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Application of Generalized Measure of 'Useful' R-norm Inaccuracy and Total Ambiguity

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Abstract: In the present paper, we introduce generalized measure of 'useful' R-norm inaccuracy having two parameters and its analogue 'useful' R-norm total ambiguity measure by merging together the concepts of probability, fuzziness, R-norm, 'useful' information and inaccuracy. Along with the basic properties, some other important properties of these two proposed measures are stated. These measures are generalizations of some well-known inaccuracy measures. Further, the monotonic behavior of the proposed 'useful' R-norm inaccuracy measures is studied, and the graphical overview is given. The measure of information improvement for both the measures is also obtained. Lastly, the application of 'useful' R-norm total ambiguity measure is presented in terms of multi-criteria decision making. For all the numerical calculations R software is used.

Index Terms: Fuzzy sets, inaccuracy measures, R-norm information measures, 'useful' information measures, total ambiguity measures, multi-criteria decision making.

1. Introduction

The concept of inaccuracy measure was first introduced by Kerridge [1] as an extension of Shannon's [2] measure of information. Kerridge [1] regarded inaccuracy as a quantity of measuring missing information. When the probabilities of the outcomes of a random experiment are stated by an experimenter, his statement may be imprecise in two ways. Firstly, his statement may be vague and secondly, he may have some incorrect information. The suitable measure for dealing with these two kinds of errors is Kerridge's [1] inaccuracy measure which is given as

$$I(P;Q) = -\sum_{i=1}^{n} p_i \log q_i \tag{1}$$

Here $P=(p_1, p_2, ..., p_n) \& Q=(q_1, q_2, ..., q_n)$ represent the true and asserted probability distributions associated with the events $\mathrm{Z}=(\mathrm{Z}_1, \mathrm{Z}_2, ..., \mathrm{Z}_n)$. Suppose the experimenter considers the importance of Z_i events (irrespective of their true and asserted probability) and assigns a non-negative number $u_i (>0)$ to each Z_i. u_i represents the importance of Z_i. In this regard, Hooda [3], defined the following 'useful' inaccuracy measure.

$$I(P;Q) = -\frac{\sum_{i=1}^{n} u_i p_i \log q_i}{\sum_{i=1}^{n} u_i p_i}. \tag{2}$$

In the context of fuzzy set theory, which was originally developed by Zadeh [4], inaccuracy measure is called total ambiguity measure. Corresponding to two fuzzy sets A & B, total ambiguity may be defined as the sum of fuzzy information measure of A and the fuzzy directed divergence measure of A from B. It is not symmetric in nature. Verma and Sharma [5] defined the fuzzy inaccuracy measure corresponding to (1) as

$$I(A;B) = -\frac{1}{n}\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_B(x_i) + (1-\mu_A(x_i))\log(1-\mu_B(x_i))\right]. \tag{3}$$

Hooda and Sharma [6] proposed the inaccuracy measure in the context of R-norm information measure (RIM) [7] as

$$I_R(P;Q) = \frac{R}{R-1}\left[\left(\sum_{i=1}^{n} p_i^R q_i^{1-R}\right)^{\frac{1}{R}} - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right]; R > 0(\neq 1)\cdot \tag{4}$$

Then, Hooda and Bajaj [8] proposed the total ambiguity measure of (4) as

$$I_R(A;B) = \frac{R}{R-1}\left[\sum_{i=1}^{n}\left\{1-\left(\mu_A^R(x_i)+(1-\mu_A(x_i))^R\right)^{\frac{1}{R}}\right\} + \sum_{i=1}^{n}\left\{\left(\mu_A^R(x_i)\mu_B^{1-R}(x_i)+(1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R}\right)^{\frac{1}{R}}-1\right\}\right];\ R>0(\neq 1). \tag{5}$$

In the present paper, we have generalized various important measures of ‘useful’ R-norm inaccuracy and ‘useful’ total ambiguity that is shown in Sub-Section *C* of Section II & Sub-Section *B* of Section III respectively. Further, the proposed ‘useful’ R-norm total ambiguity measure is successfully applied to MCDM technique.

Section wise break-up of the paper is described as: In the Section II the related work concerning the topic is given. This is followed by Section III in which we have proposed a new measure of ‘useful’ R-norm inaccuracy. Further, the properties, measure of information improvement, particular cases and the monotonic behaviour concerning the proposed measure are given in Sub-Sections *A*, *B*, *C* and *D* of Section III respectively. In Section IV, we have defined the fuzzy analogue of the measure presented in Section III along with its basic properties and particular cases that are shown in its subsequent Sub-Sections *A* and *B* respectively. Its Sub-Section *C* pertains to the introduction of R-norm fuzzy information improvement measure. In Sub-Section *D*, we have studied the monotonic behaviour of the ‘useful’ R-norm total ambiguity measure. In the last Sub-Section *E* of IV, we have presented the application of ‘useful’ R-norm total ambiguity measure. Finally, in Section V, conclusion of the paper is provided.

2. Related Work

Recently, authors like Verma and Sharma [9] proposed fuzzy inaccuracy measure and studied its application in terms of MCDM, Bhat et al. [10] developed noiseless coding theorems for generalized ‘useful’ fuzzy inaccuracy measure and in the following year, Bhat et al. [11] characterized a new generalized inaccuracy measure alongwith its average code-word length. Further, many others have proposed different measures of inaccuracy for varying situations.

3. Generalized ‘Useful’ R-Norm Inaccuracy Measure

Consider the ‘useful’ RIM defined by Sofi et al. [12]

$$H_R^{\alpha,\beta}(P;U) = \frac{R+\alpha-\beta}{R-\beta}\left[1-\left\{\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right\}^{\frac{\alpha}{R+\alpha-\beta}}\right];\ R>0(\neq 1); 0<\alpha,\beta\leq 1; R\neq\beta; u_i>0. \tag{6}$$

and the ‘useful’ R-norm directed divergence measure defined by Sofi et al. “unpublished” [13]

$$D_R^{\alpha,\beta}(P:Q;U) = \frac{R+\alpha-\beta}{\beta-R}\left[1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}} q_i^{1-\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right];\ R>0(\neq 1); 0<\alpha,\beta\leq 1; R>\beta\ \&\ u_i>0. \tag{7}$$

Corresponding to (6) and (7), we define the following ‘useful’ R-norm inaccuracy measure (RIAM) having two parameters α and β :

$$I_R^{\alpha,\beta}(P;Q;U)=\frac{R+\alpha-\beta}{R-\beta}\left[\left(\frac{\sum_{i=1}^{n}u_i p_i^{\frac{R+\alpha-\beta}{\alpha}} q_i^{1-\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n}u_i p_i}\right)^{\frac{\alpha}{R+\alpha-\beta}}-\left(\frac{\sum_{i=1}^{n}u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n}u_i p_i}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right];R>0(\neq 1);0<\alpha,\beta\leq 1;R>\beta\,\&\,u_i>0. \quad (8)$$

3.1. Properties of 'Useful' RIAM (8)

The 'useful' RIAM has the following properties:

1) Non-negativity i.e., $I_R^{\alpha,\beta}(P;Q;U)\geq 0$.
2) $H_R^{\alpha,\beta}(P;U)\leq I_R^{\alpha,\beta}(P;Q;U)$.
3) $I_R^{\alpha,\beta}(P;Q;U)$ is symmetric function of its arguments.
4) $I_R^{\alpha,\beta}(P;Q;U)$ has an infinite value if $q_i=0$, $p_i\neq 0\,\&\,u_i\neq 0$ for any i.
5) $I_R^{\alpha,\beta}(P;Q;U)$ has minimum value when $q_i=p_i\ \forall i$.
6) $I_R^{\alpha,\beta}(P;Q;U)=0$ if and only if $p_i=q_i=1$ for one value and $p_i=q_i=0$ for all other $i\,\&\,u_i\geq 0$.

With the help of following tables, the above properties are verified for the measure (8) by considering a hypothetical data.

Table 1. For Properties 1, 2 & 3

p_i	q_i	u_i	α	β	R	$H_R^{\alpha,\beta}(P;U)$	$I_R^{\alpha,\beta}(P;Q;U)$	$I_R^{\alpha,\beta}(P^S;Q^S;U^S)$
0.13	0.23	5	0.23	0.34	0.65	0.9182	1.7569	1.7569
0.03	0.11	2	0.45	0.51	70	1.1809	1.7127	1.7127
0.41	0.17	4	0.20	0.20	11	0.6002	3.0417	3.0417
0.15	0.30	1	0.92	0.85	100	0.5951	3.1142	3.1142
0.18	0.05	3	0.88	0.27	140	0.5934	3.1382	3.1382
0.10	0.14	6	0.15	0.95	13	0.5968	3.0890	3.0890

From Table 1, it is clear that

1) $I_R^{\alpha,\beta}(P;Q;U)>0$.
2) $I_R^{\alpha,\beta}(P;Q;U)>H_R^{\alpha,\beta}(P;U)$ and
3) The proposed 'useful' RIAM satisfies symmetry property, that is, $I_R^{\alpha,\beta}(P;Q;U)=I_R^{\alpha,\beta}(P^s;Q^s;U^s)$. Here, $I_R^{\alpha,\beta}(P^s;Q^s;U^s)$ represents the arrangement of elements of $I_R^{\alpha,\beta}(P;Q;U)$, in such a way that the one to one correspondence among the elements remains unchanged.

Table 2. Value of $I_R^{\alpha,\beta}(P;Q;U)$ when $q_i=0$, $p_i\neq 0\,\&\,u_i\neq 0$ for $i=3$

p_i	q_i	u_i	α	β	R	$I_R^{\alpha,\beta}(P;Q;U)$
0.13	0.23	5	0.23	0.34	0.65	∞
0.03	0.11	2				
0.41	0.00	4	0.45	0.51	70	∞
0.15	0.47	1				
0.18	0.05	3	0.92	0.85	100	∞
0.10	0.14	6				

It is clear from Table 2 that when $q_i=0$ for any i, (whatever be the values of $\alpha,\beta\,\&\,R$), we get $I_R^{\alpha,\beta}(P;Q;U)=\infty$.

Table 3. For Property 5

p_i	q_i	u_i	α	β	R	$I_R^{\alpha,\beta}(P;Q;U)$	$H_R^{\alpha,\beta}(P;U)$	$D_R^{\alpha,\beta}(P:Q;U)$
0.23	0.23	5	0.23	0.34	0.65	1.0726	1.0726	0.0
0.11	0.11	2						
0.17	0.17	4	0.45	0.51	70	0.7068	0. 7068	0.0
0.30	0.30	1						
0.05	0.05	3	0.92	0.85	100	0.7098	0. 7098	0.0
0.14	0.14	6						

We can see from the Table 3 that when $q_i = p_i \,\forall i$, divergence term becomes zero and thus $I_R^{\alpha,\beta}(P;Q;U) = H_R^{\alpha,\beta}(P;U)$. This gives the minimum value of $I_R^{\alpha,\beta}(P;Q;U)$.

Property 6: $I_R^{\alpha,\beta}(P;Q;U) = 0$ if and only if $p_i = q_i = 1$ for one value and $p_i = q_i = 0 \,\forall\, i \;\&\; u_i \geq 0$.

Suppose $p_i = q_i = 1$ for $i = 1$ and for $i = 2,3,...,n$; $p_i = q_i = 0$, we have

$$I_R^{\alpha,\beta}(P;Q;U) = \frac{R+\alpha-\beta}{R-\beta}\left[\left(\frac{u_1 p_1^{\frac{R+\alpha-\beta}{\alpha}} q_1^{1-\frac{R+\alpha-\beta}{\alpha}}}{u_1 p_1}\right)^{\frac{\alpha}{R+\alpha-\beta}} - \left(\frac{u_1 p_1^{\frac{R+\alpha-\beta}{\alpha}}}{u_1 p_1}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right] + \frac{R+\alpha-\beta}{R-\beta}\left[\left(\frac{\sum_{i=2}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}} q_i^{1-\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=2}^{n} u_i p_i}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right.$$

$$\left. - \left(\frac{\sum_{i=2}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=2}^{n} u_i p_i}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right]. \quad = \frac{R+\alpha-\beta}{R-\beta}\left[\left(\frac{u_1}{u_1}\right)^{\frac{\alpha}{R+\alpha-\beta}} - \left(\frac{u_1}{u_1}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right] + 0, \text{ since } p_1 = q_1 = 1. \Rightarrow I_R^{\alpha,\beta}(P;Q;U) = 0. \tag{9}$$

Hence, the result follows.

3.2. Measure of Information Improvement

The measure of information improvement (MII) was given by Theil [14] as

$$D(P;Q;U) - D(P;R;U). \tag{10}$$

where *P* and *Q* are the respective observed and predicted probability distributions of a random variable and *R* represents the revised probability distribution of *Q*. Corresponding to the 'useful' R-norm DDM defined in (7), we define the following 'useful' R-norm MII as

$$D_R^{\alpha,\beta}(P;Q;U) - D_R^{\alpha,\beta}(P;R;U). = \frac{R+\alpha-\beta}{\beta-R}\left[1 - \left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}} q_i^{1-\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right]$$

$$-\frac{R+\alpha-\beta}{\beta-R}\left[1 - \left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}} r_i^{1-\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right]; R > 0(\neq 1); 0 < \alpha, \beta \leq 1; R > \beta \;\&\; u_i > 0. \tag{11}$$

3.3. Particular Cases of 'Useful' RIAM Defined in (8)

- For $u_i = 1$, the proposed measure (8) reduces to the RIAM defined by Peerzada et al. [15].

- For $\alpha = 1, \beta = 1 \,\&\, u_i = 1$, the 'useful' RIAM (8) reduces to (4).
- For $\alpha = 1, \beta = 1 \,\&\, R \rightarrow 1$, the 'useful' RIAM (8) reduces to (2).
- For $\alpha = 1, \beta = 1, u_i = 1 \,\&\, R \rightarrow 1$, 'useful' RIAM (8) reduces to (1).

3.4. Monotone Behaviour of 'Useful' RIAM Defined in (8)

We study the monotonic nature of the proposed measures in the given limits of R, α and β. We take two probability distributions $P \,\&\, Q$: $P = (0.41, 0.13, 0.10, 0.18, 0.15, 0.03)$, $Q = (0.23, 0.05, 0.14, 0.30, 0.17, 0.11)$ with utility distribution $U = (5, 2, 4, 1, 3, 6)$ and $n = 6$. The results are given in the following tables by taking various values of R, α and β.

Table 4. Values of Measure (8) for Fixed $\alpha \,\&\, \beta$

R	0.95	7	20	47	62	100	120	140
$I_R^{0.96,0.20}(P;Q;U)$	1.3569	1.5086	1.8672	2.0447	2.0787	2.1201	2.1316	2.1398
$I_R^{0.59,0.59}(P;Q;U)$	1.3865	1.6502	1.9793	2.0983	2.1202	2.1465	2.1537	2.1589
$I_R^{0.62,0.81}(P;Q;U)$	1.7115	2.2621	2.4266	2.4791	2.4886	2.5000	2.5030	2.5053

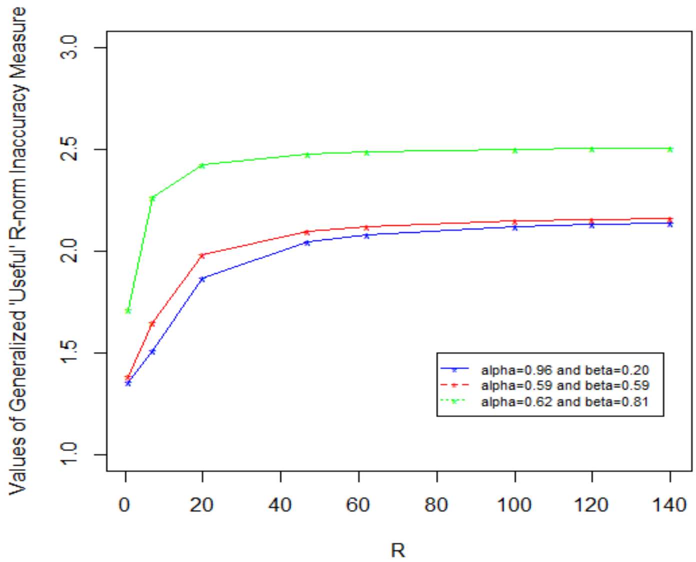

Fig.1. Graphical Overview of Measure (8) for Fixed Alpha and Beta

From Table 4, we can clearly see that as we increase the value of R and keep $\alpha \,\&\, \beta$ fixed; the 'useful' RIAM defined in (8) shows an increasing trend. Although, the value of measure (8) changes if we alter the values of $\alpha \,\&\, \beta$ but the trend (that is increasing) remains the same. This increasing nature of measure (8) with respect to varying R is depicted in the Fig. 1 by taking values of $\alpha \,\&\, \beta$ as $(0.96, 0.20), (0.59, 0.59) \,\&\, (0.62, 0.81)$ respectively.

Table 5. Values of Measure (8) for Fixed R and β

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$I_3^{\alpha,0.02}(P;Q;U)$	1.9590	1.7636	1.6182	1.5235	1.4623	1.4207	1.3910	1.3692	1.3528	1.3404
$I_8^{\alpha,0.89}(P;Q;U)$	2.0889	1.9941	1.9051	1.8223	1.7472	1.6817	1.6261	1.5798	1.5415	1.5097
$I_{23}^{\alpha,0.52}(P;Q;U)$	2.1573	2.1253	2.0939	2.0631	2.0330	2.0035	1.9745	1.9461	1.9183	1.8911

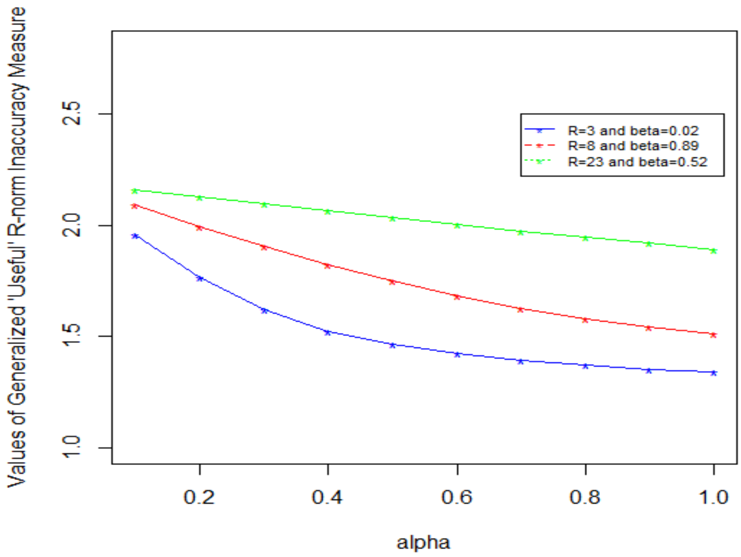

Fig.2. Graphical Overview of Measure (8) for Fixed R and Beta

From Table 5, we can easily state that as the value of α increases (R and β are fixed), measure (8) decreases. This relation exists for different possible values of R and β. Thus, there is a negative relation between α and the measure (8). This relation is depicted in the Fig. 2 by taking values of R and β as $(3,0.02)$, $(8,0.89)$ & $(23,0.52)$ respectively.

Table 6. For Fixed R and α

β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$I_{2}^{0.12,\beta}(P;Q;U)$	1.7844	1.7659	1.7459	1.7243	1.7011	1.6760	1.6492	1.6206	1.5903	1.5585
$I_{11}^{0.15,\beta}(P;Q;U)$	1.8772	1.8746	1.8720	1.8693	1.8666	1.8639	1.8611	1.8582	1.8553	1.8524
$I_{23}^{0.98,\beta}(P;Q;U)$	1.9014	1.9002	1.8991	1.8979	1.8968	1.8955	1.8943	1.8931	1.8919	1.8907

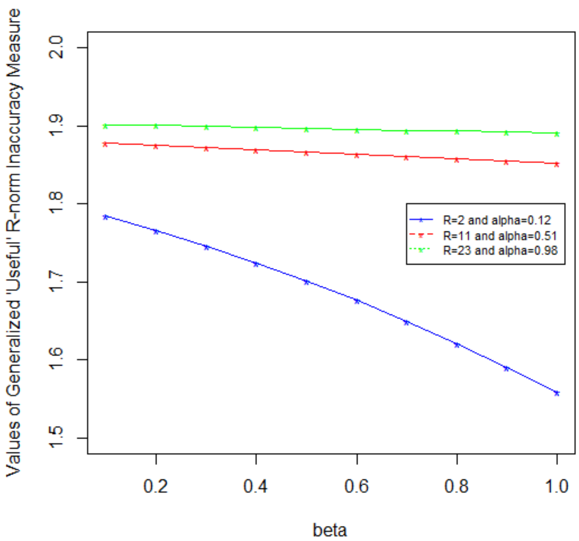

Fig.3. Graphical Overview of Measure (8) for Fixed R and Beta

From Table 6, we can easily state that as the value of β increases (R and α are fixed), the value of measure (8) decreases. Thus, there is a negative relation between β and the measure (8). This relation is depicted in the Fig. 3 by taking values of R and α as $(2,0.12),(11,0.51)$ & $(23,0.98)$ respectively.

4. Generalized Measure of 'Useful' R-Norm Total Ambiguity

Consider the 'useful' RFIM defined by Sofi et al. [16]

$$H_R^{\alpha,\beta}(A;U)=\frac{R+\alpha-\beta}{R-\beta}\left[\frac{n\sum_{i=1}^{n}u_i\left[1-\left\{\mu_A^{\frac{R+\alpha-\beta}{\alpha}}(x_i)+\left(1-\mu_A(x_i)\right)^{\frac{R+\alpha-\beta}{\alpha}}\right\}^{\frac{\alpha}{R+\alpha-\beta}}\right]}{\sum_{i=1}^{n}u_i}\right];\ R>0(\neq 1);\ 0<(\alpha,\beta)\leq 1\ \&u_i>0 \quad (12)$$

and the 'useful' R-norm fuzzy directed divergence measure defined by Sofi et al. "unpublished" [13]:

$$D_R^{\alpha,\beta}(A,B;U)=\frac{R+\alpha-\beta}{\beta-R}\left[\frac{n\sum_{i=1}^{n}u_i}{\sum_{i=1}^{n}u_i}\left\{1-\left(\mu_A^{\frac{R+\alpha-\beta}{\alpha}}(x_i)\mu_B^{1-\frac{R+\alpha-\beta}{\alpha}}(x_i)+\left(1-\mu_A(x_i)\right)^{\frac{R+\alpha-\beta}{\alpha}}\left(1-\mu_B(x_i)\right)^{1-\frac{R+\alpha-\beta}{\alpha}}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right\}\right];$$

$$R>0(\neq 1);0<\alpha,\beta\leq 1;(R+\alpha)>\beta\,\&u_i>0. \quad (13)$$

Corresponding to (12) and (13), we define the following 'useful' R-norm total ambiguity (or fuzzy inaccuracy) measure (RTAM):

$$I_R^{\alpha,\beta}(A;B;U)=n\frac{R+\alpha-\beta}{R-\beta}\left[\frac{\sum_{i=1}^{n}u_i}{\sum_{i=1}^{n}u_i}\left\{\left(\mu_A^{\frac{R+\alpha-\beta}{\alpha}}(x_i)\mu_B^{1-\frac{R+\alpha-\beta}{\alpha}}(x_i)+\left(1-\mu_A(x_i)\right)^{\frac{R+\alpha-\beta}{\alpha}}\left(1-\mu_B(x_i)\right)^{1-\frac{R+\alpha-\beta}{\alpha}}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right\}\right.$$

$$\left.-\frac{\sum_{i=1}^{n}u_i\left\{\left(\mu_A^{\frac{R+\alpha-\beta}{\alpha}}(x_i)+\left(1-\mu_A(x_i)\right)^{\frac{R+\alpha-\beta}{\alpha}}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right\}}{\sum_{i=1}^{n}u_i}\right];\ R>0(\neq 1);0<\alpha,\beta\leq 1;(R+\alpha)>\beta\,\&u_i>0. \quad (14)$$

4.1. *Properties of 'Useful' RTAM (14)*

1) $I_R^{\alpha,\beta}(A;B;U)=0$ if and only if either $\mu_A(x_i)=\mu_B(x_i)=0$ or $\mu_A(x_i)=\mu_B(x_i)=1\ \forall x_i\in X; i=1,2,...,n$.

2) $I_R^{\alpha,\beta}(A;B;U)>0$.

3) $I_R^{\alpha,\beta}(A;B;U)$ is a symmetric function of its arguments.

4) For any two fuzzy sets $A\,\&\,B$, $I_R^{\alpha,\beta}(A;B;U)\geq H_R^{\alpha,\beta}(A;U)$ with equality if and only if $\mu_A(x_i)=\mu_B(x_i)$.

For property 1, let's assume $\mu_A(x_i)=\mu_B(x_i)=0$. Thus, we have

$$I_R^{\alpha,\beta}(A;B;U)=n\frac{R+\alpha-\beta}{R-\beta}\left[\frac{\sum_{i=1}^{n}u_i\left\{(0+1)^{\frac{\alpha}{R+\alpha-\beta}}\right\}}{\sum_{i=1}^{n}u_i}-\frac{\sum_{i=1}^{n}u_i\left\{\left(0+(1)^{\frac{R+\alpha-\beta}{\alpha}}\right)\right\}^{\frac{\alpha}{R+\alpha-\beta}}}{\sum_{i=1}^{n}u_i}\right].=n\frac{R+\alpha-\beta}{R-\beta}\left[\frac{\sum_{i=1}^{n}u_i}{\sum_{i=1}^{n}u_i}-\frac{\sum_{i=1}^{n}u_i}{\sum_{i=1}^{n}u_i}\right].=0. \quad (15)$$

Similarly, if $\mu_A(x_i)=\mu_B(x_i)=1$, then $I_R^{\alpha,\beta}(A;B;U)=0$.

Conversely, suppose $I_R^{\alpha,\beta}(A;B;U)=0$, then

$$\left[\frac{\sum_{i=1}^{n}u_i\left\{\left(\mu_A^{\frac{R+\alpha-\beta}{\alpha}}(x_i)\mu_B^{1-\frac{R+\alpha-\beta}{\alpha}}(x_i)+(1-\mu_A(x_i))^{\frac{R+\alpha-\beta}{\alpha}}(1-\mu_B(x_i))^{1-\frac{R+\alpha-\beta}{\alpha}}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right\}}{\sum_{i=1}^{n}u_i}-\frac{\sum_{i=1}^{n}u_i\left\{\left(\mu_A^{\frac{R+\alpha-\beta}{\alpha}}(x_i)+(1-\mu_A(x_i))^{\frac{R+\alpha-\beta}{\alpha}}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right\}}{\sum_{i=1}^{n}u_i}\right]=0.$$

$$\Rightarrow\sum_{i=1}^{n}u_i\left\{\left(\mu_A^{\frac{R+\alpha-\beta}{\alpha}}(x_i)\mu_B^{1-\frac{R+\alpha-\beta}{\alpha}}(x_i)+(1-\mu_A(x_i))^{\frac{R+\alpha-\beta}{\alpha}}(1-\mu_B(x_i))^{1-\frac{R+\alpha-\beta}{\alpha}}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right\}=\sum_{i=1}^{n}u_i\left\{\left(\mu_A^{\frac{R+\alpha-\beta}{\alpha}}(x_i)+(1-\mu_A(x_i))^{\frac{R+\alpha-\beta}{\alpha}}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right\}. \quad (16)$$

The relation (16) holds only if $\mu_A(x_i)=\mu_B(x_i)=0$ or $\mu_A(x_i)=\mu_B(x_i)=1$.

Hence property 1 is proved.

The properties 2, 3 and 4 for the measure (14) are verified with the help of following tables by considering a hypothetical data.

Table 7. For Verification of Properties 2 and 3

$\mu_A(x_i)$	$\mu_B(x_i)$	μ_i	α	β	R	$H_R^{\alpha,\beta}(A;U)$	$I_R^{\alpha,\beta}(A;B;U)$	$I_R^{\alpha,\beta}(A^s;B^s;U^s)$
0.65	0.42	5	0.23	0.34	0.65	2.2702	11.2197	11.2197
0.23	0.28	2	0.45	0.51	70	1.5298	22.5865	22.5865
0.82	0.05	4	0.20	0.20	11	1.5481	22.1032	22.1032
0.44	0.90	1	0.92	0.85	100	1.5341	22.4718	22.4718
0.97	0.73	3	0.88	0.27	140	1.5295	22.5938	22.5938
0.31	0.61	6	0.15	0.95	13	1.5389	22.3438	22.3438

From Table 7, it is clear that

- $I_R^{\alpha,\beta}(A;B;U)>0$.
- $I_R^{\alpha,\beta}(A;B;U)>H_R^{\alpha,\beta}(A;U)$ and
- The proposed 'useful' RTAM satisfies symmetry property, that is, $I_R^{\alpha,\beta}(A;B;U)=I_R^{\alpha,\beta}(A^s;B^s;U^s)$. Here, $I_R^{\alpha,\beta}(A^s;B^s;U^s)$ represents the arrangement of elements of $I_R^{\alpha,\beta}(A;B;U)$, in such a way that the one to one correspondence among the elements remains unchanged.

Table 8. Value of $I_R^{\alpha,\beta}(A;B;U)$ when $\mu_A(x_i)=\mu_B(x_i)$

$\mu_A(x_i)$	$\mu_B(x_i)$	μ_i	α	β	R	$H_R^{\alpha,\beta}(A;U)$	$I_R^{\alpha,\beta}(A;B;U)$	$D_R^{\alpha,\beta}(A;B;U)$
0.65	0.65	5	0.23	0.34	0.65	2.2702	2.2702	0.0
0.23	0.23	2	0.45	0.51	70	1.5298	1.5298	0.0
0.82	0.82	4	0.20	0.20	11	1.5481	1.5481	0.0
0.44	0.44	1	0.92	0.85	100	1.5341	1.5341	0.0
0.97	0.97	3	0.88	0.27	140	1.5295	1.5295	0.0
0.31	0.31	6	0.15	0.95	13	1.5389	1.5389	0.0

From Table 8, we conclude that when $\mu_A(x_i)=\mu_B(x_i)$, $I_R^{\alpha,\beta}(A;B;U)=H_R^{\alpha,\beta}(A;U)$ and the error term vanishes.

4.2. Particular Cases of 'Useful' RTAM (14)

- For $u_i=1$, the 'useful' RTAM (14) reduces to the R-norm fuzzy inaccuracy measure defined by Peerzada et al. [15].
- For $\alpha=1,\beta=1\&u_i=1$, the 'useful' RTAM (14) reduces to (5).
- For $\alpha=1,\beta=1,u_i=1\&R\to 1$, the 'useful' RTAM (14) reduces to (3).

Theorem I

1) $I_R^{\alpha,\beta}(A\cup B;A\cap B;U)+I_R^{\alpha,\beta}(A\cap B;A\cup B;U)$

$$=I_R^{\alpha,\beta}(A;B;U)+I_R^{\alpha,\beta}(B;A;U). \quad (17)$$

2) $I_R^{\alpha,\beta}(A\cup B;C;U)+I_R^{\alpha,\beta}(A\cap B;C;U)$

$$=I_R^{\alpha,\beta}(A;C;U)+I_R^{\alpha,\beta}(B;C;U). \tag{18}$$

3) $I_R^{\alpha,\beta}(A;B\cup C;U)+I_R^{\alpha,\beta}(A;B\cap C;U)$

$$=I_R^{\alpha,\beta}(A;B;U)+I_R^{\alpha,\beta}(A;C;U). \tag{19}$$

For proving theorem I, we define $\mathrm{X}=\{x_1,x_2,...,x_n\}$ as universe of discourse. Any fuzzy set A is defined as $A=\{(x_i,\mu_A(x_i));x_i\in\mathrm{X}\}$ where $\mu_A(x_i)$ represents the membership function of A.

$A\cup B$ & $A\cap B$ are defined as:

- $A\cup B=\{\langle x_i,\mu_A(x_i)\Delta\mu_B(x_i)\rangle;x_i\in\mathrm{X}\}$.
- $A\cap B=\{\langle x_i,\mu_A(x_i)\nabla\mu_B(x_i)\rangle;x_i\in\mathrm{X}\}$.

where $\Delta\,\&\,\nabla$ respectively represent the maximum and minimum operators.

Also, assume $t=\dfrac{R+\alpha-\beta}{R-\beta};r=\dfrac{R+\alpha-\beta}{\alpha};\;\; x_A=\mu_A(x_i);\;x_B=\mu_B(x_i)\&\;x_C=\mu_C(x_i)$. Also, X is separated in two parts X_1 and X_2 as

$$\mathrm{X}_1=\{x:x_i\in\mathrm{X},\mu_A(x_i)\geq\mu_B(x_i)\}\&\;\;\mathrm{X}_2=\{x:x_i\in\mathrm{X},\mu_A(x_i)<\mu_B(x_i)\}. \tag{20}$$

Thus, we can write (14) as:

$$I_R^{\alpha,\beta}(A;B;U)=nt\left[\frac{\sum_{i=1}^n u_i\left\{\left(x_A^r x_B^{1-r}+(1-x_A)^r(1-x_B)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^n u_i}-\frac{\sum_{i=1}^n u_i\left\{\left(x_A^r+(1-x_A)^r\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^n u_i}\right]. \tag{21}$$

Now,

1) $I_R^{\alpha,\beta}(A\cup B;A\cap B;U)+I_R^{\alpha,\beta}(A\cap B;A\cup B;U)\;=I_R^{\alpha,\beta}(A;B;U)+I_R^{\alpha,\beta}(B;A;U)$.

Proof: Consider

$$I_R^{\alpha,\beta}(A\cup B;A\cap B;U)=nt\left[\frac{\sum_{i=1}^n u_i\left\{\left(x_{A\cup B}^r x_{A\cap B}^{1-r}+(1-x_{A\cup B})^r(1-x_{A\cap B})^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^n u_i}-\frac{\sum_{i=1}^n u_i\left\{\left(x_{A\cup B}^r+(1-x_{A\cup B})^r\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^n u_i}\right].$$

$$=nt\left[\frac{\sum_{x_i\in\mathrm{X}_1} u_i\left\{\left(x_A^r x_B^{1-r}+(1-x_A)^r(1-x_B)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in\mathrm{X}_1} u_i}-\frac{\sum_{x_i\in\mathrm{X}_1} u_i\left\{\left(x_A^r+(1-x_A)^r\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in\mathrm{X}_1} u_i}\right]$$

$$+nt\left[\frac{\sum_{x_i\in\mathrm{X}_2} u_i\left\{\left(x_B^r x_A^{1-r}+(1-x_B)^r(1-x_A)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in\mathrm{X}_2} u_i}-\frac{\sum_{x_i\in\mathrm{X}_2} u_i\left\{\left(x_B^r+(1-x_B)^r\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in\mathrm{X}_2} u_i}\right]. \tag{22}$$

Now,

$$I_R^{\alpha,\beta}\left(A\cap B;A\cup B;U\right)=nt\left[\frac{\sum_{i=1}^{n}u_i\left\{\left(x_{A\cap B}^{r}x_{A\cup B}^{1-r}+\left(1-x_{A\cap B}\right)^{r}\left(1-x_{A\cup B}\right)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}-\frac{\sum_{i=1}^{n}u_i\left\{\left(x_{A\cap B}^{r}+\left(1-x_{A\cap B}\right)^{r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}\right].$$

$$=nt\left[\frac{\sum_{x_i\in X_1}u_i\left\{\left(x_B^{r}x_A^{1-r}+\left(1-x_B\right)^{r}\left(1-x_A\right)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_1}u_i}-\frac{\sum_{x_i\in X_1}u_i\left\{\left(x_B^{r}+\left(1-x_B\right)^{r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_1}u_i}\right]$$

$$+nt\left[\frac{\sum_{x_i\in X_2}u_i\left\{\left(x_A^{r}x_B^{1-r}+\left(1-x_A\right)^{r}\left(1-x_B\right)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_2}u_i}-\frac{\sum_{x_i\in X_2}u_i\left\{\left(x_A^{r}+\left(1-x_A\right)^{r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_2}u_i}\right]. \tag{23}$$

Adding (22) and (23), we get

$$I_R^{\alpha,\beta}\left(A\cup B;A\cap B;U\right)+I_R^{\alpha,\beta}\left(A\cap B;A\cup B;U\right)=nt\left[\frac{\sum_{i=1}^{n}u_i\left\{\left(x_A^{r}x_B^{1-r}+\left(1-x_A\right)^{r}\left(1-x_B\right)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}-\frac{\sum_{i=1}^{n}u_i\left\{\left(x_A^{r}+\left(1-x_A\right)^{r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}\right]$$

$$+\,nt\left[\frac{\sum_{i=1}^{n}u_i\left\{\left(x_B^{r}x_A^{1-r}+\left(1-x_B\right)^{r}\left(1-x_A\right)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}-\frac{\sum_{i=1}^{n}u_i\left\{\left(x_B^{r}+\left(1-x_B\right)^{r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}\right].=I_R^{\alpha,\beta}\left(A;B;U\right)+I_R^{\alpha,\beta}\left(B;A;U\right). \tag{24}$$

Hence, the result.

2) $I_R^{\alpha,\beta}(A\cup B;C;U)+I_R^{\alpha,\beta}(A\cap B;C;U)=I_R^{\alpha,\beta}(A;C;U)+I_R^{\alpha,\beta}(B;C;U)$
Proof: Consider

$$I_R^{\alpha,\beta}\left(A\cup B;C;U\right)+I_R^{\alpha,\beta}\left(A\cap B;C;U\right)=nt\left[\frac{\sum_{i=1}^{n}u_i\left\{\left(x_{A\cup B}^{r}x_C^{1-r}+\left(1-x_{A\cup B}\right)^{r}\left(1-x_C\right)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}-\frac{\sum_{i=1}^{n}u_i\left\{\left(x_{A\cup B}^{r}+\left(1-x_{A\cup B}\right)^{r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}\right]$$

$$+nt\left[\frac{\sum_{i=1}^{n}u_i\left\{\left(x_{A\cap B}^{r}x_C^{1-r}+\left(1-x_{A\cap B}\right)^{r}\left(1-x_C\right)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}-\frac{\sum_{i=1}^{n}u_i\left\{\left(x_{A\cap B}^{r}+\left(1-x_{A\cap B}\right)^{r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}\right].$$

$$=\left\{nt\left[\frac{\sum_{x_i\in X_1}u_i\left\{\left(x_A^{r}x_C^{1-r}+\left(1-x_A\right)^{r}\left(1-x_C\right)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_1}u_i}-\frac{\sum_{x_i\in X_1}u_i\left\{\left(x_A^{r}+\left(1-x_A\right)^{r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_1}u_i}\right]\right.$$

$$\left.+nt\left[\frac{\sum_{x_i\in X_2}u_i\left\{\left(x_B^{r}x_C^{1-r}+\left(1-x_B\right)^{r}\left(1-x_C\right)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_2}u_i}-\frac{\sum_{x_i\in X_2}u_i\left\{\left(x_B^{r}+\left(1-x_B\right)^{r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_2}u_i}\right]\right\}$$

$$+\left\{nt\left[\frac{\sum_{x_i\in X_1}u_i\left\{\left(x_B^r x_C^{1-r}+(1-x_B)^r(1-x_C)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_1}u_i}-\frac{\sum_{x_i\in X_1}u_i\left\{\left(x_B^r+(1-x_B)^r\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_1}u_i}\right]\right.$$

$$\left.+\left[\frac{\sum_{x_i\in X_2}u_i\left\{\left(x_A^r x_C^{1-r}+(1-x_A)^r(1-x_C)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_2}u_i}-\frac{\sum_{x_i\in X_2}u_i\left\{\left(x_A^r+(1-x_A)^r\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_2}u_i}\right]\right\}.$$

$$=nt\left[\frac{\sum_{i=1}^{n}u_i\left\{\left(x_A^r x_C^{1-r}+(1-x_A)^r(1-x_C)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}-\frac{\sum_{i=1}^{n}u_i\left\{\left(x_A^r+(1-x_A)^r\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}\right]$$

$$+nt\left[\frac{\sum_{i=1}^{n}u_i\left\{\left(x_B^r x_C^{1-r}+(1-x_B)^r(1-x_C)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}-\frac{\sum_{i=1}^{n}u_i\left\{\left(x_B^r+(1-x_B)^r\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}\right].$$

$$=I_R^{\alpha,\beta}(A;C;U)+I_R^{\alpha,\beta}(B;C;U). \tag{25}$$

Hence, the result.

3) $I_R^{\alpha,\beta}(A;B\cup C;U)+I_R^{\alpha,\beta}(A;B\cap C;U)=I_R^{\alpha,\beta}(A;B;U)+I_R^{\alpha,\beta}(A;C;U)$
Consider

$$I_R^{\alpha,\beta}(A;B\cup C;U)+I_R^{\alpha,\beta}(A;B\cap C;U)=nt\left[\frac{\sum_{i=1}^{n}u_i\left\{\left(x_A^r x_{B\cup C}^{1-r}+(1-x_A)^r(1-x_{B\cup C})^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}-\frac{\sum_{i=1}^{n}u_i\left\{\left(x_A^r+(1-x_A)^r\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}\right]$$

$$+nt\left[\frac{\sum_{i=1}^{n}u_i\left\{\left(x_A^r x_{B\cap C}^{1-r}+(1-x_A)^r(1-x_{B\cap C})^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}-\frac{\sum_{i=1}^{n}u_i\left\{\left(x_A^r+(1-x_A)^r\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^{n}u_i}\right].$$

$$=\left\{nt\left[\frac{\sum_{x_i\in X_1}u_i\left\{\left(x_A^r x_B^{1-r}+(1-x_A)^r(1-x_B)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_1}u_i}-\frac{\sum_{x_i\in X_1}u_i\left\{\left(x_A^r+(1-x_A)^r\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_1}u_i}\right]\right.$$

$$\left.+nt\left[\frac{\sum_{x_i\in X_2}u_i\left\{\left(x_A^r x_C^{1-r}+(1-x_A)^r(1-x_C)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_2}u_i}-\frac{\sum_{x_i\in X_2}u_i\left\{\left(x_A^r+(1-x_A)^r\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_2}u_i}\right]\right\}$$

$$+\left\{nt\left[\frac{\sum_{x_i\in X_1}u_i\left\{\left(x_A^r x_C^{1-r}+(1-x_A)^r(1-x_C)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_1}u_i}-\frac{\sum_{x_i\in X_1}u_i\left\{\left(x_A^r+(1-x_A)^r\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_1}u_i}\right]\right.$$

$$+nt\left[\frac{\sum_{x_i\in X_2}u_i\left\{\left(x_A^r x_B^{1-r}+(1-x_A)^r(1-x_B)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_2}u_i}-\frac{\sum_{x_i\in X_2}u_i\left\{\left(x_A^r+(1-x_A)^r\right)^{\frac{1}{r}}\right\}}{\sum_{x_i\in X_2}u_i}\right]\Bigg\}.$$

$$=nt\left[\frac{\sum_{i=1}^n u_i\left\{\left(x_A^r x_B^{1-r}+(1-x_A)^r(1-x_B)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^n u_i}-\frac{\sum_{i=1}^n u_i\left\{\left(x_A^r+(1-x_A)^r\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^n u_i}\right]$$

$$+nt\left[\frac{\sum_{i=1}^n u_i\left\{\left(x_A^r x_C^{1-r}+(1-x_A)^r(1-x_C)^{1-r}\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^n u_i}-\frac{\sum_{i=1}^n u_i\left\{\left(x_A^r+(1-x_A)^r\right)^{\frac{1}{r}}\right\}}{\sum_{i=1}^n u_i}\right].$$

$$=I_R^{\alpha,\beta}(A;B;U)+I_R^{\alpha,\beta}(A;C;U). \tag{26}$$

Hence, the result follows.

4.3. Generalized 'Useful' R-Norm Fuzzy Information Improvement Measure

Suppose a fuzzy set B is used as an approximation of fuzzy set A. A revision is made and B is replaced by a new fuzzy set E. The difference between the original directed divergence measure $D(A,B)$ and the revised directed divergence measure $D(A,E)$ or the reduction achieved in ambiguity by revising the original set B by a new set E is called fuzzy information improvement. It is written as

$$D(A,B)-D(A,E). \tag{27}$$

Corresponding to 'useful' R-norm fuzzy DDM defined in (10), we propose the 'useful' R-norm fuzzy information improvement measure as

$$D_R^{\alpha,\beta}(A,B;U)-D_R^{\alpha,\beta}(A,E;U)=\frac{R+\alpha-\beta}{\beta-R}\left[\frac{n\sum_{i=1}^n u_i}{\sum_{i=1}^n u_i}\left\{1-\left(\mu_A^{\frac{R+\alpha-\beta}{\alpha}}(x_i)\mu_B^{1-\frac{R+\alpha-\beta}{\alpha}}(x_i)+\left((1-\mu_A(x_i))\right)^{\frac{R+\alpha-\beta}{\alpha}}(1-\mu_B(x_i))^{1-\frac{R+\alpha-\beta}{\alpha}}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right\}\right]$$

$$-\frac{R+\alpha-\beta}{\beta-R}\left[\frac{n\sum_{i=1}^n u_i}{\sum_{i=1}^n u_i}\left\{1-\left(\mu_A^{\frac{R+\alpha-\beta}{\alpha}}(x_i)\mu_E^{1-\frac{R+\alpha-\beta}{\alpha}}(x_i)+\left((1-\mu_A(x_i))\right)^{\frac{R+\alpha-\beta}{\alpha}}(1-\mu_E(x_i))^{1-\frac{R+\alpha-\beta}{\alpha}}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right\}\right]. \tag{28}$$

When $\alpha=1,\beta=1\,\&\,u_i=1$, (28) reduces to the measure given by Hooda and Bajaj [8].

4.4. Monotone Behaviour of 'Useful' RTAM (14)

We study the monotonic nature of the proposed measure in the given limits of R, α and β. We take two fuzzy sets $A\,\&\,B$ defined respectively as $A=(0.65,0.23,0.82,0.44,0.97,0.31)$ and $B=(0.42,0.28,0.05,0.90,0.73,0.61)$ with utility distribution $U=(5,2,4,1,3,6)$ and $n=6$.

Table 9. Values of Measure (14) for Fixed α and β

R	3	7	20	47	62	100	120	140
$I_R^{0.96,0.20}(A;B;U)$	14.6288	18.2923	20.9998	22.0255	22.2208	22.4580	22.5235	22.5704
$I_R^{0.62,0.81}(A;B;U)$	15.4946	19.3770	21.5787	22.3048	22.4372	22.5958	22.6391	22.6701
$I_R^{0.59,0.59}(A;B;U)$	16.1326	19.6114	21.6497	22.3331	22.4585	22.6088	22.6499	22.6793

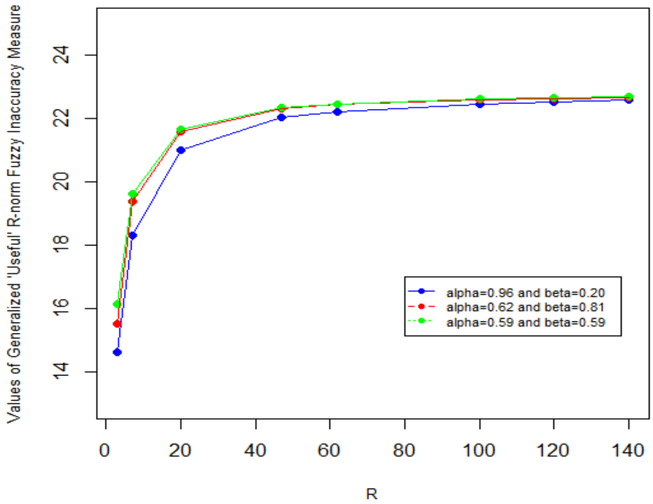

Fig.4. Graphical Overview of Measure (14) at Fixed Alpha and Beta

From Table 9, it becomes obvious that as we increase the value of R and keep $\alpha \,\&\, \beta$ fixed; the 'useful' RTAM defined in (14) shows an increasing trend. We further observe that as R increases the impact of parameters tend to vanish as the values of (14) coincide for higher values of R.

The increasing trend of measure (14) with respect to varying R is depicted in the Fig. 4 by taking values of $\alpha \,\&\, \beta$ as $(0.96, 0.20)$, $(0.62, 0.81) \,\&\, (0.59, 0.59)$ respectively.

Table 10. Values of Measure (14) for Fixed R and β

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$I_{0.72}^{\alpha,0.52}(A;B;U)$	12.9077	10.1040	8.8766	8.2124	7.8043	7.5315	7.3376	7.1933	7.0822	6.9941
$I_{5}^{\alpha,0.02}(A;B;U)$	22.0423	21.2929	20.6018	19.9626	19.3702	18.8198	18.3074	17.8292	17.3820	16.9632
$I_{23}^{\alpha,0.71}(A;B;U)$	22.6688	22.4843	22.3033	22.1257	21.9514	21.7802	21.6122	21.4473	21.2853	21.1261

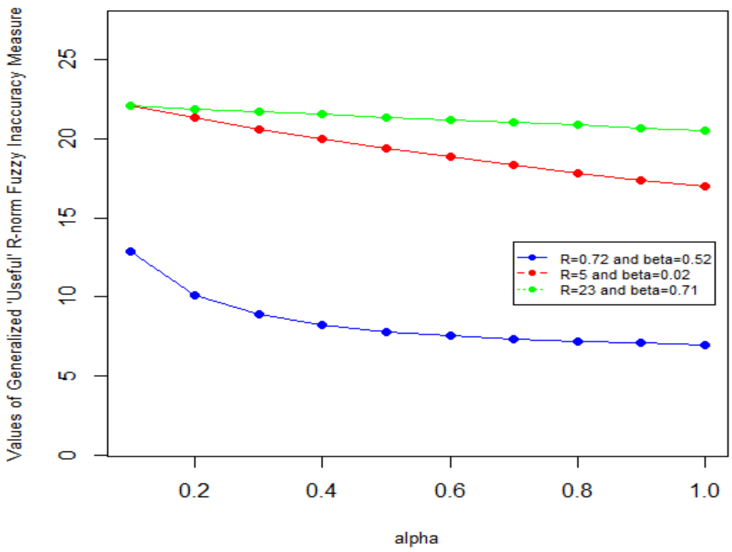

Fig.5. Graphical Overview of Measure (14) at Fixed R and Beta

From Table 10, we can easily state that as the value of α increases ($R \,\&\, \beta$ are fixed), the value of measure (14) decreases. Although, the value of measure (14) changes if we alter the values of $R \,\&\, \beta$ but the trend (that is decreasing) remains the same. Thus, there is a negative relation between α and the measure (14). This relation is presented graphically in the Fig. 5 by taking values of $R \,\&\, \beta$ as $(0.72, 0.52), (5, 0.02) \,\&\, (23, 0.71)$ respectively.

Table 11. Values of Measure (14) for Fixed α and R

β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.93
$I_{2}^{0.98,\beta}(A;B;U)$	12.7685	12.5284	12.2777	12.0157	11.7418	11.4555	11.1561	10.8431	10.5159	10.4149
$I_{0.94}^{0.16,\beta}(A;B;U)$	17.1752	16.6593	16.0419	15.2907	14.3596	13.1811	11.6573	9.6616	7.1933	6.4750
$I_{23}^{0.63,\beta}(A;B;U)$	21.7579	21.7533	21.7487	21.7441	21.7394	21.7347	21.7299	21.7252	21.7204	21.7189

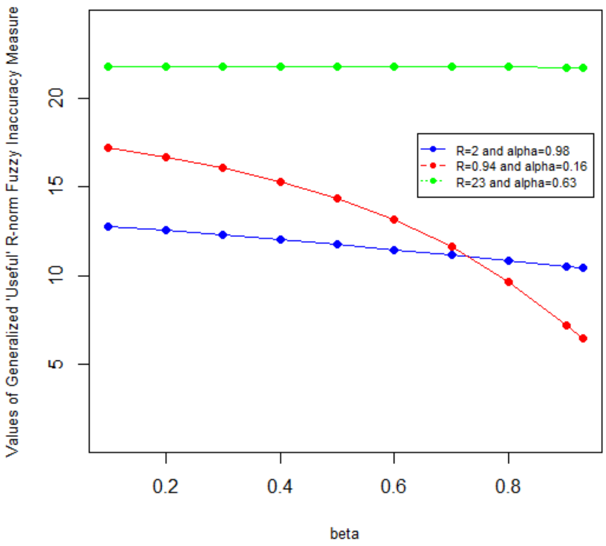

Fig.6. Graphical Overview of Measure (14) at Fixed R and Alpha

From Table 11, we infer that the measure defined in (14) decreases as the value of β increases keeping $R\&\alpha$ fixed. Thus, there is a negative relation between β and the measure (14). This relation is shown in the Fig. 6 by taking values of $R\&\alpha$ as $(2,0.98),(0.94,0.16)\&(23,0.63)$ respectively.

From the Fig. 6, we further conclude that there is a minimal decrease in the value of (14) at $R=23\&\alpha=0.63$. Also, (14) decreases sharply when we take $R<1$.

Since $R>\beta$, we have taken the value of β upto 0.93 only.

4.5. Application of 'Useful' R-Norm Total Ambiguity Measure

In this section, we demonstrate the application of the proposed 'useful' RTAM (14) in the context of multi-criteria decision making. Decision making basically concerns with making best choice from all the available choices. There are many situations where the decision makers find it hard to make the best choice since the information available is very little or vague about the alternatives. So, the decision makers present their preferences in the form of fuzzy information. Various fuzzy MCDM approaches have been established and are employed to a variety of fields.

Suppose $D=\{D_1,D_2,...,D_l\}$ be a set choice and $K=\{K_1,K_2,...,K_m\}$ be a set of criteria. Let $U=\{U_1,U_2,...,U_m\}$ represent the respective importance of each criterion. The characteristics of the choice D_i in terms of criteria K_j are symbolized by the following fuzzy sets:

$$D_i=\{\langle K_j,\zeta_{ij}\rangle;K_j\in K\},i=1,2,...,l\ \&\ j=1,2,...,m$$

where ζ_{ij} represents the extent to which D_i satisfies K_j.

The method for solving fuzzy MCDM problem in terms of the measure proposed in this paper is described in the steps given below by considering a numerical example.

Example: Suppose a person wants to admit his child in a school. He has to choose among the six options i.e., $D=\{D_1,D_2,...,D_6\}$ and take a decision based on the six criteria: 1. K_1: fee structure 2. K_2: quality education 3. K_3: status of school 4. K_4: infrastructure 5. K_5: distance from home to school 6. K_6: co-curriculum activities. Let $U=(4,3,6,1,2,5)$ be the utility distribution with $n=6$. The six possible choices under the six criteria are to be evaluated by the decision maker in the following form:

$$D_1=\{\langle K_1,0.82\rangle,\langle K_2,0.95\rangle,\langle K_3,0.76\rangle,\langle K_4,0.79\rangle,\langle K_5,0.88\rangle,\langle K_6,0.41\rangle\}\ D_2=\{\langle K_1,0.49\rangle,\langle K_2,0.51\rangle,\langle K_3,0.62\rangle,\langle K_4,0.77\rangle,\langle K_5,0.92\rangle,\langle K_6,0.83\rangle\}$$
$$D_3=\{\langle K_1,0.71\rangle,\langle K_2,0.66\rangle,\langle K_3,0.84\rangle,\langle K_4,0.56\rangle,\langle K_5,0.79\rangle,\langle K_6,0.46\rangle\}\ D_4=\{\langle K_1,0.65\rangle,\langle K_2,0.89\rangle,\langle K_3,0.55\rangle,\langle K_4,0.67\rangle,\langle K_5,0.81\rangle,\langle K_6,0.59\rangle\}$$
$$D_5=\{\langle K_1,0.87\rangle,\langle K_2,0.74\rangle,\langle K_3,0.71\rangle,\langle K_4,0.54\rangle,\langle K_5,0.69\rangle,\langle K_6,0.62\rangle\}\ D_6=\{\langle K_1,0.78\rangle,\langle K_2,0.69\rangle,\langle K_3,0.47\rangle,\langle K_4,0.61\rangle,\langle K_5,0.73\rangle,\langle K_6,0.65\rangle\} \quad (29)$$

Step 1: Obtain the positive-ideal solution D^+ and negative-ideal solution D^- as

$$D^+=\{\langle\xi_{1+}\rangle,\langle\xi_{2+}\rangle,\ldots,\langle\xi_{m+}\rangle\}\ \&\ \ D^-=\{\langle\xi_{1-}\rangle,\langle\xi_{2-}\rangle,\ldots,\langle\xi_{m-}\rangle\}. \quad (30)$$

where for each $j=1,2,\ldots,m$

$$\langle\xi_{j+}\rangle=\langle\max_i \xi_{ij}\rangle\ \ \&\ \langle\xi_{j-}\rangle=\langle\min_i \xi_{ij}\rangle\,. \quad (31)$$

Thus, D^+ and D^- are obtained respectively as:

$$D^+=\{\langle K_1,0.87\rangle,\langle K_2,0.95\rangle,\langle K_3,0.84\rangle,\langle K_4,0.79\rangle,\langle K_5,0.92\rangle,\langle K_6,0.83\rangle\}\ D^-=\{\langle K_1,0.49\rangle,\langle K_2,0.51\rangle,\langle K_3,0.47\rangle,\langle K_4,0.54\rangle,\langle K_5,0.69\rangle,\langle K_6,0.41\rangle\} \quad (32)$$

Step 2: Values of $I_R^{\alpha,\beta}(D^+;D_i;U)$ and $I_R^{\alpha,\beta}(D^-;D_i;U)$ *where* $i=1,2,\ldots,6$ are obtained respectively in the following tables as per the expression (14).

Table 12. Values of $I_R^{\alpha,\beta}(D^+;D_1;U)$

	$\alpha=0.26$ $\beta=0.37$ $R=0.97$	$\alpha=0.72$ $\beta=0.81$ $R=32$
$I_R^{\alpha,\beta}(D^+;D_1;U)$	2.4072	2.5345
$I_R^{\alpha,\beta}(D^+;D_2;U)$	2.9416	3.0413
$I_R^{\alpha,\beta}(D^+;D_3;U)$	2.6971	2.7961
$I_R^{\alpha,\beta}(D^+;D_4;U)$	2.6722	2.8549
$I_R^{\alpha,\beta}(D^+;D_5;U)$	2.1455	2.1713
$I_R^{\alpha,\beta}(D^+;D_6;U)$	2.9943	3.2245

Table 13. Values of $I_R^{\alpha,\beta}(D^-;D_i;U)$

	$\alpha=0.26$ $\beta=0.37$ $R=0.97$	$\alpha=0.72$ $\beta=0.81$ $R=32$
$I_R^{\alpha,\beta}(D^-;D_1;U)$	9.5807	15.0886
$I_R^{\alpha,\beta}(D^-;D_2;U)$	6.3635	8.6181
$I_R^{\alpha,\beta}(D^-;D_3;U)$	6.0413	8.1151
$I_R^{\alpha,\beta}(D^-;D_4;U)$	5.4506	7.3862
$I_R^{\alpha,\beta}(D^-;D_5;U)$	6.4722	8.7976
$I_R^{\alpha,\beta}(D^-;D_6;U)$	4.7101	5.7181

Step 3: Value of relative 'useful' RTAM $I_R^{\alpha,\beta}(D_i;U)$ of each D_i with respect to D^+ and D^- are computed as per the below formula

$$I_R^{\alpha,\beta}(D_i;U)=\frac{I_R^{\alpha,\beta}(D^+;D_i;U)}{I_R^{\alpha,\beta}(D^+;D_i;U)+I_R^{\alpha,\beta}(D^-;D_i;U)},\forall i\,. \quad (33)$$

The results are calculated for $\alpha=0.26,0.72$; $\beta=0.37,0.81$ & $R=0.97,32$ in the subsequent table:

Table 14. Values of $I_R^{\alpha,\beta}(D_i;U)$

	$\alpha=0.26$ $\beta=0.37$ $R=0.97$	$\alpha=0.72$ $\beta=0.81$ $R=32$
$I_R^{\alpha,\beta}(D_1;U)$	**0.2008**	**0.1438**
$I_R^{\alpha,\beta}(D_2;U)$	0.3161	0.2608
$I_R^{\alpha,\beta}(D_3;U)$	0.3086	0.2562
$I_R^{\alpha,\beta}(D_4;U)$	0.3289	0.2787
$I_R^{\alpha,\beta}(D_5;U)$	0.2489	0.1979
$I_R^{\alpha,\beta}(D_6;U)$	0.3886	0.3605

From the above Table 14, we get the following ranking order of available choices

$$D_1 \succ D_5 \succ D_3 \succ D_2 \succ D_4 \succ D_6 . \tag{34}$$

This implies D_1 is the most appropriate choice.

5. Conclusion

In this manuscript, we have presented new generalized measure of ‘useful’ R-norm inaccuracy and ‘useful’ R-norm total ambiguity. The fundamental properties of both the proposed measures are stated which validate these measures. The particular cases are also discussed for both the measures. Further, the information improvement measures are studied. The monotonic property of both the inaccuracy measures is discussed with respect to the parameters introduced. In the end, the application to multi-criteria decision making of ‘useful’ R-norm total ambiguity measure is presented.

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