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Geometrical Framework Application Directions in Identification Systems: A Review

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Abstract: The approaches review of the framework application in identification problems is fulfilled. It is showed that this concept can have different interpretations of identification problems. In particular, the framework is understood as a frame, structure, system, platform, concept, and basis. Two directions of this concept application are allocated: 1) the framework integrating the number of methods, approaches or procedures; b) the mapping describing in the generalized view processes and properties in a system. We give the review of approaches that are the basis of the second direction. They are based on the analysis of virtual geometric structures. These mappings (frameworks) differ in the theory of chaos, accidents, and the qualitative theory of dynamic systems. Introduced mappings (frameworks) are not set a priori, and they are determined based of the experimental data processing. The main directions analysis of geometrical frameworks application is fulfilled in structural identification problems of systems. The review includes following directions: i) structural identification of nonlinear systems; ii) an estimation of Lyapunov exponents; iii) structural identifiability of nonlinear systems; iv) the system structure choice with lag variables; v) system attractor reconstruction.

Index Terms: Framework, nonlinear dynamic system, phase portrait, structural identification, nonlinearity, structural identifiability, synchronizability, lag, Lyapunov exponent.

1. Introduction

The framework (FR) concept is applied in control, identification, and the analysis and data processing tasks. FR is the synonym of such concepts as a frame, structure, system, the platform, the concept, a basis, and set of approaches. The term "framework" is used in two directions in scientific research. The first direction of FR application represents the conceptual concept integrating a set of method approaches or procedures. So, FR in [1] is interpreted as the set of mathematical and technical procedures and methods for identification of the automobile battery control process. The approach to the identification is based on Bayesian framework. In [2] this concept combines the set of identification methods based on the calculation of the prediction error. Proposed methods show that such procedures allow obtaining estimations in some optimum sense. The key moment in this parametric paradigm is the choice of the sufficient reference structure. The same paradigm based on the creation of the new concept to system identification is proposed in [3, 4]. It is based on the compilation of existing approaches.

The framework can be interpreted as the theoretical model structure for the analysis of a content transmitted to video [5]. So, we have the system of some theoretical provisions which are applied for the solution of a specific problem. The hybrid system identification scheme (methodology) which is based on the application of continuous optimization is proposed in [6]. Other interpretations of the framework concept given in [7, 8, 52].

The uniform theoretical concept (framework) for the identification and the control of a nonlinear discrete dynamic system is proposed in [9]. It is based on the application of neural networks. The procedure (framework) for identification of the functional refusal is proposed in [10]. It is the basis of a new approach for functional refusal risk estimation in physical systems. The framework is based on the integration of functionality hierarchical systemic models, and results of behavioral simulation. Such interpretation of FR is dominating (see for example [11-15]).

The second interpretation of FR is based on the consideration of some mapping describing in the generalized view processes and properties in a system. Bases of such approach are proposed in the qualitative theory of dynamic systems [16-18]. Some geometrical structure corresponds to such mapping. This approach is widely applied at chaos research. The attractor is the framework in identification problems (see for example [19-21]). The equation structure is specified a priori accurate within unknown parameters in these works. Further, the identification problem is solved to obtain a required form of the attractor. Other approach is associated to design of a geometrical framework for the system under uncertainty and is directed to the solution of the structural identification problem [22, 23]. Further, we will interpret this approach as the methodology based on the design and the analysis of the geometrical framework (GF). The main difference between proposed by GF and frameworks in [19-21] consists that mathematical mapping (GF) is not postulated a priori, and is determined on the basis of data processing. The obtained GF is the main object of the analysis. GF gave the presentation on behaviour and properties the system.

Further, we consider main directions of the GF application, generalizing and developing results [22, 23]. They contain the following areas of the identification theory.

I. Structural identification of the nonlinear system.
II. The estimation of Lyapunov exponents.
III. Structural identifiability of the nonlinear system.
IV. The system structure estimation with lag variables.
V. The system phase portrait reconstruction on the time series.

The article has the following structure. Section III contains the problem statement. The methodology for geometrical frameworks design in identification problems is stated in Section IV. We will show that GF for static and dynamic systems differ significantly. The special class of mappings is applied to making decision-making on the linear dynamic system structure. The solution of this task is given on the basis of frameworks design for the Lyapunov exponents estimation. The significant geometrical framework obtaining depends on the structural identifiability of the dynamic system. The analysis of this problem is presented in Section V. It is showed that the system input should be S-synchronizing for the obtaining of significant GF. Reconstruction of the phase portrait or attractor of the system is the identification problem also. Section VI contains the analysis of identification problems giving the estimation task solution the system structure in the phase space. Proposed approaches are based on the application of the Taykens theorem. Difficulties appearing at the restoration (identification) of the phase portrait are considered. The system structure choice with lag variables is discussed in Section VII. Two approaches to the choice of the system structure are considered. The first approach is based on statistical methods application. The second approach is founded on the Lyapunov exponents estimation. The example of the proposed approach implementation is described. The conclusion contains the main inferences and results obtained in the work.

2. Related Works

Many papers are devoted to conceptual questions of various approaches creation, methods in identification problems [1-6, 9]. As a rule, the created theoretical constructions and approaches are the compilation of existing procedures and algorithms. They have directed to the accounting of system different features that to give required quality to the identification system. Such the approach to the improvement of identification methods is dominating (see for example [11-15]).

The second direction of frameworks use is associated with the analysis of the identification system quality. This direction is based on ideas of dynamic systems (DS) qualitative theory [16-18]. The mathematical mappings describing system geometrical images change are applied in this case. As a rule, GF are images of attractors or phase portraits which will are formed a priori [19-21]. Attractors recover on the basis of time series and Takens theorem [39] under uncertainty. The attractor reconstruction process is very labor-consuming. The problem of identification has specificity. Therefore, the direct transfer of DS qualitative theory methods is not applicable here. GF design methods differ from the results obtained in [19-21]. The basis of the proposed approach is stated in [23] for the problem solution of static systems structural identification. Further, obtained results are generalized on dynamic systems class in [22, 23]. Choice of the mapping describing the geometrical framework depends on the examined system class. The GF design methodology is proposed in [22, 23].

The estimation problem of Lyapunov exponents occupies a specific place. GF applied for the LE estimation differs from the frameworks used in structural identification problems. The initial data array formation for LE estimation is based on the same ideas, as the GF design for the solution of dynamic systems structural identification problem. But in this case, the subset of DS free motion trajectories [23] is allocated. The analysis of this subset allows forming the time series describing the LE change. Time series is the basis for the design of the mapping describing the LE change dynamics in a special structural space [23, 27]. The change calculation of Lyapunov exponents is fulfilled on the standard formula [26].

GF is the basis for the structural identifiability (SI) estimation of nonlinear systems. The SI estimation is based on the analysis of frameworks proposed in [22, 23]. The property of a system S-synchronizability [25] plays the main role in the SI analysis. The considered SI concept significantly differs from parametrical identifiability [28-32].

The system structure estimation problem with the distributed lag is considered in [25, 33-37]. Its decision is based on the application of parametric schemes for the system identification with lag variables. Such concept to the parameter estimation is based on the available a priori identification. Various statistical hypotheses [33-36] are used in identification procedures. The implementation of this approach is very laborious under uncertainty. The approach based on the GF analysis is proposed for the system structure estimation with lags. Geometrical frameworks describe the dynamics of LE change. The approach develops results [23, 24] on the case of static systems with lags.

3. Problem Statement

Consider dynamic system

$$\begin{aligned} \dot{X} &= AX + \varphi(y)I + Bu \\ y &= C^T X \end{aligned} \quad (1)$$

where $u \in R$, $y \in R$ are the input and the output; $A \in R^{m\times m}$, $B \in R^m$, $I \in R^m$ $C \in R^m$ are matrices of corresponding dimensions; $\varphi(y)$ is a scalar nonlinear function. A is the Hurwitz matrix.

ξ is the linear combination of state variable X. We suppose that $\chi = \varphi(y)$ belongs to the set

$$\chi \in \mathcal{F}_{\varphi} = \{\upsilon_1\xi^2 \le \varphi(\xi)\xi \le \upsilon_2\xi^2,\ \xi \ne 0, \\ \varphi(0) = 0,\ \upsilon_1 \ge 0, \upsilon_2 < \infty\}. \quad (2)$$

The system (1) nonlinear part is described by static (algebraic) equations often. Therefore, further, we consider the case when $\varphi(y)$ describe by the algebraic equation.

Let the informational set be known for the system (1)

$$\mathrm{I}_o = \{u(t), y(t),\ t \in J = [t_0, t_k]\}. \quad (3)$$

Problem: evaluate the class of nonlinear function $\varphi(y)$ in (1) and characteristics the matrix A on the basis of the data processing (3).

4. Geometrical Frameworks in Dynamic Systems Structural Identification Problem

4.1. $\mathcal{S}_{ey}$-frameworks

The geometrical framework $\mathcal{S}_{ey}$ design is one of the main stages in the structural identification problem. The method for the framework $\mathcal{S}_{ey}$ design is defined by the estimation possibility of system structural parameters. The framework $\mathcal{S}_{ey}$ is derivative from a phase portrait $\mathcal{S}$. $\mathcal{S}$ is the starting point for further researches on the formation $\mathcal{S}_{ey}$ under uncertainty.

The GF design approach depends on system properties and the considered problem of structure identification. The synthesis $\mathcal{S}_{ey}$ method is proposed in [21] and generalized on dynamic systems in [20, 22]. The approach essence consists in the formation of a subset I_{GF} which allows obtaining a mapping for the design $\mathcal{S}_{ey}$. I_{GF} is the result of the set I_o analysis. I_{GF} may contain data on the transient process or the steady motion in the system. In the estimation problem of the nonlinear dynamic system (1) structure, at first, the set $\mathrm{I}_{GF} = \mathrm{I}_{N,g}$ which contains the information about system nonlinear properties is formed.

The set $\mathrm{I}_{N,g}$ is identified as follows. Apply to $y(t)$ the differentiation operation and designate by the obtained variable as x_1. Determine the model

$$\hat{x}_1^l(t) = H^T [1\ u(t)\ y(t)]^T, \quad (4)$$

where $\hat{x}_1^l$ is the estimation of the linear component in x_1 on the time gape $J_g = J \setminus J_{tr}$ corresponding to the steady motion in the system (1); $H \in R^3$ is the vector of model (4) parameters; J_{tr} is the time gap corresponding to transient process in the system. Determine by the vector H having applied a least square method.

Determine the forecast for the variable x_1 used the model (4) and form the error $e(t) = \hat{x}_1^l(t) - x_1(t)$. $e(t)$ depends on the nonlinearity $\varphi(y)$ in the system (1). So, we obtain the set $\mathrm{I}_{N,g} = \{y(t), e(t)\ t \in J_g\}$. Further, we apply the designation $y(t)$ assuming that $y(t) \in \mathrm{I}_{N,g}$.

Construct the phase portrait $\mathcal{S}$ and GF $\mathcal{S}_{ey}$ described by functions $\Gamma:\{y\}\to\{y'\}$, $\Gamma_{ey}:\{y\}\to\{e\}$. $\mathcal{S}_{ey}$ is the basis for the further analysis and the identification system design. The framework $\mathcal{S}_{ey}$ should have specified properties [22]. Properties of structural identifiability and S-synchronizability [25] are basic. The obtained GF correctness sign is a regularity of its presentation and the condition $|e(t)|>\delta_e$ for $\forall t\ge t_q>t_{tr}$, where t_{tr} is the end time of the transient process, $\delta_e>0$ is some number. The application of the described approach gives how significant $\mathcal{S}_{ey}$, and insignificant $\mathcal{NS}_{ey}$ frameworks ($\mathcal{S}_{ey}=\mathcal{NS}_{ey}$). Decision-making on the significance $\mathcal{S}_{ey}$ is based on the results obtained in [22]. The framework $\mathcal{NS}_{ey}$ is the result of the condition S-synchronizability (SS) non-fulfillment of the system (1). S-synchronizability of the system (1) (framework $\mathcal{S}_{ey}$) depends on the fulfillment of the excitation constancy condition for the input $u(t)$. The significance $\mathcal{S}_{ey}$ estimation algorithm is based on the sector set properties analysis for $\mathcal{S}_{ey}$ [23] if the SS-condition of is satisfied.

Definition 1. The framework $\mathcal{S}_{ey}$ is called the regular if the condition S-synchronizability is satisfied for the system (1).

The example of the regular framework $\mathcal{S}_{ey}$ for the system with a static hysteresis is presented in Fig. 1 [23].

If the function $\varphi(y)$ has the complex law of change, the application of the approach described above can give to "false" $\mathcal{NS}_{ey}$-framework.

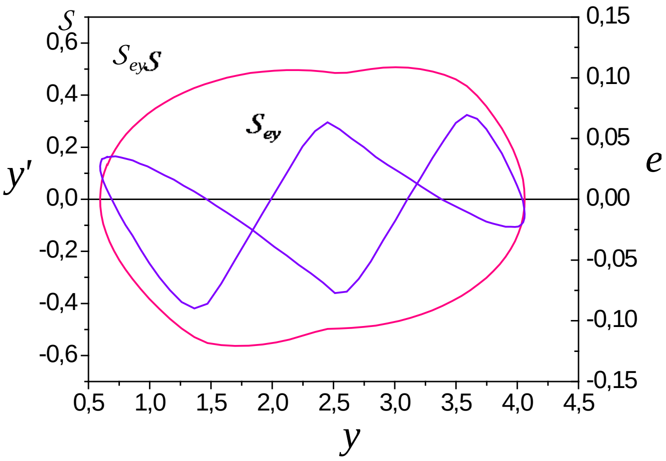


Fig.1. Frameworks $\mathcal{S}$, $\mathcal{S}_{ey}$ for the second order system (1) with static hysteresis

The example of such framework for the system describing processes in RC-OTA the chaotic oscillator

$$\begin{aligned}&\ddot{x}-0.1\dot{x}+x=\varphi\\&\dot{\varphi}=10\left(-\varphi+\operatorname{sgn}\left(x+\operatorname{sgn}(\varphi)\right)\right)\end{aligned}\tag{5}$$

is shown in Fig. 2. RC-OTA is applied to the design of electronic and control systems [50].

The obtaining of the regular framework gives to the application of the hierarchical immersion method [23] in state-space. This method provides the model (4) structure choice for each layer of hierarchy. The example of the regular framework $\mathcal{S}_{ey}$ for the system (5) is shown in Fig. 3. Designations showed in Fig. 3 given in [21 23].

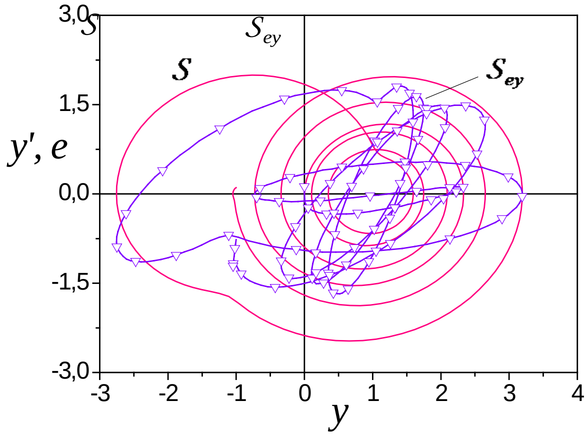


Fig.2. Frameworks S , S_{ey} for the second order system (5) with dynamic hysteresis

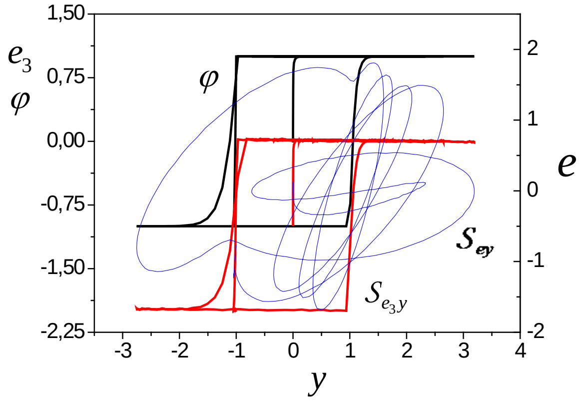


Fig.3. Regular structure for the system (5)

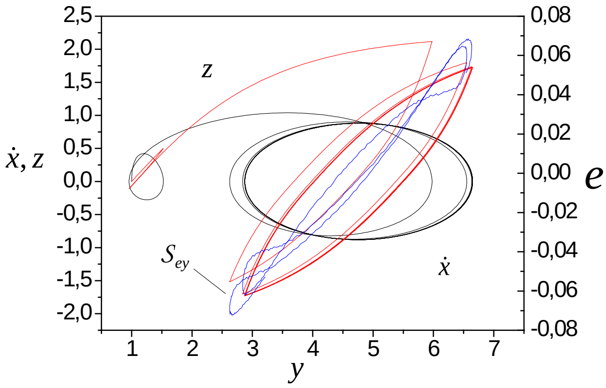


Fig.4. Framework S_{ey} , phase portrait and z

Example 1. Consider a mechanical system with Bouc-Wen hysteresis [51]. This has the form

$$
\begin{aligned}
& m\ddot{x}+c\dot{x}+F(x,z,t)=f(t),\ y=x \\
& F(x,z,t)=\alpha kx(t)+(1-\alpha)kdz(t) \\
& \dot{z}=d^{-1}\left(a\dot{x}-\beta|\dot{x}||z|^{n}\,sign(z)-\gamma\dot{x}|z|^{n}\right)
\end{aligned} \tag{6}
$$

where $m>0$ is weight, $c>0$ is damping, $F(x,z,t)$ is the restoring force, $d>0$, $n>0$, $k>0$, $\alpha\in(0,1)$, $f(t)$ is exciting force, a,β,γ are some numbers. Denote by the system (1)-(3) as SBW.

Parameters the SBW-system are equal: $d=a=m$, $n=1.5$, $\beta=0.5$, $\alpha=1.5$, $k=0.6$, $m=1$, $c=2$. The exciting force $f(t)=2-2\sin(0.15\pi t)$.

The model (4) has the form: $\dot{x} = -0.199x + 0.471f$. The application of the proposed method gives $\mathcal{S}_{ey}$-frameworks (Fig. 4). Ranges of definition R and F match. Analysis of the $\mathcal{S}_{ey}$-structure shows that the system (6) is nonlinear.

4.2. $\mathcal{S}_{k_{s,\rho}}$, $\mathcal{SK}_{\Delta k^i_{s,\rho}}$ *-frameworks*

Another class of framework $\mathcal{S}_{k_{s,\rho}}$ is designed on the basis of the system (1) general solution analysis. $\mathcal{S}_{k_{s,\rho}}$ apply to the structure choice for the system (1) linear part. This task differs from the problem considered above. Therefore, mappings allowing making decisions should have other form [24, 25]. They are based on the analysis of the Lyapunov exponent dynamics change. Apply the model

$$\hat{X}_q(t) = \hat{A}_q W(t) \quad \forall t \in J_q \tag{7}$$

to the particular solution estimation of the system (1) on the output y, where $\hat{A}_q \in R^{2\times 2}$ is the parameter matrix, $W = [u\ u']^T$, $\hat{X}_q \in R^2$ is the estimation output of the system output and its derivative. The choice of the interval $J_q \subset J$ depends on the system (1) properties.

Further, we obtain the estimation for the system (1) general solution on the basis $\hat{X}_q$

$$\hat{X}_g(t) = X(t) - \hat{X}_q(t) \quad \forall t \in J_g$$

where $\hat{X}_g(t) = [\hat{y}_g(t)\ \dot{\hat{y}}_g(t)]^T$. This approach can be generalized on the case $m > 2$.

Functions

$$\rho\left(\hat{y}_g\right) = \rho_g = \ln\left|\hat{y}_g(t)\right| \quad \forall t \in \bar{J}_g \subset J_g$$

$$k_s(t,\rho) = \frac{\rho\left(\hat{y}_g\right)}{t} \tag{8}$$

are basis of the mapping describing $\mathcal{S}_{k_{s,\rho}}$ where $\bar{J}_g = [t_0, \bar{t}]$ is determined on the basis by the LE theory [26]. $k_s(t,\rho)$ is the basis for the Lyapunov exponent calculation.

Remark 1. The framework $\mathcal{S}_{k_{s,\rho}}$ use simplifies the choice of the upper bound for time at the calculation of LE.

Perform the analysis of sets

$$\begin{aligned} \mathrm{I}_{k_s} &= \left\{k_s\left(t, \rho\left(\hat{y}_g(t)\right)\right),\ t \in \bar{J}_g\right\} \\ \mathrm{I}_{k_s'} &= \left\{k_s\left(t, \rho\left(\dot{\hat{y}}_g(t)\right)\right),\ t \in \bar{J}_g\right\}, \end{aligned} \quad \bar{J}_g \subset J_g \tag{9}$$

for the LE determination.

On sets I_{k_s}, $\mathrm{I}_{k_s'}$ the framework $\mathcal{S}_{k_{s,\rho}}$ described by the function $\Gamma_{k_{s,\rho}} : \mathrm{I}_{k_s} \to \mathrm{I}_{k_s'}$ is introduced. The framework $\mathcal{S}_{k_{s,\rho}}$ reflects the change dynamics of indexes depending on LE. Consider also the function describing the first difference $k_s(t, \rho(\dot{\hat{y}}_g(t))$ change

$$\Delta k_s'(t) = k_s\left(t, \rho\left(\dot{\hat{y}}_g(t+\tau)\right)\right) - k_s\left(t, \rho\left(\dot{\hat{y}}_g(t)\right)\right) \tag{10}$$

where $\tau > 0$.

Form the set $\mathrm{I}_{\Delta k_s'} = \left\{\Delta k_s\left(t, \rho\left(\dot{\hat{y}}_g(t)\right)\right),\ t \in \bar{J}_g\right\}$ and introduce the framework $\mathcal{SK}_{\Delta k_{s,\rho}'}$ which function $\Gamma_{\Delta k_{s,\rho}'} : \mathrm{I}_{k_{s,\rho}} \to \mathrm{I}_{\Delta k_{s,\rho}'}$ corresponds.

Consider the framework $\mathcal{LSK}_{\Delta k'_{s,\rho}}$ with $\Gamma_{\Delta k'_{s,\rho}} : \mathrm{I}_{k_{s,\rho}} \to B\left(\mathrm{I}_{\Delta k'_{s,\rho}}\right)$, where $B\left(\mathrm{I}_{\Delta k'_{s,\rho}}\right) \subset \{-1;1\}$. Define by elements of the binary set $B\left(\mathrm{I}_{\Delta k'_{s,\rho}}\right)$ as

$$b(t) = \begin{cases} 1, \text{ если } \Delta k'_s(t) \geq 0 \\ -1, \text{ если } \Delta k'_s(t) < 0 \end{cases} \quad t \in \bar{J}_g . \tag{11}$$

Frameworks $\mathcal{SK}_{\Delta k^i_{s,\rho}}$ which are based on the change $\Delta k^i_s(t)$ ($i > 1$) analysis are formed similarly. $\Delta k^i_s(t)$ is determined by analogy with (10), and i designates i-th derivative $\hat{y}_g(t)$.

The application (8)-(11) allows to obtain the LE set and to estimate their type. The generalization of the proposed approach on periodic dynamic systems is given in [27].

5. Structural Identifiability of Nonlinear Dynamic System

In the previous section, it is noted that the nonlinear DS structure estimation depends on the system identifiability.

Many publications (see for example [28-30]) are devoted to the problem of dynamic systems parametric identifiability. The structural identifiability of nonlinear dynamic systems reduced to the parametrical identifiability on the basis of various approximation methods application [29-31]. Modifications of the proposed approaches are considered in case when not all system parameters can be identified.

In [25] structural identifiability is considered in the following aspect: determine by conditions under which the nonlinear system structure estimation is possible under uncertainty. The solution to this problem for the system (1) is given in [25] when the nonlinear function $\varphi(y)$ satisfies the condition (2). Decision-making is based on the analysis of the framework $\mathcal{S}_{ey}$ which describes the system (1) behavior in the steady state. It is showed that the system should satisfy to h-identifiability property.

Let conditions be satisfied.

B1. The initial set I_o gives to the parametrical identification problem solution of the model (1). It means that the input $u(t)$ is constantly excited on the interval J.

B2. The input $u(t)$ use gives to the informative framework $\mathcal{S}_{ey}\left(\mathrm{I}_{N,g}\right)$. It means that the analysis $\mathcal{S}_{ey}$ gives the estimation problem solution of the system (1) nonlinear properties.

Remark 2. Property of the excitation constancy on which the parametric identifiability estimation is based has features at the solution of the task h-identifiability problem.

Let the framework $\mathcal{S}_{ey}$ be closed and the area $\mathcal{S}_{ey}$ is not zero. Designate by height $\mathcal{S}_{ey}$ as $h\left(\mathcal{S}_{ey}\right)$ where the height is the distance between two points of opposite sides of the framework $\mathcal{S}_{ey}$.

Statement 1 [23]. Let i) the linear part of the system (1) is stable, and the nonlinearity $\varphi(\cdot)$ satisfies the condition (2); ii) the input $u(t)$ is limited, piecewise continuous and constantly excited; iii) $\delta_S > 0$ exists what $h\left(\mathcal{S}_{ey}\right) \geq \delta_S$. Then the framework $\mathcal{S}_{ey}$ is identifiable on the set $\mathrm{I}_{N,g}$.

Definition 2. The framework $\mathcal{S}_{ey}$ having the specified properties in the statement 1 is h-identified.

The statement 1 conditions fulfillment can give "insignificant" $\mathcal{S}_{ey}$-framework ($\mathcal{NS}_{ey}$-framework). Therefore, h-identifiability is a sufficient, but necessary condition of SI. S-synchronizability ensuring for the system [25] is such condition.

Introduce designations: $\mathcal{D}_y = \mathrm{dom}\left(\mathcal{S}_{ey}\right)$ is the domain, $D_y = D_y\left(\mathcal{D}_y\right) = \max_t y(t) - \min_t y(t)$ is the diameter $\mathcal{D}_y$. Let $u(t) \in \mathrm{U}$ is admissible set of inputs for the system (1).

Definition 3 [25]. The input $u(t) \in \mathrm{U}$ S-synchronizes the system (1) if the framework $\mathcal{S}_{ey}$ domain has the maximum diameter D_y on the set $\left\{y(t), t \in J\right\}$.

Synchronization $u(t) \in \mathrm{U}$ is understood as the choice of such input $u_h(t) \in \mathrm{U}$ which allows reflecting all features $\mathcal{S}_{ey}$ characterizing $\varphi(y)$. It is possible only in case when $u(t)$ ensures $\max_{u_h} D_y$. Synchronization allows obtaining the framework $\mathcal{S}_{ey} \neq \mathcal{NS}_{ey}$. Such selection $u_h(t) \in \mathrm{U}$ can be interpreted as the synchronization between the model and the system. Therefore, fulfillment of the condition $\mathcal{d}_{h,y} = \max_{u_h} D_y$ ensures the system h_{δ_h}-identifiability.

Let the input $u_h(t)$ synchronize the set $\mathcal{D}_y$. If $u(t)$ is S-synchronizing, then we will write $u_h(t) \in \mathrm{S}$. Let's notice that the finite set $\{u_h(t)\} \in \mathrm{S}$ exists for the system (1). The choice of the optimum input $u_h(t)$ depends from $\mathcal{d}_{h,y}$. Ensuring this condition is one of the prerequisites for the system (1) structural identifiability.

Consider the reference structure $\mathcal{S}_{ey}^{ref}$. $\mathcal{S}_{ey}^{ref}$ reflects all properties of the function $\varphi(y)$. Denote by diameter $D_y\left(\mathcal{S}_{ey}^{ref}\right)$ as D_y^{ref}. If $u_h(t) \in \mathrm{S}$, that D_y^{ref} exists for the system (1).

Corollary from definitions 2, 3. If $\mathcal{S}_{ey} \cong \mathcal{S}_{ey}^{ref}$ then $\left|D_y - D_y^{ref}\right| \leq \varepsilon_y$ where $\varepsilon_y \geq 0$, $\cong$ is the sign of proximity. Elements of a subset $\mathrm{U_S}$ have property

$$\left|D_y\left(\mathcal{S}_{ey}\left(u(t)\big|_{u\in \mathrm{U_S}}\right)\right) - D_y^{ref}\right| \leq \varepsilon_y,$$

and

$$\left|D_y\left(\mathcal{S}_{ey}\left(u(t)\big|_{u\in \mathrm{U}\backslash\mathrm{U_S}}\right)\right) - \mathcal{d}_{h,y}\right| > \varepsilon_y$$

is the appearance condition $\mathcal{NS}_{ey}$.

Let $\mathcal{S}_{ey}$ is h-identifiable and $\mathcal{S}_{ey} = F_{\mathcal{S}_{ey}}^{l} \cup F_{\mathcal{S}_{ey}}^{r}$, where $\mathcal{F}_{\mathcal{S}_{ey}}^{l}, \mathcal{F}_{\mathcal{S}_{ey}}^{r}$ are the left and right fragments $\mathcal{S}_{ey}$. Secants for $\mathcal{F}_{\mathcal{S}_{ey}}^{l}, \mathcal{F}_{\mathcal{S}_{ey}}^{r}$ are described by equations

$$\gamma_{\mathrm{S}}^{r} = a^r y,\ \gamma_{\mathrm{S}}^{l} = a^l y \tag{12}$$

where a^l, a^r are numbers determined by the least squares method (LSM).

Definition 4. If the framework $\mathcal{S}_{ey}$ is h-identifiable and the condition $\left\| \left|a^l\right| - \left|a^r\right| \right\| \leq \delta_h$ is satisfied, the framework $\mathcal{S}_{ey}$ (the system (1)) is structurally identifiable or h_{δ_h}-identifiable.

Definition 4 shows if the system (1) is h_{δ_h}-identifiable, then the structure $\mathcal{S}_{ey}$ have the maximum the area $\mathcal{D}_y$ diameter, and the system is S-synchronizable.

Let the structure $\mathcal{S}$ have m features. We understand features of the function $\varphi(y)$ as loss of continuity, inflection points or extremes. These features are signs of the function nonlinearity.

Definition 5. If the framework $\mathcal{S}_{ey}$ is h_{δ_h}-identifiable, then the model (4) is SM-identifying.

Theorem 1 [22]. Let 1) the input $u(t)$ is constantly excited and ensures the system (1) S-synchronization; 2) the phase portrait $\mathcal{S}$ of the system (1) contains features; 3) the $\mathcal{S}_{ey}$-framework is h_{δ_h}-identifiable and contains fragments corresponding to features of the phase portrait $\mathcal{S}$. Then the model (4) is SM-identifying.

Remark 3. According to results of Section 4, the process design of the model (4) structure can have a hierarchical form. It is rightly for nonlinearities, which do not satisfy the condition (2).

Consider the framework $\mathcal{S}_{ey}$. Designate by the center $\mathcal{S}_{ey}$ on the set $J_y = \{y(t)\}$ as $c_{\mathcal{S}}$, and the center of the area $\mathcal{D}_y$ as c_{D_y}.

Theorem 2 [25]. Let on the set U of representative inputs $u(t)$ of the system (1): i) such $\varepsilon \geq 0$ exists what $\left|c_S - c_{D_y}\right| \leq \varepsilon$; ii) the condition $\left\| a^l \right| - \left| a^r \right\| \leq \delta_h$ is satisfied. Then the system (1) is h_{δ_h} -identifiable, and the input $u_h(t) \in \mathrm{S}$.

Some subset $\{u_{h,i}(t)\} \subset \mathrm{U}_h \subseteq \mathrm{U}$ ($i \geq 1$) which elements have the S-synchronizability property exists. Everyone $u_{h,i}(t)$ correspond to the framework $\mathcal{S}_{ey,i}(u_{h,i})$ with the diameter $D_{y,i}$ of the domain $\mathcal{D}_{y,i}$. As $u_{h,i}(t) \in \mathrm{S}$, diameters $D_{y,i}$ will have the feature $\mathcal{d}_{h,\Sigma}$ -optimality.

Let the hypothetical framework $\mathcal{S}_{ey}$ of the system (1) have the diameter $\mathcal{d}_{h,\Sigma}$.

Definition 6. The framework $\mathcal{S}_{ey,i}$ has the feature $\mathcal{d}_{h,\Sigma}$ -optimality on the set U_h if such $\varepsilon_\Sigma > 0$ exists that $\left|\mathcal{d}_{h,\Sigma} - D_{y,i}\right| \leq \varepsilon_\Sigma \quad \forall i = \overline{1,\# \mathrm{U}_h}$.

Definition 7. If the subset of inputs $\{u_{h,i}(t)\} = \mathrm{U}_h \subset \mathrm{U}$ ($i \geq 1$) which elements $u_{h,i}(t) \in \mathrm{S}$, and frameworks $\mathcal{S}_{ey,i}(u_{h,i})$ corresponding it having property $\mathcal{d}_{h,\Sigma}$ -optimality, frameworks $\mathcal{S}_{ey,i}(u_{h,i})$ are indiscernible on sets $\{u_{h,i}(t)\}$, exists.

Definitions 6, 7 show that the h_{δ_h} -identifiability estimation can be obtained on any input $u(t) \subset \mathrm{U}_h$. The approach to the estimation of the system (1) h_{δ_h} -identifiability is proposed in [25]. It is based on the application of an integral indicator for the framework $\mathcal{S}_{ey}$ analysis and is based on the development of results obtained in [23].

Example 2. Consider the system (6). The structure $\mathcal{S}_{ey}$ is shown in Fig. 4. The model approximating $\mathcal{S}_{ey}$ has the form

$$\gamma_{ey} = 0.033y - 0.153 \,, \; r_{ey}^2 = 0.983 \tag{13}$$

where $\gamma_{ey} = \hat{e}$ is the secant framework $\mathcal{S}_{ey}$, r_{ey}^2 is determination coefficient.

The structural identifiability of the system follows from theorem 3, $\delta_h = 0.002$. S_{BW} -system is S-synchronized, and the model (4) for obtaining $\mathcal{S}_{ey}$ is *SM* -identifying. The centre of the framework $\mathcal{S}_{ey}$ $c_{\mathrm{S}} = ?\ .001$. Secants (12) have the form

$$\begin{aligned} \gamma_e^l &= 0.0313y - \ 0.146, \; r_{ye,l}^2 = 0.912, \\ \gamma_e^r &= 0.032y - \ 0{,}15, \; r_{ye,r}^2 = 0.926 \end{aligned} \quad , \tag{14}$$

Models (13) structurally coincide with (14). These results confirm the fulfillment of the condition

$$\left| D_y \left(\mathcal{S}_{ey} \left(u(t) \big|_{u \in \mathrm{U_S}} \right) \right) - D_y^{ref} \right| \leq \varepsilon_y \,.$$

Example 2. Consider the system consisting of a nonlinear actuator and an object. The object has dry and quadratic friction. The actuator described by the nonlinear function with saturation (system S_{ST})

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -c_1 \varphi_1(x_2) \end{bmatrix} + \begin{bmatrix} 0 \\ c\varphi_2(u) \end{bmatrix}, \\ y &= x_1, \end{aligned}$$

where $\varphi_1(x_2) = -c_1 x_2^2 \mathrm{sign}(x_2)$ is quadratic friction, $\varphi_2(u) = \mathrm{sat}(u)$ is dry friction, $x = x_1$ is the rotation angle of the object shaft, u is excitation current of the actuator winding, y is output, $c_1 = 2$, $c = 1$, $u(t) = 3\sin(0.1\pi t)$. Set of measurements $\mathrm{I}_o = \{u(t), y(t), t = [0, t_k]\}$, $t_k < \infty$.

The frameworks $\mathcal{S}, \mathcal{S}_{ey}$ presented in Fig. 5. Apply the proposed approach to SI estimation and obtain the structural identifiability of the system S_{ST}. The conclusion about the structure of nonlinearity cannot be base on $\mathcal{S}, \mathcal{S}_{ey}$. A nonlinear input complicates the task. Analysis of the structure $\mathcal{S}_{ey}$ shows that the input $\varphi_2(u)$ is constant on the interval $\bar{J}_y = [4;8.5]$ and the constancy excitation condition for input not hold.

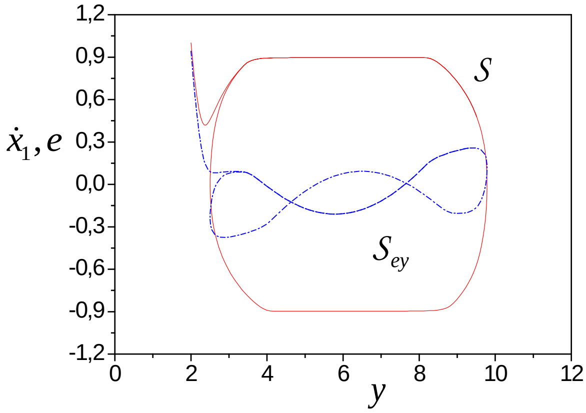


Fig.5. Frameworks $\mathcal{S}, \mathcal{S}_{ey}$

Fig. 5 shows that you can set $\hat{\varphi}_2(u) = \text{sat}(u)$. $J_y = [2;4] \vee [8.5;10]$ is an interval the decision-making about the nonlinearity. The application of the model (4) (framework $\mathcal{S}_{ey}$) is inefficient. Therefore, go to the analysis of $\dot{x}_2$ dependence on available variables.

Coefficients of determination between $\dot{x}_2$ and x_2, y are respectively equal $r^2_{x_2\dot{x}_2} = 0.995$, $r^2_{y\dot{x}_2} = 0.916$. We see that there is a relationship between $\dot{x}_2$ and x_2. Use the hierarchical immersion (HI) method to refine structural relationships. HI allows to step by step eliminating relationships in the system S_{ST} and gives the final estimate for nonlinearity. We found that the influence degree of the $|x_2|x_2$ on system properties is 97%. The framework $\mathcal{S}_{\varepsilon,|x_2|x_2}$ (Fig. 6) confirms the properties of the system S_{ST}.

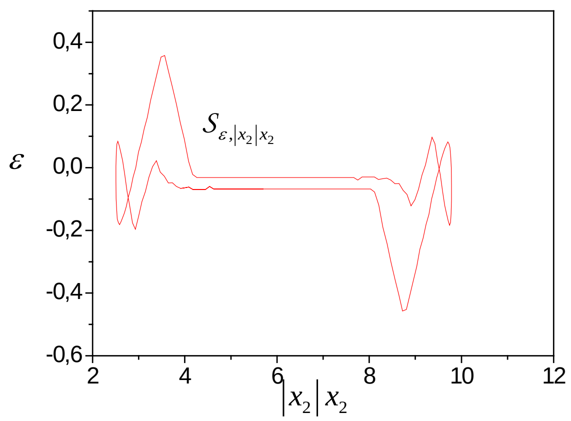


Fig.6. Framework $\mathcal{S}_{\varepsilon,|x_2|x_2}$

So, the analysis confirms the possibility of the system S_{ST} structural identification and its identifiability at the interval J_y. The model (4) application depends on the system structure (framework $\mathcal{S}_{ey}$). The general approach to the choice of the model structure not succeeds. The nonlinearity structure depends on the specifics of the system. This conclusion illustrates this example. It confirms the versatility and complexity of the considered problem. The system with several nonlinearities requires the development of proposed approaches.

Example 3. System for generating self-oscillations

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -gy_2 + k_0 y_5, \\ \dot{y}_3 = -\dfrac{1}{T_1} y_3 + \dfrac{1}{T_1} f_1(y_1), \\ \dot{y}_4 = -\dfrac{1}{T_2} y_4 + \dfrac{k_2}{T_2} y_2, \\ \dot{y}_5 = -\dfrac{1}{T_3} u - \dfrac{1}{T_3} f_3(y_3 + y_4), \end{cases}$$

where $[y_1, y_2]^T$ is state vector of a object; y_3, y_4 are output of gauges; y_5 is the output of a linear transducer amplifier with a linear actuator (feedback) (TA); $f_1(\cdot), f_3(\cdot)$ are saturation functions with dead zone; T_1, T_2, T_3 are time constants of elements; k_0, k_2 is gain; $g > 0$. The function $f_i(x)$ has the form

$$f_i(x) = \begin{cases} c, & if\ x \geq d_{2,i}, \\ 2(x - d_{1,i}), & if\ d_{1,i} < x < d_{2,i}, \\ 0, & if\ -d_{1,i} \leq x \leq d_{1,i}, \\ 2(x + d_{1,i}), & if\ -d_{1,i} < x, \\ -c, & if\ x < -d_{2,i}, \end{cases}$$

where $i = 1;3$, $c = 2$, $d_{1,1} = 0.5$, $d_{2,1} = 1.5$, $d_{1,3} = 0.25$, $d_{2,3} = 1.25$.

Difficulties in SI evaluating.

1. The signal $y_5(t)$ presence which is the actuator output of and the object input. R affects all processes in the system.

2. The indirect effect of variables on each other. This is a fundamental feature of systems with multiple nonlinearities. This feature levels the influence of some variables on system properties. Estimation of leveling is not always possible under uncertainty.

First, build a tree of relationships. The example relationships y_1, y_2 tree with other variables shown in Fig. 7. Markers highlight significant relationships that exceed the 80% level. Such the layered tree obtained for the system state vector.

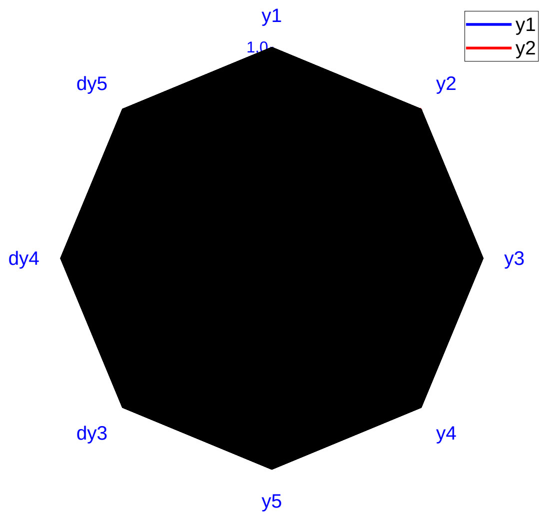


Fig.7. Layers graph for y_1 and y_2

Apply the approach described in section 4. The analysis showed that the object described by the linear equation (variables y_1, y_2). Variables y_1, y_5 effect on the variable y_3 (the amplifier-gauge 1 output), and variables y_2, y_4, $\dot{y}_5$ effected on variable y_3. The phase portrait the amplifier-gauge 1 showed in Fig. 8. We see that the amplifier-gauge 1 is nonlinear.

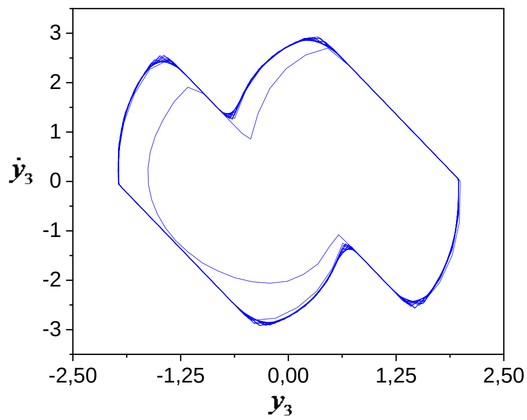


Fig.8. Phase portrait of the first gauge

Choose the model similar to (4), and variables to estimate the nonlinear function. Analyze the relationships for this element and obtain the model

$$\hat{\dot{y}}_3 = -0{,}778\dot{y}_5 - 0{,}0928\,,\ r^2_{\dot{y}_3,\dot{y}_5} = 0.69\,.$$

Introduce the error $\varepsilon_3 = \dot{y}_3 - \hat{\dot{y}}_3$ and the framework $S_{\varepsilon_3 y_1}$ described by the function $\gamma_{\varepsilon_3 y_1} : y_1 \to \varepsilon_3$ (Fig. 9).

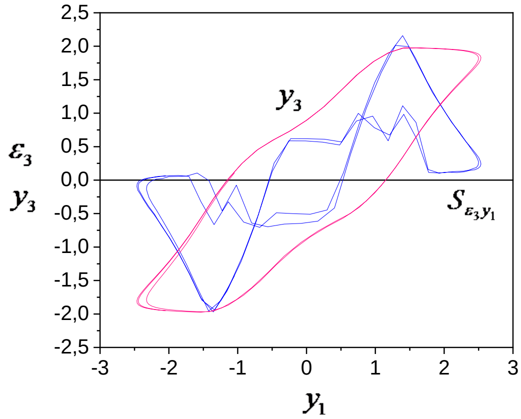


Fig.9. $S_{\varepsilon_3 y_1}$-framework

We see that the framework $S_{\varepsilon_3 y_1}$ is h_{δ_h}-identifiable. Diameters of the framework $S_{\varepsilon_3 y_1}$ are almost equal. $S_{\varepsilon_3 y_1}$ has a dead zone in the range [–0.5; 0.5] and growth in the segment [0.5; 1. 5]. Therefore, the nonlinearity has the form $f_1(x)$.

The next element is an amplifier-gauge 3 with the output y_4. Variables y_2 and $\dot{y}_3$ influence on y_4. $\dot{y}_3$ reflects the variable y_2 influence of object. The structural analysis showed that the framework $S_{\dot{y}_3 y_2}$ does not contain features, and S_{ey}-analog is an insignificant framework. Therefore, amplifier-gauge three does not contain nonlinearities.

Consider the last element with the output y_5. Variables y_3 and $\dot{y}_4$ effect on $\dot{y}_5$, and variables y_2, $\dot{y}_3$ influence on $\dot{y}_5$. Applying the model (secant) $\hat{\dot{y}}_5 = a_{53}\dot{y}_3 + b_{53}$ to the framework $S_{\dot{y}_5 \dot{y}_3}$ and the introduction of the misalignment $\varepsilon_5 = \dot{y}_5 - \hat{\dot{y}}_5$ gives the framework $S_{\varepsilon_5 y_4}$ that is described by the function $\gamma_{\varepsilon_5 y_4} : y_4 \to \varepsilon_5$ (Fig. 10). Fig. 10 shows the phase portrait S_{y_5} of this element.

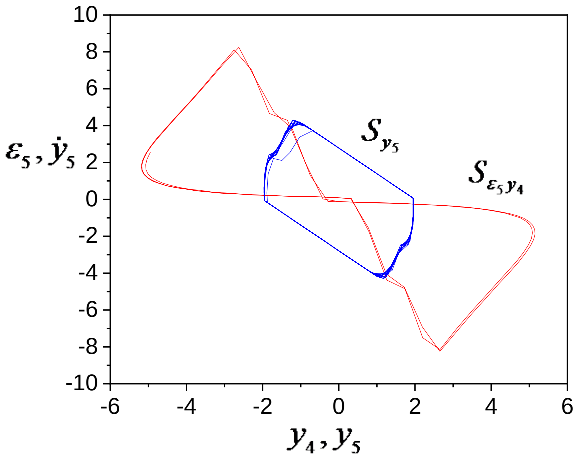


Fig.10. Phase portrait S_{y_5} and framework $S_{\varepsilon_5 y_4}$

We see (Fig. 10) that $S_{\varepsilon_5 y_4}$ is zero in the interval (-0.25; 0.25), then there is a linear growth of $S_{\varepsilon_5 y_4}$ by [0.25; 1.25], which coincides with f_3. This element is structurally identifiable by y_4. But this element is not identifiable by y_3.

So, we see that the possibility of structural identifiability of a nonlinear system depends on the interaction of its elements. Just the structural organization of the system determines the ability to solve the structural identifiability problem.

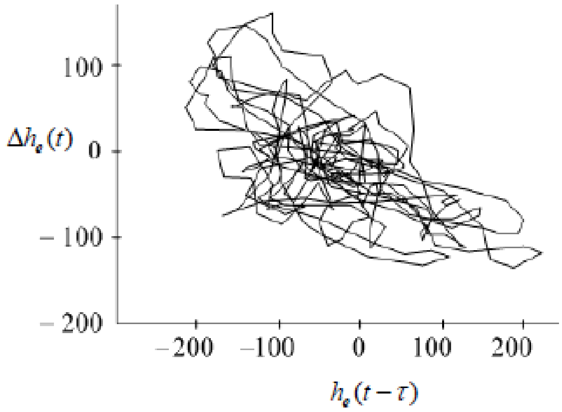
6. System Attractor Reconstruction

Reconstruction (restoration) of the phase portrait (PP) or a system attractor can be performed on the basis of time series analysis. The proof of this approach is given in [33], and the practical application is based on Wolf and Rosenstein algorithms [34, 35]. This problem can be interpreted as the system structure restoration task in the phase space. Many authors (see e.g. reviews in [36-38]) have studied this problem. Many procedures are heuristic [37]. The phase portrait construction depends on the choice of reconstruction optimum parameters. The main parameter is the choice of the time delay for new variables obtaining on the basis of the available time series. To solve this problem, various approaches (see references in [38, 39] are used: the autocorrelation and cross-correlation, the choice of the attractor shape, the method about the neighbor, and also the prediction statistics based on various models. Recommendations about the choice of the delay value estimation method are not provided. This is explained by the complexity and the variety of considered objects. The second problem is concatenated to the quality criteria choice [37] for the estimation of the PP reconstruction. Unfortunately, this problem has not obtained the final solution. Some recommendations for solving this problem are given in [37].

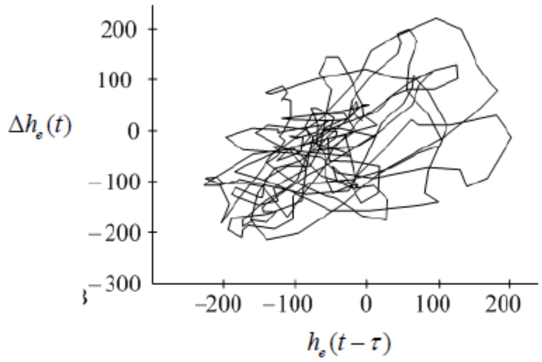
The choice of such structural parameter as the attractor dimension [36, 40, 41] is also an important task. An attempt to resolve the existing problems is made in [38]. In [38], a statistic for the choice of the delay value and the attractor dimension is proposed. It is shown that these statistics can be applied to the attractor creation for multidimensional systems. The problem of constructed attractor further analysis is not completed at this stage. As a rule, the designed attractor not always satisfies requirements of the researcher. It is not smooth. Therefore, smoothing various methods [36, 42] apply to obtain the smooth mapping.

Identification of the dynamic system was considered in [36, 37] on the basis of the obtained set of state space variables. This issue is discussed in the review [43] in more detail. Various approximation methods of the desired operator describing the system state are applied to the model design. The basis of identification methods is interpolation procedures decomposition of a nonlinear function on the specified basis, the application a spline-functions and neural networks, and also many other approaches [43].

Remark 4. As noted in [37], none of the considered identification methods is efficient. The major role is played by heuristics, the researcher experience and the prior information. This remark is true also for PP restoration methods [36]. As a rule, at first the data approximation is performed on the given class of basic functions. Then the phase portrait, topologically the equivalent to an initial system, is under construction. Further, unknown parameters are introduced in the obtained model that properties of the obtained mapping to improve. Various heuristics and procedures of additional information accounting on a system are applied for this purpose. Obtained models are very unwieldy and inconvenient for the application. Therefore, in [36] it is noted that the use of the complex models is not always justified in practical applications.



(a) left hemisphere



(b) right hemisphere

Fig.11. Attractors of native EEG for the "neuralgia" diagnosis

Example 3. Consider the native analysis process of electroencephalographic data (EEG). Experimental data are represented by a one-dimensional time series. Construct the phase portrait for the estimation of a patient condition. Apply the Takens theorem and the method for finding nearest false neighbors. Time series analysis showed that the time delay τ varies within certain limits [1] for different groups of patients. The choice of the phase space dimension is based on the analysis of the attractor correlation dimension D . D changes in the interval [5, 11] for all groups of patients. Apply these methods and get time series for constructing attractors that reflect the symmetric leads C3, C4-A2. The abscissa axis of the phase plane contains experimental values $h_e(t-\tau)$, where h_e is a voltage level of a native EEG; the axis of ordinates reflects the change $\Delta h_e(t)=h_e(t+\tau)-h_e(t-\tau)$. Such representation eliminates the delay τ effect of on attractor properties. Examples of the attractor reconstruction shown for a woman in Fig.11.

7. System Structure Choice with Lag Variables

Models with distributed lags (DL) are widely applied in various areas [44-48]. Independent and dependent variables can have the delay. The distributed lag accounting activates autocorrelation between variables [45] and the parameter identification process complicates. Various schemes of parameters approximation at DL [44, 47] apply to system parameters identification. The prior information is considered at the same time. Such an approach reduces the estimated parameter number of the system. Parametric schemes minimize the number of unknown parameters. The least-squares procedure and its modifications apply to the parameter estimation. Methods of the maximum length lag choice are considered. Statistics based on the analysis of residuals [45, 48] are the basis of the applied approaches. The Akai criteria and Bayes information criteria are used for decision-making on the model structure. The identification of the system structure and parameters was not examined under uncertainty.

Scheme choice of the model parameters approximation is bound with the performance of labour-consuming calculations under uncertainty. In [49] the approach to the structure DL choice based on the analysis of properties framework $\mathcal{S}_{k,e}^{v}$ is applied. Therefore, previously considered methods are not applicable for its analysis. The structure estimation of the system with DL is based on the analysis by means of secants [38].

Further, the estimation method of the DL system structure based on Lyapunov exponent identification is stated. This method is the development of the approach described in subsection 4.B. The direct transfer of results [24, 25] on the considered system class is impossible since these systems have the specifics.

Consider the system

$$y_n = A^T U_n + B^T X_n + \xi_n \tag{15}$$

where $y_n \in R$ is output; $U_n \in R^k$ is input vector which elements are limited extremely nondegenerate functions; $n \in J_N = [0, N]$ is discrete time, $N < \infty$; $X_n \in R^m$, $X_n = X\left(u_{i,n} \in U_n\right) = \left[u_{i,n-1}, u_{i,n-2}, \ldots, u_{i,n-m}\right]^T$ is the vector of distributed lags on $u_{i,n} \in U_n$; $A \in R^k$, $B \in R^m$ are constant parameter vectors; $\xi_n \in R$ is external disturbance, $|\xi_n| < \infty$ for all $n \in J_N$.

Let the informational set I_o for the system (13) containing the information on measured inputs and output on an interval J_N has the form

$$\mathrm{I}_o = \left\{U_n, y_n, n \in J_N\right\}. \tag{16}$$

Problem: estimate the vector X_n dimension based on data (14) analysis.

Remark 5. Here the case of lags availability on the input U_n is considered. If the output y_n contains lags, then the proposed approach allows to estimate the DL structure and in this case.

Analyze the effect of $u_{j,n}$, $j = \overline{1,k}$ on the output y_n. Determine by determination coefficient $r^2_{u_j,y}$ for everyone $u_{j,n-1}$. Introduce the number $\delta > 0$. Find such j that $r^2_{u_j,y} \geq \delta$ satisfied and designate $i = j$. So, the element of the vector U_n is determined. Form the vector $\tilde{U}_n \in R^{m-1}$ which does not contain the element $u_{i,n}$, for the lag estimation on $u_{i,n}$ and apply model

$$\hat{\tilde{y}}_n = \tilde{B}^T \tilde{U}_n \tag{7}$$

where $\tilde{B}^T \in R^{m-1}$ is the parameter vector.

The system (13) is not dynamic in the standard sense.

Assumption 1. Let the system (13) contain the variable $\pi_n = u_{j,n}$ which changes on the dynamic law

$$S_\pi : \pi_n = \sum_{i=1}^{h} \alpha_i \pi_{n-i} + \kappa \zeta_n \tag{18}$$

where α_i, κ are some numbers, $h < \infty$, $\zeta_n \in R$ is some limited function for all $n \in J_N$.

Let the system (16) be stable, i.e. $\alpha_i < 1$.

Definition 8. The systems (13) have π-steady state or π-state if such $j \geq 1$ exists that the variable $u_{j,n} \in U_n$ satisfies the equation (16).

Select the transient process (the system (16) general solution) for the application of LE to the S_π-system. Localise in (13) a space which the variable $\pi_n = u_{j,n}$ belongs, and π-steady state eliminate on the interval $J_g = J_N \setminus J_{\pi,N}$, where the interval corresponding the π-state in the S_π-system.

Consider the set I_o (14). Apply the model (15) on the interval $J_{\pi,N}$, where $J_{\pi,N}$ choose so that the coefficient of determination was maximal between $\hat{\tilde{y}}_n$ and $\hat{y}_n$. Next, calculate the error $e_n = y_n - \hat{\tilde{y}}_n$. Note that the variable e_n contains information about $u_{j,n}$.

Now the analysis reduced to the study of the discrete dynamic system properties with the output e_n. We obtain the system $\mathcal{SD}_e$ that is a prototype of the system (16).

The problem is reduced to LE estimation on the basis of the set $\mathrm{I}_e = \left\{e_n, n \in J_g\right\}$ analysis. It is the close to the attractor reconstruction problem of the dynamic system by the time set I_e. Reconstruction of the phase portrait (attractor) is based on the application of the Takens theorem [33]. F. Takens has proved that the new row $e_{d,n}$ based on lagging values e_n gives to the PP reconstruction problem solution. The obtained row $e_{d,n}$ describes the change the dynamics of the derivative variable e_n. Many procedures are proposed for choice of the delay interval [36]. It is supposed that trajectories of the dynamic system belong to the smooth manifold. It is necessary to notice that the delay interval choice problem did not obtain the final solution. Heuristic procedures, algorithms of approximation and smoothing time series are often used in practical applications. A priori information is important. After obtaining of the set $\left\{e_n,\ e_{d,n}, n=\overline{1,n_k}\right\}$, the problem solution of the design phase portrait which also is nontrivial [36, 37, 40] is necessary.

Remark 6. Smoothing algorithms are widely used in the attractor reconstruction problem. Smoothing procedures application to set I_e elements at the LE identification for the system (13) can lead to the loss of valuable information. Residual errors caused by disturbances ξ_n in (13) effect on properties of obtained LE estimations.

Use the formula (8) for the calculation of Lyapunov exponents. Considering remark 6, the detection of LE, but not their values we will evaluate.

Consider analogues of frameworks $\mathcal{S}_{k_{s,\rho}}$, $\mathcal{SK}_{\Delta k'_{s,\rho}}$ and $\mathcal{LSK}_{\Delta k'_{s,\rho}}$ defined at $t_n = n\tau$, where τ is the data measurement interval. Introduce the discrete analogue of the function (11)

$$b_n = \begin{cases} 1, if \;\; \Delta k'_{s,n} \geq 0 \\ -1, if \;\; \Delta k'_{s,n} < 0 \end{cases}. \tag{19}$$

where $b_n = b(n\tau)$, $\Delta k'_{s,n} = \Delta k'_s(n\tau)$.

Theorem 3 [25]. If the function b_n on the interval $\left[t_0, t^*\right] \subset \bar{J}_g$ $\left(t^* \leq \bar{t}\right)$ changes the sign h times, that the system (16) have the order h.

In [25] it is shown if the theorem 3 conditions are satisfied, then local minima of the framework $\mathcal{SK}_{\Delta k'_{s,\rho}}$ correspond to LE estimations of the system (16) in space $\left(k_{s,\rho}, \Delta k'_{s,\rho}\right)$.

Theorem 4. If conditions of the theorem 3 are satisfied and the framework $\mathcal{SK}_{\Delta k'_{s,\rho}}$ described by the function $\Gamma_{\Delta k'_{s,\rho}} : \mathrm{I}_{k_{s,\rho}} \to \mathrm{I}_{\Delta k'_{s,\rho}}$ has local minima on the plane $\left(k_{s,\rho}, \Delta k'_{s,\rho}\right)$, then S_π-system have π-state.

The proof of the theorem 4 is obvious. The local minima quantity corresponds to the lag structure of the system (13) on the variable $u_{j,n}$.

So, we have shown that the discrete informational set I_o modification based on the approach [25] allows extending the methodology of geometrical frameworks application to systems with the distributed lags of input variables.

Consider the identifiability problem of Lyapunov exponents. Let the vector U_n is limited constantly excited

$$\mathcal{PE}_\alpha : \; U_n U_n^T \geq \alpha I_k \tag{20}$$

for some $\alpha > 0$ and $\forall n \geq 0$ on the interval J_N, where $I_k \in R^{k \times k}$ is the unit matrix.

If (18) is satisfied, then we will write $U_n \in \mathcal{PE}_\alpha$. As shown in section 5, the fulfilment (18) is sufficient for the S_π-system π-state estimation. Parameters of the model (14) are identifiable since $\tilde{U}_n = U_n \setminus u_{i,n}$ and $\tilde{U}_n \in \mathcal{PE}_{\bar{\alpha}}$, $\bar{\alpha} > 0$. S_π-system with π-state corresponds to the system (12). Let the framework $\mathcal{LSK}_{\Delta k'_{s,\rho}}$ and the function b_n which on the interval $\left[t_0, t^*\right] \subset \bar{J}_g$ changes the sign h of times exist. Then the system has h Lyapunov exponents. Therefore, S_π-system is identifiable on the set $\mathcal{M}_{S_\pi}$ LE. So, it is true

Theorem 5. Let: 1) the vector U_n of the system (13) have property $U_n \in \mathcal{PE}_\alpha$; 2) the vector $\tilde{B}^T \in R^{m-1}$ of the model (15) is identifiable with $\tilde{U}_n \in \mathcal{PE}_{\bar{\alpha}}$; 3) the framework $\mathcal{LSK}_{\Delta k'_{s,\rho}}$ and the function b_n (17) satisfying theorems 3 conditions exist; 4) the S_π-system (16) have the π-state. Then the dynamic S_π-system (16) corresponding to the system (13) is identifiable on the Lyapunov exponent set.

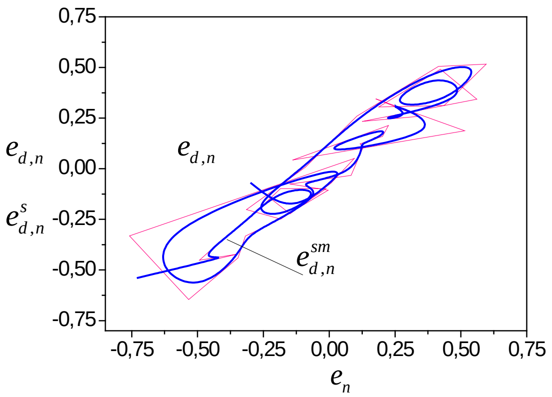


Fig.12. System (13) phase portrait of the with $k=3$ and $h=2$

Example 4. Consider the system with $k=3$ and $h=2$, $A=[0,7;3;3,5]^T$, $B=[0,4;0,45]^T$, $X_n=[u_{1,n-1},u_{1,n-2}]^T$. $u_{1,n}$ is obtained as the system (16) output with the input ζ_n, distributed to the normal law with the zero average and final dispersion. The set I_o (14) is generated for $n\in[1;60]$. The analysis of the set I_o has shown that lags are had by the variable $u_{1,n}$. Time series $\{e_n\}_{n=\overline{1;60}}$, $\{e_{d,n}\}_{n=\overline{1;60}}$ are formed. Apply the model (15)

$$\hat{\hat{y}}_n=[3;3,52][u_{2,n};u_{3,n}]^T+7,35, \tag{21}$$

which is obtained on the basis of LSM for $n\in[30;60]$. The determination coefficient of the model (19) is 0.99.

The system (16) phase portrait and its smoothed analogue (variable $e^{sm}_{d,n}$) are shown in Fig.12.

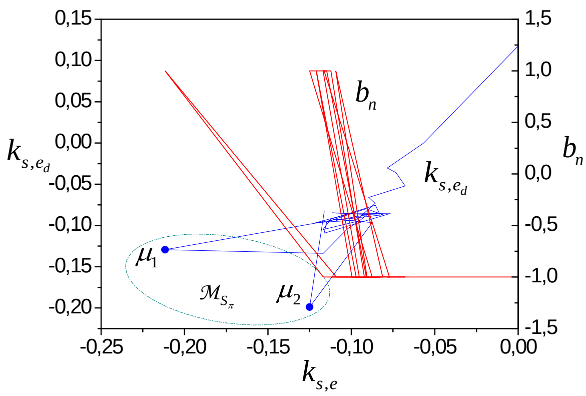


Fig.13. Lyapunov exponents estimations

Fig.12 is shown, processes in the S_π-system have non-smooth. Results of the lag structure estimation are presented in Fig. 7 where frameworks $\mathcal{S}_{k_{s,\rho}}$ and $\mathcal{LSK}_{\Delta k'_{s,\rho}}$ are shown. Designations in Fig. 13: μ_1,μ_2 are estimations of Lyapunov exponents, k_s are calculated on the basis of (8)

$$k_{s,e,n}=\frac{\rho(e_n)}{n\tau},\ k_{s,e_d,n}=\frac{\rho(e_{d,n})}{n\tau}.$$

The set $\mathcal{M}_{S_\pi}$ of Lyapunov exponent is showed also in Fig. 13.

The analysis of results shows that the system (16) describing the change $u_{1,n}$ have the order 2.

Example 5. Consider the control system for supplying cars to the Vladivostok transport hub (Russia). Study the case of 6 cars simultaneous giving from railway tracks on berth tracks. The maximum capacity of the hub is 175 cars. Let N_4 be the number of cars from the railway; N_5 be the number of cars received on the railway lines of the port. Determine

$\omega = N_5 - N_4$. The variable R reflects the current status of a hub and influences on the process of cars giving. The mathematical model for decision-making has the form

$$\hat{N}_{5,n} = f\left(\hat{N}_{5,n-1}, N_{4,n}, \omega_n\right) \tag{22}$$

where $\hat{N}_{5,n}$ is a model output in an instant n. The model (20) structure is described by an autoregressive equation of the first order. Apply the approach stated above and evaluate the effect ω. The system (16) has the first order to ω. Apply algorithms from section IV.B and evaluate the autoregression order. The model (20) has the form

$$\hat{N}_{5,n} = 1.06\hat{N}_{5,n-1} - 0.13\omega_{n-1} - 0,08N_{4,n} - 4.59\,. \tag{23}$$

The determination coefficient of the model (21) is 0.964. The simulation showed good predictive properties of the model (21).

So, modelling results confirm efficiency of the proposed approach to the lag structure estimation of the system (13).

8. Conclusion

The analysis of the concept “framework” application in identification problems is realized. It is showed that this concept is widely used in parametrical estimation problems. Concept “framework” can be interpreted as a frame, a structure, the system, a platform, the concept, the basis, the system of approaches. It is showed that framework can be used in two directions: i) the conceptual concept integrating the number of methods, approaches or procedures; ii) the mapping describing in the generalised form processes and properties in the system. The second direction is closer to methods which are applied in the qualitative theory of dynamic systems. In work, this approach is interpreted as the methodology based on the analysis of virtual geometrical frameworks. The main difference of geometrical frameworks from the approaches applied in the theory of chaos, accidents and so on consists that mathematical mapping (structure) is not postulated a priori, and it is determined on the basis of the available experimental data processing. The obtained geometrical framework is the analysis main object which allows the make the decision about properties and features of the system. The present review contains the analysis of virtual geometrical frameworks application directions in structural identification systems. There are five areas of the identification theory where this approach is applicable:

1. Structural identification of the nonlinear system.
2. Lyapunov exponent estimation of the system.
3. Structural identifiability of the nonlinear system.
4. The system phase portrait reconstruction on the time series.
5. The system structure estimation with lag variables.

We show that Lyapunov exponents can be applied to the decision-making on static systems structure with lag variables. The structural identifiability estimation of the system with lag variables is given.

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