2} > \alpha_1 > \frac{1}{4}$ and $\alpha_2 < \frac{1}{4}$, and the scheme (27) is conditionally stable if

1) $0 \leq \alpha_1 < \frac{1}{4}, 0 < \alpha_2 < \frac{1}{4}, r < \frac{1 - 4\alpha_2}{1 - 4\alpha_1}$.

2) $0 \leq \alpha_1 < \frac{1}{4}, \alpha_2 > \frac{1}{4}, \frac{4\alpha_2 - 1}{4\alpha_1} < r < \frac{1 - 4\alpha_2}{1 - 4\alpha_1}$.

3) $\alpha_1 > \frac{1}{4}, \alpha_2 > \frac{1}{4}, r < \frac{1 - 4\alpha_2}{1 - 4\alpha_1}$.

V A CLASS OF METHODS

By choosing suitable values of parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ and γ, we obtain the following a class of methods:

1) If we choose $\alpha_1 = 0, \beta_1 = \frac{1}{2}, \alpha_2 = \frac{1}{6}$ and $\beta_2 = \frac{1}{3}$ in (27), we get formula of Rashidinia et al. [5] with truncation error of $0(k^2 + h^2)$, which is conditionally stable for $0 < \gamma < \frac{1}{3}$.

2) For $\alpha_1 = \alpha_2 = \frac{1}{6}$ and $\beta_1 = \beta_2 = \frac{1}{3}$ in (27), we obtain the conditionally stable formula of accuracy $0(k^2 + h^2)$, and $0(k^4 + h^4)$ with $\gamma = 1$.

3) For $\alpha_1 = \frac{1}{6}, \beta_1 = \frac{1}{3}, \alpha_2 = \frac{1}{12}$ and $\beta_2 = \frac{5}{12}$ in (27), we get formula with truncation error of $0(k^2 + h^4)$, which is conditionally stable for $0 < \gamma < 2$.

4) For the choice $\alpha_1 = 0, \beta_1 = \frac{1}{2}, \alpha_2 = \frac{1}{12}$ and $\beta_2 = \frac{5}{12}$ in (27), we get formula Rashidinia et al.[5] with truncation error of $0(k^2 + h^4)$, which is conditionally stable for $0 < \gamma < \frac{2}{3}$.

5) If we choose $\alpha_1 = \frac{1}{3}, \beta_1 = \frac{1}{6}, \alpha_2 = \frac{1}{12}$ and

$\beta_2 = \frac{5}{12}$ in (27), we arrive at the unconditionally stable scheme with accuracy $0(k^2 + h^4)$.

6) For $\alpha_1 = \alpha_2 = \frac{1}{12}$ and $\beta_1 = \beta_2 = \frac{5}{12}$ in (27), we obtain formula of accuracy $0(k^4 + h^4)$, which is conditionally stable for $0 < \gamma < 1$.

VI NUMERICAL EXAMPLE AND DISCUSSION

Example: Consider the hyperbolic equation (1) with the initial boundary value conditions (2-4) in which $f_1(t) = f_2(t) = 1$, $g_1(x) = \sin(\pi x) + 1$ and $g_2(x) = 0$. The average relative error percentage (AREP) is tabulated in Table 1-4 at $t = 2.0$ with $\gamma = 1/3, 0.4, 2/3, 1$. The exact solution $u(x,t)$ and the solution at the first time level $u(x,k)$ are respectively given by

$$u(x,t) = 1 + \sin \pi x \cos \pi t,$$

and

$$u(x,k) = g_1(x) + kg_2(x) + \frac{k^2}{2} g_{1xx}(x) + \frac{k^3}{6} g_{2xx}(x) + 0(k^4).$$

Table 1. The observed average relative error in present methods $(r = 1/3)$

$(\alpha_1, \beta_1, \alpha_2, \beta_2)$	$h = \frac{1}{16}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$	$h = \frac{1}{128}$
(1/6,1/3,1/6,1/3)	1.76E(-4)	4.84E(-5)	5.15E(-6)	3.76E(-7)
(1/6,1/3,1/12,5/12)	9.81E(-5)	2.48E(-5)	2.62E(-6)	1.91E(-7)
(1/3,1/6,1/12,5/12)	3.11E(-4)	7.56E(-5)	7.93E(-6)	5.77E(-7)
(1/12,5/12,1/12,5/12)	4.86E(-7)	3.16E(-8)	8.36E(-10)	1.52E(-11)
(0,1/2,1/6,1/3)[5]	3.27E(-4)	9.51E(-5)	1.02E(-5)	7.46E(-7)
(0,1/2,1/12,5/12)[5]	9.08E(-5)	2.44E(-5)	2.59E(-6)	1.89E(-7)

Table 2. The observed average relative error in present methods $(r = 1/4)$

$(\alpha_1, \beta_1, \alpha_2, \beta_2)$	$h = \frac{1}{16}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$	$h = \frac{1}{128}$
(1/6,1/3,1/6,1/3)	2.48E(-4)	1.04E(-5)	2.78E(-6)	7.09E(-7)
(1/6,1/3,1/12,5/12)	1.77E(-4)	7.68E(-6)	1.90E(-6)	4.76E(-7)
(1/3,1/6,1/12,5/12)	5.58E(-6)	2.48E(-5)	5.82E(-6)	1.43E(-6)
(1/12,5/12,1/12,5/12)	1.82E(-6)	7.46E(-9)	4.75E(-10)	3.00E(-11)
(0,1/2,1/6,1/3)[5]	3.83E(-3)	-	-	-
(0,1/2,1/12,5/12)[5]	4.44E(-4)	7.06E(-6)	1.86E(-6)	4.73E(-7)

Table 3. The observed average relative error in present methods $(r = 2/3)$

$(\alpha_1, \beta_1, \alpha_2, \beta_2)$	$h = \frac{1}{16}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$	$h = \frac{1}{128}$
(1/6,1/3,1/6,1/3)	5.66E(-5)	1.54E(-5)	3.99E(-6)	3.53E(-7)
(1/6,1/3,1/12,5/12)	1.33E(-4)	3.21E(-5)	8.06E(-6)	7.11E(-7)
(1/3,1/6,1/12,5/12)	4.71E(-4)	1.01E(-4)	2.45E(-5)	2.15E(-6)
(1/12,5/12,1/12,5/12)	1.92E(-7)	1.26E(-8)	8.05E(-10)	1.78E(-11)
(0,1/2,1/6,1/3)[5]	-	-	-	-
(0,1/2,1/12,5/12)[5]	1.07E(-4)	3.05E(-5)	7.96E(-6)	7.05E(-7)

Table 4. The observed average relative error in present methods $(r = 1)$

$(\alpha_1, \beta_1, \alpha_2, \beta_2)$	$h = \frac{1}{16}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$	$h = \frac{1}{128}$
(1/6,1/3,1/6,1/3)	1.17E(-16)	1.18E(-15)	5.28E(-15)	1.45E(-14)
(1/6,1/3,1/12,5/12)	2.83E(-5)	1.90E(-6)	1.23E(-7)	7.97E(-9)
(1/3,1/6,1/12,5/12)	2.52E(-4)	1.70E(-5)	1.10E(-6)	7.01E(-8)
(1/12,5/12,1/12,5/12)	1.17E(-16)	1.18E(-15)	5.28E(-15)	1.45E(-14)
(0,1/2,1/6,1/3)[5]	-	-	-	-
(0,1/2,1/12,5/12)[5]	-	-	-	-

It can be seen from Table 1-4 that the results by the present method with accuracy $O(k^4 + h^4)$ is much better than that obtained by method in [5], and the schemes in [5] are conditionally stable only.

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References

[1] M. Chawla, M.A. Al-Zanaidi, D. J. Evans, Generalized trapezoidal formulas for parabolic equations, Int. J. Comput. Math. 70 (1998) 429-443.

[2] U. Dale von Rosenberg, Methods for the Numberical Solution of Partial Differential Equations, American Elsevier Publishing Company, Inc., 1969.

[3] K. George, E.H. Twizell, Stable second-order finite-difference methods for linear initial-boundary-value problems, Appl. Math. Lett. 19 (2006) 146-154.

[4] R. Hiberman, Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, fourthed., Pearson Prentice Hall, New Jersey, 2004.

[5] J. Rashidinia, R. Jalilian, V. Kazemi, Spline methods for the solutions of hyperbolic equations, Appl. Math. Comput. 190 (2007) 882-886.

[6] Tariq Aziz, Arshad Khan, Jalil Rashidinia, Spline methods for the solution of fourth-order parabolic partial differential equations, Appl. Math. Comput. 167 (2005) 153-166.

[7] S. Sallam, M. Naim Anwar, M.R. Abdel-Aziz, Unconditioally stalbe C^1-cubic spline collocation method for solving parabolic equations, Int. J. Comput. Math. 81 (2004) 813-821.

[8] J. Rashidinia, Applications of spline to numerical solution of differential equations, AMU India (1990), M.phil. Dissertation.

[9] J.J.H. Miller, On the location of zeros of certain classes of polynomials with application to numerical analysis, J. Inst. Math. Appls. 8 (1971) 394-406.

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