A Partial Backlogging Inventory Model with Time-Varying Demand During Shortage Period

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Abstract—Harris’s classic square root economic order quantity (EOQ) model forms the basis for many other models that relax one or more of its assumptions. A key assumption of the basic EOQ model is that stockouts are not permitted. Due to the excess demands, stock-out situations may arise occasionally. Sometimes, shortages are permitted and they are backordered and satisfied in the very next replenishment. Therefore the objective of this paper is to develop a partial backlogging inventory model, and proposes a new algorithm to minimize the total cost, at the same time also propose the prediction method and algorithm of ordering period. Finally, a practical example of the numerical analysis is given.

Index Terms — EOQ model, partial backlogging, shortage

I. INTRODUCTION

The two basic questions any inventory control system must answer are when and how much to order. Over the years, hundreds of papers and books have been published presenting models for doing this under various conditions and assumptions. The best known of these models is Harris’s classic square root economic order quantity (EOQ) model that appears in every basic textbook covering inventory management. While this model has been criticized for the unreasonableness of its assumptions, surveys have shown that it is widely used. Further, it forms the basis for many other models that relax one or more of its assumptions. See Jaggieta and Michenzi [3] for examples of extensions to the classic EOQ model that relax one or more of its traditional assumptions.

A key assumption of the basic EOQ model is that stockouts are not permitted. Assuming that the lead time and demand are known and constant, this means that an order will be placed when the inventory available is exactly sufficient to cover the demand during that lead time. Under conditions of demand certainty, however, it is possible to prove that, assuming customers are always willing, although not necessarily happy, to wait for delivery, planned backorders can make economic sense, even if they incur some actual or implied cost. Relaxing the basic EOQ that stockouts are not permitted led to the development of EOQ models with partial backordering, most of them incorporate considerable more complicated assumption sets than the classic EOQ model do. Therefore the objective of this paper is to develop a partial backlogging inventory model, and proposes a new algorithm to minimize the total cost, and finds the possible research direction of inventory control problems in the future.

II. LITERATURE REVIEW

Relaxation of the basic assumption that stockouts are not permitted led to the development of EOQ models for the two basic cases that occur when inventory is stocked out: backorders and lost sales. What took longer to develop was a model that recognized that, while some customers are willing to wait for delivery, others are not. Either these customers will cancel their orders or the supplier will have to fill them within the normal delivery time by using more expensive supply methods.

While there have been a number of models developed for the EOQ model with partial backordering, most of them incorporate considerably more complicated assumption sets than the classic EOQ model do. Therefore the objective of this paper is to develop a partial backlogging inventory model, and proposes a new algorithm to minimize the total cost, and finds the possible research direction of inventory control problems in the future.
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III. MODEL ASSUMPTIONS

In order to simplify the problem, the reverse logistics inventory model must satisfy the following assumptions:

1. Inventory system with the only species;
2. During the plan period \([0, H]\), the purchase costs are same and fixed, namely the cost is nothing to do with the order frequency and quantity;
3. The demand rate is a known determinate strictly monotonic function;
4. The initial time of shortage \(S_0 = 0\), its inventory level must be zero;
5. Lead time is zero, and replenishment rate is infinity;
6. There is no degradation in product;
7. Shortage is allowed and happen in the beginning of any period;
8. Consider the case of partial backlogging.

IV. MODEL DERIVED

Concerning the allowed shortage inventory model, we assume that inventory level must be zero when \(S_0 = 0\), and the ordering behavior appears in time \(t_i < s_i\). Once ordering, products will be immediately sent. In addition, during the plan period, the shortage time in last shortage period also meet accurately to the plan period \((S_n = H)\). The relationship between inventory levels and time is shown in Fig.1 in the allowed shortage inventory system.

![Figure 1. the relationship between inventory levels and time](image)

In any shortage period \(ST_i(= [s_{i-1}, t_i])\), backorder rate is defined as \(1/1 + \alpha(t_i - t)\) and \(s_{i-1} \leq t < t_i\), where \(\alpha\) is expressed as backlogging parameter. Backorder rate means that the longer wait time \((t_i - t)\), the fewer consumers will wait. Thus, in the \(i\)th shortage period, the time variable of the inventory level considering the partial backlogging can be expressed as

\[
\frac{dQ(t)}{du} = -\frac{f(u)}{1 + \alpha(t_i - u)}, \quad s_{i-1} \leq u \leq t_i
\]

where \(f(t)\) is expressed as instantaneous demand rate at time \(t\). Since the boundary condition satisfies permanently for \(Q(s_{i-1}) = 0\) in any period \(i = 1, 2, ..., n\), the shortage quantity is defined by (1) as
According to the \( i \)th shortage period, the shortage cost can be defined as

\[
SC_i(s_{i-1}) = c_s \int_{s_{i-1}}^{s_i} \frac{(t_i - u)}{1 + \alpha(t_i - u)} f(u)du
\]

(3)

where \( c_s \) is expressed as unit shortage cost per unit time.

As this inventory model considers partial backlogging, therefore once shortage takes place, the loss cost of sale is defined as

\[
RQ_i(u) = \int_u^{s_i} f(t)dt, \quad t_i \leq u \leq s_i
\]

(6)

\[
\frac{dQ_i(u)}{du} = -f(u), \quad t_i \leq u \leq s_i
\]

(5)

Since the boundary condition satisfies permanently for \( Q(s_i) = 0 \) in any period \( i = 1, 2, ..., n \), the shortage quantity is defined by (5) as

\[
RQ_i(u) = \int_{s_i}^{u} f(t)dt, \quad t_i \leq u \leq s_i
\]

(6)

\[
\frac{dQ_i(u)}{du} = -f(u), \quad t_i \leq u \leq s_i
\]

(5)

**Lemma 1** Suppose the demand rate \( f(t) \) is a strictly increasing function, period time will reduce with the ordering frequency \( i \) increasing, however the ordering quantity \( RQ_i(t_i) \) will increase as ordering frequency \( i \) increases.

**Prove** We apply mean value theorem and assume that \( f(t) \) is a strictly increasing function, while the ordering quantity is

\[
RQ_i(t_i) = \int_{t_i}^{s_i} f(u)du = (s_i - t_i) f(s_i) < (s_{i+1} - t_{i+1}) f(\theta_i)
\]

\[
= \int_{s_{i+1}}^{s_i} f(u)du = RQ_{i+1}(t_{i+1})
\]

(7)

where \( t_{i+1} < \theta_i < s_{i+1} \). According to Lemma 2, period \( (s_{i+1} - t_{i+1}) \) will be less than the previous period \( (s_i - t_i) \), so we get that \( RQ_i(t_i) \) will increase as \( i \) increases.

In ordering period \( RT_i(= [t_i, s_i]) \), the holding cost \( RC_i(t_i) \) is defined as

\[
RC_i(t_i) = c_2 \int_{s_i}^{t_i} (u - t_i) f(u)du
\]

(8)

where \( c_2 \) is expressed as unit the holding cost per unit time. Based on the above definition, the total cost can be described as

\[
W = nc_i + c_2 \sum_{i=0}^{n-1} \int_{t_i}^{s_i} (u - t_i) f(u)du
\]

\[
+ (c_3 + \alpha c_4) \sum_{i=0}^{n-1} \int_{s_i}^{t_{i+1}} \frac{(t_{i+1} - u)}{1 + \alpha(t_{i+1} - u)} f(u)du
\]

(9)

where \( c_1 \) is expressed as the single ordering cost, \( n \) is the total ordering frequency.

Dealing with the allowed shortage inventory problem, the scholars often use the total cost of (9) divided by the holding cost per unit time \( c_2 \), so the above equation can be rewritten as

\[
\overline{W} = nM + \sum_{i=0}^{n-1} \int_{t_i}^{s_i} (u - t_i) f(u)du
\]

\[
+ (N + \alpha O) \sum_{i=0}^{n-1} \int_{s_i}^{t_{i+1}} \frac{(t_{i+1} - u)}{1 + \alpha(t_{i+1} - u)} f(u)du
\]

(10)

Where

\[
\overline{W} = \frac{W}{c_2}, \quad M = \frac{c_1}{c_2}, \quad N = \frac{c_3}{c_2}, \quad O = \frac{c_4}{c_2}.
\]

**V. THE NECESSARY CONDITION OF THE OPTIMAL SOLUTION**

We take (9) respectively to do \( s_i \) and \( t_i \) partial differential calculation, assuming under the condition of the given ordering frequency \( n \), the necessary condition which minimizes the total cost must satisfy

\[
\frac{\partial W}{\partial t_i} = (c_3 + c_4 \alpha) \int_{s_i}^{t_i} \frac{1}{1 + \alpha(t_i - u)} f(u)du
\]

\[
- c_2 \int_{s_i}^{t_i} (u - t_i) f(u)du = 0, \quad i = 1, 2, ..., n
\]

(11)
\[ \frac{\partial W}{\partial s_i} = (c_3 + c_4 \alpha) \frac{(t_{i+1} - s_i)}{1 + \alpha(t_{i+1} - s_i)} 
- c_2(s_i - t_i) 
= 0, \quad i = 1, 2, \ldots, n \]  

(12)

**Lemma 2** Suppose the demand rate \( f(t) \) is a strictly increasing function, period time will reduce with ordering frequency \( t_i \) increasing.

**Prove** Suppose \( f(t) \) is a strictly increasing function, we use mean value theorem to solve (11) and can get as follows

\[
(c_3 + c_4 \alpha) \frac{(t_{i+1} - s_i)}{1 + \alpha(t_{i+1} - s_i)} = c_2(s_i - t_i) 
\]

(13)

where \( s_{i-1} < \theta_i < t_i \) and \( t_i < \theta_2 < s_i \). According to (12), we know

\[
(c_3 + c_4 \alpha) \frac{(t_{i+1} - s_i)}{1 + \alpha(t_{i+1} - s_i)} = c_2(s_i - t_i) 
\]

(14)

Taking above formula into (13), we can obtain

\[
\frac{(t_{i+1} - s_i)}{1 + \alpha(t_{i+1} - s_i)} f(\theta_1) = \frac{(t_{i+1} - s_i)}{1 + \alpha(t_{i+1} - s_i)} f(\theta_2) 
\]

Since \( f(t) \) is a strictly increasing function, then \( f(\theta_1) < f(\theta_2) \), we can rewrite above formula as

\[
\frac{(t_{i+1} - s_i)}{1 + \alpha(t_{i+1} - s_i)} > \frac{(t_{i+1} - s_i)}{1 + \alpha(t_{i+1} - s_i)} 
\]

(15)

Suppose \( g(u) = 1/(1 + \alpha u) \), where \( \alpha \geq 0 \), and take it first-ordering differential

\[
\frac{d}{du} g(u) = \frac{1}{(1 + \alpha u)^2} > 0 
\]

(17)

For any \( u \geq 0 \), \( g(u) \) is permanent a strictly increasing function, therefore period time \( (t_{i+1} - s_i) > (t_{i+1} - s_i) \), which proves that period time will reduce with the ordering frequency \( t_i \) increasing.

**VI. DETERMINE ORDERING PERIOD \( t_1 \)**

Appropriate ordering period \( t_1 \) conduces to reduce calculation time, but past research has not proposed the determination mode of ordering period \( t_1 \) for the partial backlogging problem. Therefore, this paper proposes the determination mode of the ordering period \( t_1 \) for the partial backlogging inventory model. The paper tries to use the relative total cost minimization to derive the optimal ordering period \( t_1 \). First, the relative total cost is defined as

\[
TRCUT(T) = \frac{1}{T} \left( M + \int_0^T (u-t_i)f(u)du \right) + (N + \alpha O) \left( t_i - u \right) \frac{f(u)du}{1 + \alpha(t_i - u)} 
\]

(18)

where shortage period is \( ST_i = [0, t_i] \), and ordering period is \( RT_i = [t_i, T] \). Therefore we take (18) to do period time \( T \) differential calculation, to minimize the relative total cost period time must satisfy

\[
\frac{dTRCUT(T)}{dT} = 0 
\]

(19)

Expanding the above formula, we can get

\[
M + (N + \alpha O) \int_{t_i}^T u \frac{f(u)du}{1 + \alpha u} + \int_{t_i}^T (u-T)f(u)du = T \int_{t_i}^T f(u)du 
\]

(20)

**VII. ALGORITHM**

The proposed algorithm in this section can be used in the linear demand rate, also in the nonlinear or transcendental function demand rate. The calculation process is as follows:

1. Make \( r = 0 \), initial time \( s_0 = 0 \), plan time \( H \), tolerate error \( tol = 10^{-6} \), at the same time sets ordering time \( t_1 \) and controls step size;
2. Rewrite ordering frequency as \( r \leftarrow r + 1 \), and reset the added ordering frequency as \( n_{add} = 0 \);
3. Substituted into the equation \( c_3 + c_4 \alpha \int_{t_{i+1}}^{s_i} f(u)du/\left[1 + \alpha(t_i - u)\right] \) to solve \( s_i \leftarrow s_i^* \); then resubstituted into \( (c_3 + c_4 \alpha)(t^* - s_i)/(1 + \alpha(t^* - s_i)) = c_2(s_i - t_i) \)
to solve \( t_{i+1} \leftarrow t^* \). Furthermore we change ordering frequency \( n_{\text{add}} = n_{\text{add}} + 1 \) to calculate shortage and ordering time sequentially. If \((H - s_n) > tol\), then we adjust \( t_i^* = t_i + (H - s_n) / \text{step size} \) until \((H - s_n) \leq tol\) to calculate the total cost \( W^* \);

(4) To determine the direction of the total cost reducing and amend ordering time \( t_i^* \), return to step 2;

(5) To calculate shortage, ordering time and the total cost sequentially.

VIII. NUMERICAL EXAMPLES

According to the above proposed method, this paper uses mathematical software MATLAB7.0 as the calculation and analysis tool. The example in this research derived from Donaldson[13], and add shortage cost, opportunity cost and backlogging parameter. The parameters are as follows:

(1) non-linear increasing demand function ;
(2) single ordering cost \( = 9 \);
(3) unit holding cost per unit time \( = 2 \);
(4) unit shortage cost per unit time \( = 7 \);
(5) unit opportunity cost per unit time \( = 1 \);
(6) backlogging parameter \( = 20 \);
(7) plan period \( = 1 \).

Based on the proposed method of the section, the total order number is 6 times which the minimum total cost is 117.4323.

| TABLE I. COMPUTATIONAL RESULTS |
|------------------|------------------|------------------|------------------|
| total order number | total cost       |
| 7                | 117.4409         |
| 6                | 117.4323         |
| 5                | 120.8574         |

According to the numerical results in Table 2, shortage period and order cycle have shown a regular decrease and also prove Lemma 1. shortage interval is expressed as \( ST_i = t_i - s_{i-1} \), holding interval is expressed as \( RT_i = s_i - t_i \).

| TABLE II. ORDERING TIME AND CYCLE TIME |
|----------------|----------------|----------------|----------------|
| Ordering Number | Ordering time | cycle time     |
| \( t_i \)       | \( s_i \)     | \( ST_i \)     | \( RT_i \)     |
| 1               | 0.1245        | 0.1245         | 0.1905         |
| 2               | 0.3347        | 0.0197         | 0.1513         |
| 3               | 0.5004        | 0.0144         | 0.1319         |
| 4               | 0.6445        | 0.0121         | 0.1197         |
| 5               | 0.7749        | 0.0108         | 0.1109         |
| 6               | 0.8957        | 0.0098         | 0.1043         |

Since the inventory model adopts afterwards replenishment mode, therefore the ordering quantity \( RQ_i' = (SQ_i + RQ_i) \) needs to replenish the shortage, in addition to purchase current inventory. The results in table3 also shows that quantity appears the regular increasing tendency.

| TABLE III. CALCULATION RESULTS OF SHORTAGE AND ORDERING QUANTITY |
|----------------|----------------|----------------|----------------|
| Ordering Frequency | \( S_i \), \( SQ_i \) | \( RQ_i \) | \( RQ_i' \) |
| \( i \)          | \( t_i \)     | \( s_i \)     | \( ST_i \)     | \( RT_i \)     |
| 1               | 4.2136        | 37.6777       | 41.8913        |
| 2               | 4.8551        | 55.8777       | 60.7328        |
| 3               | 5.6366        | 67.2469       | 72.8835        |
| 4               | 6.2495        | 75.8515       | 82.1010        |
| 5               | 6.7588        | 82.9127       | 89.6715        |
| 6               | 7.1987        | 88.9717       | 96.1703        |

Sensitivity analysis can help us to observe the influence that the diversifications of inventory cost, demand and time affect the total cost. The paper analyze the parameters of this example, including \( H, c_1, c_2, c_3, c_4 \). The sensitivity of optimal ordering frequency is defined as follows.

\[
S_o = \frac{n^* - n}{n} \times 100 \%
\]  

(21)

While the sensitivity of the optimal total cost is defined as follows.

\[
S_w = \frac{W^* - W}{W} \times 100 \%
\]  

(22)

where \( n^* \) and \( W^* \) respectively expresses the total ordering frequency and the minimum total cost the after parameters being changed.

The results of sensitivity analysis indicate that the plan period \( H \) is more sensitive parameter in all parameters; Otherwise, backlogging parameter \( \alpha \), shortage cost \( c_3 \) and the opportunity cost \( c_4 \) are less sensitive. In addition to ordering costs, the order number of all parameters increased with the parameter.
TABLE IV. SENSITIVITY ANALYSIS

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Change Rate</th>
<th>Order Number</th>
<th>Minimum total cost</th>
<th>( S_n^% ) (%)</th>
<th>( S_W^% ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H )</td>
<td>+50%</td>
<td>12</td>
<td>214.2290</td>
<td>100.00%</td>
<td>82.43%</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>6</td>
<td>117.4323</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-50%</td>
<td>2</td>
<td>42.0303</td>
<td>-66.67%</td>
<td>-64.21%</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>+50%</td>
<td>5</td>
<td>143.3575</td>
<td>-16.67%</td>
<td>22.08%</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>6</td>
<td>117.4323</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-50%</td>
<td>9</td>
<td>82.8446</td>
<td>50.00%</td>
<td>-29.45%</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>+50%</td>
<td>8</td>
<td>139.3573</td>
<td>33.33%</td>
<td>18.67%</td>
</tr>
<tr>
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<td>0%</td>
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<td>117.4323</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-50%</td>
<td>5</td>
<td>86.0051</td>
<td>-16.67%</td>
<td>-26.76%</td>
</tr>
<tr>
<td>( c_3 )</td>
<td>+50%</td>
<td>7</td>
<td>118.2377</td>
<td>16.67%</td>
<td>0.69%</td>
</tr>
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<td>0%</td>
<td>6</td>
<td>117.4323</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-50%</td>
<td>6</td>
<td>116.1250</td>
<td>0.00%</td>
<td>-1.11%</td>
</tr>
<tr>
<td>( c_4 )</td>
<td>+50%</td>
<td>7</td>
<td>119.3310</td>
<td>16.67%</td>
<td>1.62%</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>6</td>
<td>117.4323</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-50%</td>
<td>6</td>
<td>112.3916</td>
<td>0.00%</td>
<td>-4.29%</td>
</tr>
<tr>
<td>( b )</td>
<td>+50%</td>
<td>8</td>
<td>143.4732</td>
<td>33.33%</td>
<td>22.18%</td>
</tr>
<tr>
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<td>0%</td>
<td>6</td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-50%</td>
<td>4</td>
<td>83.0195</td>
<td>-33.33%</td>
<td>-29.30%</td>
</tr>
</tbody>
</table>

As this paper focuses on partial backlogging inventory model, so we specifically discussed backorder parameters. We can conclude the following from Table 5:

1. With the increase in backorder parameter, the total cost and ordering number will also increase;
2. When backorder parameter approaches zero, the total cost will be close to completely backorder model;
3. When backorder parameter approaches infinity, the total cost will be close to not allowed out of stock model.

TABLE V. SENSITIVITY ANALYSIS OF BACKORDER PARAMETER

<table>
<thead>
<tr>
<th>completely backorder</th>
<th>Sensitivity analysis of backorder parameter ( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>114.5741</td>
</tr>
<tr>
<td>20</td>
<td>117.4323</td>
</tr>
<tr>
<td>30</td>
<td>118.7454</td>
</tr>
<tr>
<td>50</td>
<td>120.1319</td>
</tr>
<tr>
<td>not allowed backorder</td>
<td>125.2604</td>
</tr>
</tbody>
</table>

IX. CONCLUSIONS

The purpose of this study is to simulate more reasonable market competition condition, and relax the previous research assumptions. The partial backlogging inventory model proposed in this paper makes the inventory model more reasonable during shortage period, at the same time also propose the prediction method and algorithm of ordering period \( t^*_1 \). Finally, the paper proposes a practical example of the numerical analysis. The inventory model proposed in this paper can provide the reverse logistics enterprise a appropriate assessment method dealing with the shortage problem, so they can work out a more reasonable inventory strategy.

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REFERENCES


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