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Topological Characterization, Measures of Uncertainty and Rough Equality of Sets on Two Universal Sets

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Abstract— The notion of rough set captures indiscernibility of elements in a set. But, in many real life situations, an information system establishes the relation between different universes. This gave the extension of rough set on single universal set to rough set on two universal sets. In this paper, we introduce rough equality of sets on two universal sets and rough inclusion of sets employing the notion of the lower and upper approximation. Also, we establish some basic properties that refer to our knowledge about the universes.

Index Terms— Rough Set, Solitary Set, Boolean Matrix, Rough Equality, Rough Inclusion

I. Introduction

In modern era of computing, there is a need of development in data analysis and knowledge representation. Many new mathematical modeling tools are emerging to the thrust of the real world task. Fuzzy set by Zadeh [1], rough set theory [2, 3], soft set by Molodtsov [4] are such mathematical models gained its popularity in past few decades. Development of these techniques and tools are studied under different domains like knowledge discovery in database, computational intelligence, knowledge engineering, granular computing etc. [5, 6, 7, 8, 9, 10].

The rough set [2, 3] philosophy specifies about the depth understanding of the object and its attributes influencing the object with a depicted value. So, there is a need to classify objects of the universe based on the indiscernibility relation between them. The basic idea of rough set is based upon the approximation of sets by pair of sets known as lower approximation and upper approximation. Here, the lower approximation and upper approximation operators are based on equivalence relation. However, the requirement of equivalence relation is a restrictive condition that may limit the application of rough set model. Therefore, rough set is generalized to some extent. For instance, the equivalence relation is generalized to binary relations [11, 12, 13, 14, 15, 16], neighborhood systems [17], coverings [18], Boolean algebras [19, 20], fuzzy lattices [21], and completely distributive lattices [22].

On the other hand, rough set is generalized to fuzzy environment such as fuzzy rough set [23], and rough fuzzy set [24]. Further, the indiscernibility relation is generalized to almost indiscernibility relation to study many real life problems. The concept of rough set on fuzzy approximation spaces based on fuzzy proximity relation is studied by Acharjya and Tripathy [25, 26]. Further it is generalized to intuitionistic fuzzy proximity relation, and the concept of rough set on intuitionistic fuzzy approximation space is studied by Tripathy [27]. The different applications are also studied by the authors [28, 29, 30]. Further rough set of Pawlak is generalized to rough set on two universal sets with generalized approximation spaces and interval structure [31]. We continue a further study in the same direction.

The rest of the paper is organized as follows: Section 2 presents the foundations of rough set based on two universal sets and its topological characterization. In Section 3, we study the measures of uncertainty due to rough sets on two universal sets. Rough equality of sets on two universal sets and its properties are studied in Section 4. In Section 5, we introduce rough inclusion of sets on two universal sets. This is further followed by a conclusion in Section 6.

II. Rough Set Based on Two Universal Sets

An information system is a table that provides a convenient way to describe a finite set of objects called the universe by a finite set of attributes thereby representing all available information and knowledge. But, in many real life situations, an information system establishes the relation between different universes. This gave the extension of rough set on single universal set to rough set on two universal sets. Wong et. al [31] generalized the rough set models using two distinct but related universal sets. Let U and V be two universal sets and $R \subseteq (U\times V)$ be a binary relation. By a knowledge base, we understand the relational system (U, V, R) an approximation space. For an element $x \in U$, we define the right neighborhood or the R-relative set of x in U,

$r(x)$ as $r(x) = \cup\{y \in V : (x, y) \in R\}$. Similarly for an element $y \in V$, we define the left neighborhood or the R-relative set of y in V, $l(y)$ as $l(y) = \cup\{x \in U : (x, y) \in R\}$.

For any two elements $x_1, x_2 \in U$, we say x_1 and x_2 are equivalent if $r(x_1) = r(x_2)$. Therefore, $(x_1, x_2) \in E_U$ if and only if $r(x_1) = r(x_2)$, where E_U denote the equivalence relation on U. Hence, E_U partitions the universal set U into disjoint subsets. Similarly for any two elements $y_1, y_2 \in V,$ we say y_1 and y_2 are equivalent if $l(y_1) = l(y_2)$. Thus, $(y_1, y_2) \in E_V$ if and only if $l(y_1) = l(y_2)$, where E_V denote the equivalence relation on V and partitions the universal set V into disjoint subsets. Therefore for the approximation space (U, V, R), it is clear that $E_V \circ R = R = R \circ E_U$, where $E_V \circ R$ is the composition of R and E_V.

For any $Y \subseteq V$ and the binary relation R, we associate two subsets $\underline{R}Y$ and $\overline{R}Y$ called the R-lower and R-upper approximations of Y respectively, which are given by:

$$\underline{R}Y = \cup\{x \in U : r(x) \subseteq Y\} \tag{1}$$

$$\overline{R}Y = \cup\{x \in U : r(x) \cap Y \neq \phi\} \tag{2}$$

The R-boundary of Y is denoted as $BN_R(Y)$ and is given as $BN_R(Y) = \overline{R}Y - \underline{R}Y$. The pair $(\underline{R}Y, \overline{R}Y)$ is called as the rough set of $Y \subseteq V$ if $\underline{R}Y \neq \overline{R}Y$ or equivalently $BN_R(Y) \neq \phi$. Further, if U and V are finite sets, then the binary relation R from U to V can be represented as $R(x, y)$, where

$$R(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R \\ 0 & \text{if } (x, y) \notin R \end{cases}$$

The characteristic function of $X \subseteq U$ is defined for each $x \in U$ as follows:

$$X(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

Therefore, the R-lower and R-upper approximations can be also presented in an equivalent form as shown below, where $\wedge$ and $\vee$ denotes the minimum and maximum operators respectively.

$$(\underline{R}Y)x = \underset{y \in V}{\wedge}((1 - R(x, y)) \vee Y(y)) \tag{3}$$

$$(\overline{R}Y)x = \underset{y \in V}{\vee}(R(x, y) \wedge Y(y)) \tag{4}$$

Example 2.1 Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $V = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. Consider the relation R given by its Boolean matrix:

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

From the above relation R it is clear that, $r(x_1) = \{y_1, y_2, y_5\}$; $r(x_2) = \{y_3, y_6\}$; $r(x_3) = \{y_2, y_4\}$; $r(x_4) = \{y_1, y_3, y_4, y_5, y_6\}$ and $r(x_5) = \{y_1, y_2, y_5\}$. Therefore, we get $U / E_U = \{\{x_1, x_5\}, \{x_2\}, \{x_3\}, \{x_4\}\}$. Similarly, $V / E_V = \{\{y_1, y_5\}, \{y_3, y_6\}, \{y_2\}, \{y_4\}\}$ .Let us consider the target set $Y = \{y_1, y_2, y_4, y_5\}$. Therefore, the R-lower approximation, $\underline{R}Y$ is given as $\underline{R}Y = \{x_1, x_3, x_5\}$ whereas the R-upper approximation, $\overline{R}Y$ is given as $\overline{R}Y = \{x_1, x_3, x_4, x_5\}$. The R-boundary of Y is given as $BN_R(Y) = \{x_4\}$.

Definition 2.1 Let U and V be two universal sets. Let R be a binary relation from U to V. If $x \in U$ and $r(x) = \phi$, then we call x is a solitary element with respect to R. The set of all solitary elements with respect to the relation R is called as solitary set and is denoted as S. Mathematically,

$$S = \{x \in U : r(x) = \phi\} \tag{5}$$

2.1 Algebraic Properties of Rough Set based on Two Universal Sets

In this section, we list the algebraic properties as established by Guilong Liu [32] that are interesting and valuable in the theory of rough sets as below. Let R be an arbitrary binary relation from U to V. Let S be a solitary set with respect to the relation R. For subsets X, Y, in V

(*i*) $\overline{R}Y = \underset{y \in Y}{\cup} l(y)$ (6)

(*ii*) $\underline{R}\phi = S, \overline{R}\phi = \phi, \underline{R}V = U$ and $\overline{R}V = S'$, where S' denotes the complement of S in U. (7)

(*iii*) $S \subseteq \underline{R}X$ and $\overline{R}X \subseteq S'$ (8)

(*iv*) $\underline{R}X - S \subseteq \overline{R}X$ (9)

(*v*) $\underline{R}X = U$ if and only if $\underset{x \in U}{\cup} r(x) \subseteq X$; $\overline{R}X = \phi$ if and only if $X \subseteq (\underset{x \in U}{\cup} r(x))'$ (10)

(*vi*) If $S \neq \phi$, then $\underline{R}X \neq \overline{R}X$ for all $X \in P(V)$, where $P(V)$ denotes the power set of *V*. (11)

(*vii*) For any given index set *I*, $X_i \in P(V)$,
$\underline{R}(\bigcap_{i \in I} X_i) = \bigcap_{i \in I} \underline{R}X_i$ and $\overline{R}(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} \overline{R}X_i$. (12)

(*viii*) If $X \subseteq Y$, then $\underline{R}X \subseteq \underline{R}Y$ and $\overline{R}X \subseteq \overline{R}Y$ (13)

(*ix*) $\underline{R}X \cup \underline{R}Y \subseteq \underline{R}(X \cup Y)$, and
$\overline{R}(X \cap Y) \subseteq \overline{R}X \cap \overline{R}Y$ (14)

(*x*) $(\underline{R}X)' = \overline{R}X'$, and $(\overline{R}X)' = \underline{R}X'$; (15)

(*xi*) There exists some $X \in P(U)$ such that $\underline{R}X = \overline{R}X$ if and only if *R* is serial. (16)

(*xii*) If G is another binary relation from *U* to *V* and $\overline{R}X = \overline{G}X$ for all $x \in P(V)$, then $R = G$. (17)

(*xiii*) If G is another binary relation from *U* to *V* and $\underline{R}X = \underline{G}X$ for all $x \in P(V)$, then $R = G$. (18)

2.2 Topological Characterization of Rough Set based on Two Universal Sets

In this section, we introduce an interesting topological characterization of rough set on two universal sets employing the notion of the lower and upper approximation. It results four important and different types of rough sets on two universal sets as discussed by Acharjya and Tripathy [33].

Type 1: If $\underline{R}Y \neq \phi$ and $\overline{R}Y \neq U$, then we say that *Y* is *roughly R-definable* on two universal sets.

Type 2: If $\underline{R}Y = \phi$ and $\overline{R}Y \neq U$, then we say that *Y* is *internally R-undefinable* on two universal sets.

Type 3: If $\underline{R}Y \neq \phi$ and $\overline{R}Y = U$, then we say that *Y* is *externally R-undefinable* on two universal sets.

Type 4: If $\underline{R}Y = \phi$ and $\overline{R}Y = U$, then we say that *Y* is *totally R-undefinable* on two universal sets.

2.3 Table of Union

In this section, we discuss the set theoretic operations such as union on types of rough sets on two universal sets. We state the corresponding tables for the set theoretic operation union. From the Table 1, it is clear that seven cases consist of ambiguous. In one case it can be any one of the four types. These ambiguities are due to inclusion $\underline{R}X \cup \underline{R}Y \subseteq \underline{R}(X \cup Y)$. The necessary proofs of ambiguity cases are thoroughly studied by Acharjya and Tripathy [33].

Table 1: Table of union

$\bigcup$	**Type 1**	**Type 2**	**Type 3**	**Type 4**
Type 1	Type 1 / Type 3	Type 1 / Type 3	Type 3	Type 3
Type 2	Type 1 / Type 3	Type 1 / Type 2 / Type 3 / Type 4	Type 3	Type 3 / Type 4
Type 3	Type 3	Type 3	Type 3	Type 3
Type 4	Type 3	Type 3 / Type 4	Type 3	Type 3 / Type 4

2.4 Table of Intersection

Unlike union operation, it is interesting to see from the Table 2 that, out of sixteen cases for intersection, seven cases are ambiguous. Also it is observed that, in one case it can be any one of the four types. These ambiguities are due to inclusion $\overline{R}(X \cap Y) \subseteq \overline{R}X \cap \overline{R}Y$. The necessary proofs of ambiguity cases are thoroughly studied by Acharjya and Tripathy [33].

Table 2: Table of intersection

$\bigcap$	**Type 1**	**Type 2**	**Type 3**	**Type 4**
Type 1	Type 1 / Type 2	Type 2	Type 1 / Type 2	Type 2
Type 2	Type 2	Type 2	Type 2	Type 2
Type 3	Type 1 / Type 2	Type 2	Type 1 / Type 2 / Type 3 / Type 4	Type 2 / Type 4
Type 4	Type 2	Type 2	Type 2 / Type 4	Type 2 / Type 4

III. Measures of Uncertainty

The rough set [2, 3] philosophy specifies about the understanding of the objects and their attributes influencing the objects with a depicted value. So, there is a need to classify objects of the universe based on the indiscernibility relation between them. The basic idea of rough set is based upon the approximation of sets by a pair of sets known as the lower approximation and upper approximation of the set. Here, the lower and upper approximation operators are based on equivalence relation. However, the requirement of equivalence relation is a restrictive condition that may limit the application of rough set model. Therefore, rough set is generalized by Guilong Liu [32] to rough set on two universal sets. Because we are interested in classifications based on binary relation, it is interesting to have the idea of approximation of classifications. It is because classifications of universes play central roles in rough set theory. Recently Acharjya and Tripathy [34] have established important results and measures of uncertainty such as accuracy and quality of approximation employing the binary relation R and discuss on properties of classifications. However, for completeness of the paper, we state the basic definitions and notions of measures of uncertainty.

Definition 3.1 Let $F=\{Y_1, Y_2, \cdots, Y_n\}$, where $n>1$ be a family of non empty sets defined over V. We say that F is a classification of V if and only if $(Y_i \cap Y_j)=\phi$ for $i \neq j$ and $\sum_{k=1}^{n} Y_k = V$.

Definition 3.2 Let $F=\{Y_1, Y_2, \cdots, Y_n\}$ be a family of non empty classification of V and let R be a binary relation from $U \to V$. Then the R-lower and R-upper approximation of the family F is given as $\underline{R}F=\{\underline{R}Y_1, \underline{R}Y_2, \cdots, \underline{R}Y_n\}$ and $\overline{R}F=\{\overline{R}Y_1, \overline{R}Y_2, \cdots, \overline{R}Y_n\}$ respectively.

Definition 3.3 The accuracy of approximation of F that expresses the percentage of possible correct decisions when classifying objects employing the binary relation R is defined as

$$\alpha_R(F)=\frac{\sum card\ (\underline{R}Y_i)}{\sum card\ (\overline{R}Y_i)} \quad \text{for } i=1, 2, 3, \cdots, n.$$

Definition 3.4 The quality of approximation of F that expresses the percentage of objects which can be correctly classified to classes of F by the binary relation R is defined as

$$\nu_R(F)=\frac{\sum card\ (\underline{R}Y_i)}{card\ (V)} \quad \text{for } i=1, 2, 3, \cdots, n$$

Definition 3.5 We say that $F=\{Y_1, Y_2, \cdots, Y_n\}$ is R-definable if and only if $\underline{R}F=\overline{R}F$; that is $\underline{R}Y_i=\overline{R}Y_i$ for $i=1, 2, 3, \cdots, n$.

Theorem 3.1 Let R be a binary relation from $U \to V$ and let $F=\{Y_1, Y_2, \cdots, Y_n\}$, where $n>1$ be a classification of V. For any R-definable classification F in U, $\alpha_R(F)=\nu_R(F)=1$. Hence, if a classification F is R-definable then it is totally independent on R.

Theorem 3.2 Let R be a binary relation from $U \to V$ and let $F=\{Y_1, Y_2, \cdots, Y_n\}$, where $n>1$ be a classification of V. If $\alpha_R(F)=\nu_R(F)=1$, then F is R-definable in V.

Theorem 3.3 Let R be a binary relation from $U \to V$ and for any classification $F=\{Y_1, Y_2, \cdots, Y_n\}$, $n>1$ in V, $0 \le \alpha_R(F) \le \nu_R(F) \le 1$.

IV. Rough Equality of Sets on Two Universal Sets

The concept of rough set differs essentially from the ordinary concept of the set in that for the rough sets we are unable to define uniquely the membership relation. In set theory, two sets are said to be equal if they have same elements. However it is not true in case of rough sets. Therefore, the concept of rough (approximate) equality is introduced by Novotny and Pawlak [35]. Thus two sets can be unequal in set theory, but can be approximately equal. This is an important feature and according to our state of knowledge, the sets have close features which are enough to be assumed approximately equal. This is due to the indiscernibility relation between the objects of the universe. But, the indiscernibility relation is a restrictive relation that may limit the application of rough set. Therefore, rough set has extended to the settings of rough set on two universal sets based on binary relation. Hence the above concept of rough equality of sets can be extended to the settings of rough equality of sets on two universal sets. In fact we introduce three kinds of rough equality of sets on two universal sets. Now we present the formal definitions.

Definition 4.1 Let U and V be two universal sets and $R \subseteq (U \times V)$ be a binary relation. Let the relational system (U, V, R) be a knowledge base, and $Y_1, Y_2 \subseteq V$. We say that

(i) Sets Y_1 and Y_2 are bottom R-equal in V if $\underline{R}Y_1=\underline{R}Y_2$. We write it as $Y_1 \approx_B Y_2$.

(ii) Sets Y_1 and Y_2 are top R-equal in V if $\overline{R}Y_1=\overline{R}Y_2$. We write it as $Y_1 \approx_T Y_2$.

(iii) Sets Y_1 and Y_2 are R-equal in V if $Y_1 \approx_B Y_2$ and $Y_1 \approx_T Y_2$. We write it as $Y_1 \approx Y_2$.

We associate the following physical interpretations, with the above notion of rough equality of sets on two universal sets. If $Y_1 \approx_B Y_2$, this means that positive examples of the sets Y_1 and Y_2 in V are equal. If $Y_1 \approx_T Y_2$, then the negative examples of the sets Y_1 and Y_2 in V are equal. If $Y_1 \approx Y_2$, this means that both positive and negative examples of the sets Y_1 and Y_2 in V are the same.

Example 4.1 Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $V = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}$. Consider the binary relation R as $R = \{(x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_4), (x_2, y_5), (x_3, y_3), (x_3, y_6), (x_3, y_7), (x_4, y_1), (x_4, y_7), (x_4, y_8), (x_5, y_2), (x_5, y_3)\}$. Thus R can be written in its Boolean matrix form as:

$$R = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From the above relation R it is clear that, $r(x_1) = \{y_2, y_3\}$; $r(x_2) = \{y_1, y_4, y_5\}$; $r(x_3) = \{y_3, y_6, y_7\}$; $r(x_4) = \{y_1, y_4, y_5\}$; and $r(x_5) = \{y_2, y_3\}$. Therefore, we get $U \mid E_U = \{\{x_1, x_5\}, \{x_2\}, \{x_3\}, \{x_4\}\}$.

For sets $Y_1 = \{y_1, y_2, y_3\}$ and $Y_2 = \{y_2, y_3, y_7\}$ we have $\underline{R}Y_1 = \{x_1, x_5\} = \underline{R}Y_2$, therefore $Y_1 \approx_B Y_2$. Hence Y_1 and Y_2 are bottom R-equal in V. Again, on considering $Y_1 = \{y_1, y_2, y_7\}$ and $Y_2 = \{y_2, y_3, y_4, y_8\}$ we have $\overline{R}Y_1 = \{x_1, x_2, x_3, x_4, x_5\} = \overline{R}Y_2$, therefore $Y_1 \approx_T Y_2$. Hence Y_1 and Y_2 are top R-equal in V. Similarly on taking $Y_1 = \{y_2, y_4, y_6\}$ and $Y_2 = \{y_3, y_4, y_6\}$ we have $\underline{R}Y_1 = \phi = \underline{R}Y_2$, and $\overline{R}Y_1 = \{x_1, x_2, x_3, x_5\} = \overline{R}Y_2$, therefore $Y_1 \approx Y_2$. Hence Y_1 and Y_2 are R-equal in V.

Proposition 4.1 The following properties of relations $\approx_B$, $\approx_T$, and $\approx$ are immediate consequences of the definitions. Let U and V be two universal sets and $R \subseteq (U \times V)$ be a binary relation. Then for $Y_1, Y_2 \subseteq V$, the following properties holds.

(a) $Y_1 \approx_B Y_2$ if and only if $(Y_1 \cap Y_2) \approx_B Y_1$ and $(Y_1 \cap Y_2) \approx_B Y_2$.

(b) $Y_1 \approx_T Y_2$ if and only if $(Y_1 \cup Y_2) \approx_T Y_1$ and $(Y_1 \cup Y_2) \approx_T Y_2$.

(c) If $Y_1 \approx_T Y_1'$ and $Y_2 \approx_T Y_2'$, then $(Y_1 \cup Y_2) \approx_T (Y_1' \cup Y_2')$.

(d) If $Y_1 \approx_B Y_1'$ and $Y_2 \approx_B Y_2'$, then $(Y_1 \cap Y_2) \approx_B (Y_1' \cap Y_2')$.

(e) If $Y_1 \subseteq Y_2$ and $Y_2 \approx_T \phi$, then $Y_1 \approx_T \phi$.

(f) If $Y_1 \subseteq Y_2$ and $Y_1 \approx_T V$, then $Y_2 \approx_T V$.

(g) If $Y_1 \approx_B \phi$ or $Y_2 \approx_B \phi$, then $(Y_1 \cap Y_2) \approx_B \phi$.

(h) If $Y_1 \approx_T V$ or $Y_2 \approx_T V$, then $(Y_1 \cup Y_2) \approx_T V$.

Proof (*a*) Assume that $(Y_1 \cap Y_2) \approx_B Y_1$ and $(Y_1 \cap Y_2) \approx_B Y_2$. It implies that $\underline{R}(Y_1 \cap Y_2) = \underline{R}(Y_1)$ and $\underline{R}(Y_1 \cap Y_2) = \underline{R}(Y_2)$. Therefore, $\underline{R}(Y_1) = \underline{R}(Y_2)$ and hence $Y_1 \approx_B Y_2$. Conversely assume that $Y_1 \approx_B Y_2$. It implies that $\underline{R}(Y_1) = \underline{R}(Y_2)$. But, $\underline{R}(Y_1 \cap Y_2) = \underline{R}(Y_1) \cap \underline{R}(Y_2) = \underline{R}(Y_1)$ and $\underline{R}(Y_1 \cap Y_2) = \underline{R}(Y_1) \cap \underline{R}(Y_2) = \underline{R}(Y_2)$. Therefore, we have $(Y_1 \cap Y_2) \approx_B Y_1$ and $(Y_1 \cap Y_2) \approx_B Y_2$.

Proof (*b*) Assume that $(Y_1 \cup Y_2) \approx_T Y_1$ and $(Y_1 \cup Y_2) \approx_T Y_2$. It implies that $\overline{R}(Y_1 \cup Y_2) = \overline{R}(Y_1)$ and $\overline{R}(Y_1 \cup Y_2) = \overline{R}(Y_2)$. Therefore, $\overline{R}(Y_1) = \overline{R}(Y_2)$ and hence $Y_1 \approx_T Y_2$. Conversely assume that $Y_1 \approx_T Y_2$. It implies that $\overline{R}(Y_1) = \overline{R}(Y_2)$. But, $\overline{R}(Y_1 \cup Y_2) = \overline{R}(Y_1) \cup \overline{R}(Y_2) = \overline{R}(Y_1)$ and $\overline{R}(Y_1 \cup Y_2) = \overline{R}(Y_1) \cup \overline{R}(Y_2) = \overline{R}(Y_2)$. Therefore, we have $(Y_1 \cup Y_2) \approx_T Y_1$ and $(Y_1 \cup Y_2) \approx_T Y_2$.

Proof (*c*) Assume that $Y_1 \approx_T Y_1'$ and $Y_2 \approx_T Y_2'$. It implies that $\overline{R}Y_1 = \overline{R}Y_1'$ and $\overline{R}Y_2 = \overline{R}Y_2'$. But, $\overline{R}(Y_1 \cup Y_2) = \overline{R}Y_1 \cup \overline{R}Y_2 = \overline{R}Y_1' \cup \overline{R}Y_2' = \overline{R}(Y_1' \cup Y_2')$. Therefore, $(Y_1 \cup Y_2) \approx_T (Y_1' \cup Y_2')$.

Proof (*d*) Assume that $Y_1 \approx_B Y_1'$ and $Y_2 \approx_B Y_2'$. Thus we have $\underline{R}Y_1 = \underline{R}Y_1'$ and $\underline{R}Y_2 = \underline{R}Y_2'$. But, $\underline{R}(Y_1 \cap Y_2) = \underline{R}Y_1 \cap \underline{R}Y_2 = \underline{R}Y_1' \cap \underline{R}Y_2' = \underline{R}(Y_1' \cap Y_2')$. Therefore, $(Y_1 \cap Y_2) \approx_B (Y_1' \cap Y_2')$.

Proof (*e*) Assume that $Y_1 \subseteq Y_2$ and $Y_2 \approx_T \phi$. It implies that $\overline{R}Y_1 \subseteq \overline{R}Y_2$ and $\overline{R}Y_2 = \overline{R}\phi$. Therefore, $\overline{R}Y_1 \subseteq \overline{R}Y_2 =$

$\overline{R}\phi = \phi$, i.e., $\overline{R}Y_1 \subseteq \phi$. So definitely, $\overline{R}Y_1 = \phi = \overline{R}\phi$. Hence, $Y_1 \approx_T \phi$.

Proof (*f*) Assume that $Y_1 \subseteq Y_2$ and $Y_1 \approx_T V$. It implies that $\overline{R}Y_1 \subseteq \overline{R}Y_2$ and $\overline{R}Y_1 = \overline{R}V = S'$. But $\overline{R}Y_2 \supseteq \overline{R}Y_1 = S'$, i.e., $\overline{R}Y_2 \supseteq S'$. Since S' is the complement of the solitary set, definitely $\overline{R}Y_2 = S'$. Therefore, $\overline{R}Y_2 = S' = \overline{R}V$. It implies that $Y_2 \approx_T V$.

Proof (*g*) Let us assume $Y_1 \approx_B \phi$. It implies that $\underline{R}Y_1 = \underline{R}\phi = S$. Therefore, $\underline{R}(Y_1 \cap Y_2) = \underline{R}Y_1 \cap \underline{R}Y_2 = S \cap \underline{R}Y_2$. Since *S* is a solitary set, definitely $S \cap \underline{R}Y_2$ is equal to *S*. Thus, $\underline{R}(Y_1 \cap Y_2) = S = \underline{R}\phi$, i.e., $\underline{R}(Y_1 \cap Y_2) = \underline{R}\phi$. Hence, $(Y_1 \cap Y_2) \approx_B \phi$. Similarly it can be verified that if $Y_2 \approx_B \phi$, then $(Y_1 \cap Y_2) \approx_B \phi$.

Proof (*h*) Let us assume $Y_1 \approx_T V$. It implies that $\overline{R}Y_1 = \overline{R}V = S'$. But, $\overline{R}(Y_1 \cup Y_2) = \overline{R}Y_1 \cup \overline{R}Y_2 = S' \cup \overline{R}Y_2 = S' = \overline{R}V$. It indicates that, $(Y_1 \cup Y_2) \approx_T V$. Similarly it can be verified that if $Y_2 \approx_T V$, then $(Y_1 \cup Y_2) \approx_T V$.

V. Rough Inclusion of Sets on Two Universal Sets

Inclusion relation is one of the fundamental concepts in set theory. An analogous notion in rough set is introduced by Pawlak [2]. Hence, rough inclusion of sets can be extended to the settings of rough inclusion of sets on two universal sets. We define the rough inclusion of sets on two universal sets in the same way as rough equality of sets on two universal sets. The formal definition of rough inclusion of sets on two universal sets is as follows.

Definition 5.1 Let *U* and *V* be two universal sets and $R \subseteq (U \times V)$ be a binary relation. Let the relational system (*U*, *V*, *R*) be a knowledge base, and $Y_1, Y_2 \subseteq V$. We say that

(i) Set Y_1 is bottom *R*-included in Y_2 if and only if $\underline{R}Y_1 \subseteq \underline{R}Y_2$. We denote it as $Y_1 \square_B Y_2$.

(ii) Set Y_1 is top *R*-included in Y_2 if and only if $\overline{R}Y_1 \subseteq \overline{R}Y_2$. We denote it as $Y_1 \square_T Y_2$.

(iii) Set Y_1 is said to be *R*-included in Y_2 if and only if $Y_1 \square_B Y_2$ and $Y_1 \square_T Y_2$. We denote it as $Y_1 \square Y_2$.

Example 5.1 Let us consider the knowledge base as in Example 4.1. In this knowledge base for sets $Y_1 = \{y_2, y_3, y_4\}$ and $Y_2 = \{y_2, y_3, y_6, y_7\}$ we have $\underline{R}Y_1 = \{x_1, x_5\}$ and $\underline{R}Y_2 = \{x_1, x_3, x_5\}$. Therefore, $\underline{R}Y_1 \subseteq \underline{R}Y_2$. It implies that Y_1 is bottom *R*-included in the set Y_2. Again on taking $Y_1 = \{y_2, y_3, y_6\}$ and $Y_2 = \{y_2, y_7\}$ we have $\overline{R}Y_1 = \{x_1, x_3, x_5\}$ and $\overline{R}Y_2 = \{x_1, x_3, x_4, x_5\}$. Thus, we get $\overline{R}Y_1 \subseteq \overline{R}Y_2$. It indicates that Y_1 is top *R*-included in the set Y_2. Similarly on taking $Y_1 = \{y_2, y_3\}$ and $Y_2 = \{y_2, y_3, y_6, y_7\}$ we have $\underline{R}Y_1 = \{x_1, x_5\}$; $\underline{R}Y_2 = \{x_1, x_3, x_5\}$; $\overline{R}Y_1 = \{x_1, x_5\}$ and $\overline{R}Y_2 = \{x_1, x_3, x_4, x_5\}$. Therefore, we get $\underline{R}Y_1 \subseteq \underline{R}Y_2$ and $\overline{R}Y_1 \subseteq \overline{R}Y_2$. It indicates that Y_1 is *R*-included in the set Y_2.

Proposition 5.1 The following properties of relations $\square_B$, $\square_T$, and $\square$ are immediate consequences of the definitions. Let *U* and *V* be two universal sets and $R \subseteq (U \times V)$ be a binary relation. Then for $Y_1, Y_2 \subseteq V$, the following properties holds.

(i) If $Y_1 \subseteq Y_2$, then $Y_1 \square_B Y_2$, $Y_1 \square_T Y_2$, and $Y_1 \square Y_2$.

(j) $Y_1 \square_B Y_2$ and $Y_2 \square_B Y_1$, then $Y_1 \approx_B Y_2$.

(k) $Y_1 \square_T Y_2$ and $Y_2 \square_T Y_1$, then $Y_1 \approx_T Y_2$.

(l) If $Y_1 \square Y_2$ and $Y_2 \square Y_1$, then $Y_1 \approx Y_2$.

(m) $Y_1 \square_T Y_2$ if and only if $(Y_1 \cup Y_2) \approx_T Y_2$.

(n) $Y_1 \square_B Y_2$ if and only if $(Y_1 \cap Y_2) \approx_B Y_1$.

(o) If $Y_1 \subseteq Y_2$, $Y_1 \approx_B Y_1'$ and $Y_2 \approx_B Y_2'$, then $Y_1' \square_B Y_2'$.

(p) If $Y_1 \subseteq Y_2$, $Y_1 \approx_T Y_1'$ and $Y_2 \approx_T Y_2'$, then $Y_1' \square_T Y_2'$.

(q) If $Y_1 \subseteq Y_2$, $Y_1 \approx Y_1'$ and $Y_2 \approx Y_2'$, then $Y_1' \square Y_2'$.

(r) If $Y_1' \square_T Y_1$ and $Y_2' \square_T Y_2$, then $(Y_1' \cup Y_2') \square_T (Y_1 \cup Y_2)$.

(s) If $Y_1' \square_B Y_1$ and $Y_2' \square_B Y_2$, then $(Y_1' \cap Y_2') \square_B (Y_1 \cap Y_2)$.

(t) $(Y_1 \cap Y_2) \square_B Y_1 \square_T (Y_1 \cup Y_2)$.

(u) If $Y_1 \square_B Y_2$ and $Y_1 \approx_B Y_3$, then $Y_3 \square_B Y_2$.

(v) If $Y_1 \Box_T Y_2$ and $Y_1 \approx_T Y_3$, then $Y_3 \Box_T Y_2$.

(w) If $Y_1 \Box Y_2$ and $Y_1 \approx Y_3$, then $Y_3 \Box Y_2$.

Proof (*a*) Assume that $Y_1 \subseteq Y_2$. It implies that $\underline{R}Y_1 \subseteq \underline{R}Y_2$ and $\overline{R}Y_1 \subseteq \overline{R}Y_2$. Therefore, $Y_1 \Box_B Y_2$; $Y_1 \Box_T Y_2$ and thus $Y_1 \Box Y_2$.

Proof (*b*) Suppose that $Y_1 \Box_B Y_2$ and $Y_2 \Box_B Y_1$. It implies that $\underline{R}Y_1 \subseteq \underline{R}Y_2$ and $\underline{R}Y_2 \subseteq \underline{R}Y_1$. Therefore, $\underline{R}Y_1 = \underline{R}Y_2$. It indicates that $Y_1 \approx_B Y_2$.

Proof (*c*) Suppose that $Y_1 \Box_T Y_2$ and $Y_2 \Box_T Y_1$. It implies that $\overline{R}Y_1 \subseteq \overline{R}Y_2$ and $\overline{R}Y_2 \subseteq \overline{R}Y_1$. Therefore, $\overline{R}Y_1 = \overline{R}Y_2$. It indicates that $Y_1 \approx_T Y_2$.

Proof (*d*) Suppose that $Y_1 \Box Y_2$ and $Y_2 \Box Y_1$. It implies that $Y_1 \Box_B Y_2$; $Y_1 \Box_T Y_2$; $Y_2 \Box_B Y_1$ and $Y_2 \Box_T Y_1$. Therefore, by above property (*b*) and (*c*) we get $Y_1 \approx_B Y_2$ and $Y_1 \approx_T Y_2$. Hence, we have $Y_1 \approx Y_2$.

Proof (*e*) Assume that $Y_1 \Box_T Y_2$. It implies that $\overline{R}Y_1 \subseteq \overline{R}Y_2$. But, $\overline{R}(Y_1 \cup Y_2) = \overline{R}Y_1 \cup \overline{R}Y_2 = \overline{R}Y_2$. Thus we get, $(Y_1 \cup Y_2) \approx_T Y_2$. Conversely, suppose that $(Y_1 \cup Y_2) \approx_T Y_2$. It implies that $\overline{R}(Y_1 \cup Y_2) = \overline{R}Y_2$, i.e., $\overline{R}Y_1 \cup \overline{R}Y_2 = \overline{R}Y_2$. It is possible only when $\overline{R}Y_1 \subseteq \overline{R}Y_2$ or $\overline{R}Y_1 = \phi$. Therefore, in either case $Y_1 \Box_T Y_2$.

Proof (*f*) Assume that $Y_1 \Box_B Y_2$. It implies that $\underline{R}Y_1 \subseteq \underline{R}Y_2$. But, $\underline{R}(Y_1 \cap Y_2) = \underline{R}Y_1 \cap \underline{R}Y_2 = \underline{R}Y_1$. Thus we get, $(Y_1 \cap Y_2) \approx_B Y_1$. Conversely, suppose that $(Y_1 \cap Y_2) \approx_B Y_1$. It implies that $\underline{R}(Y_1 \cap Y_2) = \underline{R}Y_1$, i.e., $\underline{R}Y_1 \cap \underline{R}Y_2 = \underline{R}Y_1$. It is possible only when $\underline{R}Y_1 \subseteq \underline{R}Y_2$. Therefore, in either case $Y_1 \Box_B Y_2$.

Proof (*g*) Assume that $Y_1 \subseteq Y_2$; $Y_1 \approx_B Y_1'$ and $Y_2 \approx_B Y_2'$. It implies that $\underline{R}Y_1 \subseteq \underline{R}Y_2$; $\underline{R}Y_1 = \underline{R}Y_1'$ and $\underline{R}Y_2 = \underline{R}Y_2'$. Therefore, $\underline{R}Y_1' \subseteq \underline{R}Y_2'$. It indicates that $Y_1' \Box_B Y_2'$.

Proof (*h*) Assume that $Y_1 \subseteq Y_2$; $Y_1 \approx_T Y_1'$ and $Y_2 \approx_T Y_2'$. It implies that $\overline{R}Y_1 \subseteq \overline{R}Y_2$; $\overline{R}Y_1 = \overline{R}Y_1'$ and $\overline{R}Y_2 = \overline{R}Y_2'$. Therefore, $\overline{R}Y_1' \subseteq \overline{R}Y_2'$. It indicates that $Y_1' \Box_T Y_2'$.

Proof (*i*) Assume that $Y_1 \subseteq Y_2$; $Y_1 \approx Y_1'$ and $Y_2 \approx Y_2'$. It implies that $\overline{R}Y_1 \subseteq \overline{R}Y_2$; $\underline{R}Y_1 \subseteq \underline{R}Y_2$; $\underline{R}Y_1 = \underline{R}Y_1'$; $\overline{R}Y_1 = \overline{R}Y_1'$; $\underline{R}Y_2 = \underline{R}Y_2'$ and $\overline{R}Y_2 = \overline{R}Y_2'$. Therefore, we have $\underline{R}Y_1' = \underline{R}Y_1 \subseteq \underline{R}Y_2 = \underline{R}Y_2'$, i.e., $\underline{R}Y_1' \subseteq \underline{R}Y_2'$. It indicates that $Y_1' \Box_B Y_2'$. Similarly, $\overline{R}Y_1' = \overline{R}Y_1 \subseteq \overline{R}Y_2 = \overline{R}Y_2'$, i.e., $\overline{R}Y_1' \subseteq \overline{R}Y_2'$. It indicates that $Y_1' \Box_T Y_2'$. Thus on combining the results we get $Y_1' \Box Y_2'$.

Proof (*j*) Suppose that $Y_1' \Box_T Y_1$ and $Y_2' \Box_T Y_2$. It implies that $\overline{R}Y_1' \subseteq \overline{R}Y_1$ and $\overline{R}Y_2' \subseteq \overline{R}Y_2$. But, $\overline{R}(Y_1' \cup Y_2') = \overline{R}Y_1' \cup \overline{R}Y_2' \subseteq \overline{R}Y_1 \cup \overline{R}Y_2 = \overline{R}(Y_1 \cup Y_2)$. Thus, $\overline{R}(Y_1' \cup Y_2') \subseteq \overline{R}(Y_1 \cup Y_2)$ and hence $(Y_1' \cup Y_2') \Box_T (Y_1 \cup Y_2)$.

Proof (*k*) Suppose that $Y_1' \Box_B Y_1$ and $Y_2' \Box_B Y_2$. It implies that $\underline{R}Y_1' \subseteq \underline{R}Y_1$ and $\underline{R}Y_2' \subseteq \underline{R}Y_2$. But, $\underline{R}(Y_1' \cap Y_2') = \underline{R}Y_1' \cap \underline{R}Y_2' \subseteq \underline{R}Y_1 \cap \underline{R}Y_2 = \underline{R}(Y_1 \cap Y_2)$. Thus, $\underline{R}(Y_1' \cap Y_2') \subseteq \underline{R}(Y_1 \cap Y_2)$ and hence $(Y_1' \cap Y_2') \Box_B (Y_1 \cap Y_2)$.

Proof (*l*) We know that $(Y_1 \cap Y_2) \subseteq Y_1$. Therefore, $\underline{R}(Y_1 \cap Y_2) \subseteq \underline{R}Y_1$. It indicates that $(Y_1 \cap Y_2) \Box_B Y_1$. Similarly, $Y_1 \subseteq (Y_1 \cup Y_2)$. Therefore, $\overline{R}Y_1 \subseteq \overline{R}(Y_1 \cup Y_2)$ and hence $Y_1 \Box_T (Y_1 \cup Y_2)$. On combining the results we get $(Y_1 \cap Y_2) \Box_B Y_1 \Box_T (Y_1 \cup Y_2)$.

Proof (*m*) Suppose that $Y_1 \Box_B Y_2$ and $Y_1 \approx_B Y_3$. It implies that $\underline{R}Y_1 \subseteq \underline{R}Y_2$ and $\underline{R}Y_1 = \underline{R}Y_3$. Hence, we have $\underline{R}Y_3 = \underline{R}Y_1 \subseteq \underline{R}Y_2$, i.e., $\underline{R}Y_3 \subseteq \underline{R}Y_2$. It indicates that $Y_3 \Box_B Y_2$.

Proof (*n*) Suppose that $Y_1 \Box_T Y_2$ and $Y_1 \approx_T Y_3$. It implies that $\overline{R}Y_1 \subseteq \overline{R}Y_2$ and $\overline{R}Y_1 = \overline{R}Y_3$. Hence, we have $\overline{R}Y_3 = \overline{R}Y_1 \subseteq \overline{R}Y_2$, i.e., $\overline{R}Y_3 \subseteq \overline{R}Y_2$. It indicates that $Y_3 \Box_T Y_2$.

Proof (*o*) Suppose that $Y_1 \Box Y_2$ and $Y_1 \approx_T Y_3$. It implies that $Y_1 \Box_B Y_2$; $Y_1 \Box_T Y_2$; $Y_1 \approx_B Y_3$ and $Y_1 \approx_T Y_3$. Therefore, by property (*m*) and (*n*)

VI. Conclusion

In this paper, we compare the concepts of classical set with that of rough set on two universal sets. Basic properties of rough set on two universal sets like topological characteristics, measures of uncertainty, equality and inclusion of sets are expressed in terms of binary relation. It is also observed that equality and inclusion of sets can not be decided in the absolute sense, but depend on what we know about the sets. It is also clear that, all properties of rough set on two universal sets are not absolute, but are related to what we know about them. Therefore, rough set on two universal set approach could be viewed as a subjective counterpart of the classical set theory. These results obtained are important for their application in the design of knowledge bases.

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