

Quantum Wavelet Transforms Generated by the Product of the Sine Polynomial and the Gaussian Envelope on the Tetrahedral Graph

Jean Bosco Mugiraneza

Mila Global Campus, P.O Box 408 Kigali, Rwanda
Email: jbmug@milaglobalcampus.com

Received: 28 July 2017; Accepted: 20 April 2018; Published: 08 July 2018

Abstract—In this paper we present a novel technique that permits to extract the essential information embedded in the product of sine polynomial and Gaussian envelope by simply knowing the vertices of the tetrahedral graph. The study proves that the matrix of vertices of the tetrahedral graph and its variants are the building block of both Haar wavelets, Hadamard-Walsh transform, wavelets sets and tight frames. We also prove that the Berkeley B Gate is a function of the degree matrix and the adjacency matrix of the tetrahedral graph. The latter is the Hermitian part of the unitary polar decomposition in terms of elementary gates for quantum computation [68] which reveals interesting properties of the tetrahedral graph in both quantum group, Lie group and Pauli group for wavelets sets, quantum image processing and quantum data compression. We explore the connection existing among graphs theory, wavelets, tight frames and quantum logic gates.

Index Terms—Graphs Theory, Schrödinger Equation, Sine Polynomial, Haar Wavelets, Quantum Gates, Spectral Analysis, Yang-Baxter Equation, Gamma Matrices, Hamiltonian, Hadamard-Walsh Transform.

I. INTRODUCTION

Graphs are mathematical objects that allow representing complex structures and facilitate their study [1]. In the last decade graph signal processing has been a hot subject in signal and information processing for different reasons [2]. Graphs are modeling tool suitable to many applications such as social and economic networks, epidemiology and biological networks, transportation networks, information networks, internet blog data, power grids arising in large variety of applications, generate large sets of raw data from which a detailed analysis may extract useful information [3]. Some data on this graphs can be modeled as scalar (or vectors) functions on each of its nodes, forming what is called signal graph [4]. A first step in modelling a graph signal consists of the construction of the appropriate signal transforms [5]; this would require the modeling of signals on graphs characterized by a connectivity or adjacency matrix that captures dependencies in the data [7], [6]. The

eigenvalues of the Laplacian matrix give the discrete spectral frequencies [6]. In [7] Sandryhaila and Moura defined a shift or translation of the graph using adjacency matrix and arrived at notions of graph linear filtering and graph Fourier transform (GFT). In this context then the graph Fourier transform is defined as a projection on the generalized eigenvectors of the Laplacian matrix. This operator point of view allows not only to generalize the notion of transforms but also the notions of filtering and other general linear operations on graph signals [5]. There is an extensive literature that explored algebraic graph theory, spectral graph theory and wavelets transform on graphs [8], [9], [10] and [11] and references therein.

Graphs formally represent a network, which is basically a collection of interconnected objects. A network picture has been applied to various physical and biological systems to understand their governing mechanisms intuitively. Utilizing discretization schemes, both electrical and optical materials can also be interpreted as abstract ‘graph’ networks composed of couplings (edges) between local elements (vertices), which define the correlation between material structures and wave flows [12]. Chensheng Wu et al. [13] used a plenoptic wave front sensor to image the distorted beam into its 4D phase space. A fast reconstruction algorithm based on graph theory was applied to recognize the phase distortion of a laser beam and command the adaptive optics (AO) device to perform phase compensation. In [14] authors provided an experimental demonstration of how a fiber network can be treated as an optical oracle for the Hamiltonian path problem, the famous mathematical complexity problem of finding whether a set of towns can be travelled via a path in which each town is visited only once. In order to solve this Hamiltonian path problem, the graph was implemented as a network consisting of optical fibers (roads) that connected all of the five nodes (towns), and the network was probed using a short optical pulse.

Quantum computing is at the interface of quantum physics, mathematics and computer science. Quantum information theory has been investigated in connection with quantum algorithms and communication protocols using density matrices and operators associated with

tensor-product of Hilbert vector spaces. In [15] authors showed a strong link between quantum experiments and graph theory. In the experimental setups to realize high dimensional multipartite quantum states each of the setup corresponded to an undirected graph, and every undirected graph corresponded to an experimental set up. A link between quantum physics and graph theory has been drawn before, but for different reasons [16]. For example, in Graph states [17], [18], which are related to the resources for measurement-based quantum computation [19], the vertices of the Graph correspond to qubits in a quantum state, and the edges correspond to correlations between two qubits. In different works, the Laplacian of a graph has been interpreted as the density matrix of a quantum state [20] with the Hamiltonian of such system approximated by the adjacency matrix of the graph, and the energy levels and states represented by eigenvalues and eigenvectors [21], this allowed to investigate new entanglement criteria [22].

The objective of this paper is to investigate the product of the sine polynomial and the Gaussian envelope which yields the matrix of vertices of the tetrahedral graph. The spectral analysis shows interesting properties of the Laplacian matrix, adjacency matrix and the eigenvalues matrix of the tetrahedral graph with applications in matrix representation of quantum gates. This paper is organized as follows: section II overviews the background and concepts from quantum graph, quantum logic gates and quantum wavelet transforms. Section III discusses the like Schrödinger Equation for the wavelet function consisting of a sine polynomial of four sinusoidal waves into a matrix of vertices of the tetrahedral graph. Section IV discusses the results out of this paper that include the modelling of quantum logic gates using the matrices derived from the tetrahedral graph. Finally, section V winds up with the conclusion highlighting an open up for future research directions that will arise out of this work linking the graph theory, the wave propagation theory, quantum theory and molecular orbital structures. This paper will obviously serve as a foundation for a variety of useful applications of graph theory to quantum image processing and quantum data compression and related areas.

II. RELATED WORKS

Quantum computation and quantum information are modern developments taking advantage from the Quantum Mechanics features to propose technological applications. Quantum mechanics deals with microscopic objects like atoms, molecules, etc. Here the consideration is given to only those quantities which may be measured. The evolution of quantum systems is completely determined by its Hamiltonian. The latter describing the quantum system exponentiated to produce a unitary operator. In the case of time-evolution, it is possible to achieve the same unitary transformation with different Hamiltonians, since the former is only concerned with the input and output of the transformation and not the intermediate state of the system. This is one reason for

why there are so many different proposals for performing quantum computation in different physical settings [23].

The evolution of quantum systems is completely determined by its Hamiltonian. Pauling introduced quantum graphs of connected one-dimensional wires some decades ago [24]. Later Kuhn used Pauling's idea [25] to describe organic molecules by free electron models. Some of the physical systems modeled by quantum graphs include electromagnetic waveguides [26], [27], mesoscopic systems [28], quantum wires [29], [30], excitation of fractons in fractal strictures [31], [32]. Quantum graphs with external leads (antennas) have been analyzed in detail in [33], [34]. In [35] authors investigated Wigner's reaction matrix using tetrahedral microwave networks that correspond to graph with the time reversal symmetry $\beta=1$. In [36] the author attempted to model electrical network using graph theory. The most fundamental concept of classical computation and classical information is Bit (binary digit). This is a system that can take on one of two values, such as true and false or 0 and 1. Qubit (Quantum bit) is the quantum analog of a bit. Classical computer circuits consist of wires and logic gates. In a similar way, quantum computer has a quantum gates from which quantum computing devices are designed. Quantum gates on a n -qubit can be described by a 2^n by 2^n matrices. Because of the normalization condition that requires

$$\sum_{i=0}^{2^n-1} |\alpha_i|^2 = 1 \quad (1)$$

there constrain for matrices which can be used as quantum gates. The matrix U describing the quantum gate should be unitary that is, $U^\dagger U = I$, where I is the identity matrix [37]. The action of quantum logic gates is always logically reversible [38]. Construction of quantum gates from the elementary braid matrices of the stocks prices closely follows the work of L. S. Geogiev [45], [46], [47]. Hadamard gate, Pauli gates or Controlled-Z gate are some of the elementary quantum gates that are identified in the stock market structure [48]. The Fourier and Walsh-Hadamard transforms have been the ones studied most extensively by the quantum computing community [39], [40], [41]. The quantum Fourier transform (QFT) is now recognized as being pivotal in many known quantum algorithms [42]. Amir and Colin derived efficient, complete, quantum circuits for two representative quantum wavelet transforms, the quantum Haar and quantum Daubechies $D^{(4)}$ transforms. Their approach was to factor the classical operators for these transforms into direct sums, direct products and dot products of unitary matrices. They found that permutation matrices play a pivotal role in the development of wavelets transform and arise explicitly in the packet and pyramid but also in the factorization of wavelet kernels. they considered the particular set of permutation matrices arising in quantum wavelet transforms and developed efficient quantum circuits that implement them. Wavelet transforms are useful for quantum image processing and

quantum data compression. It is natural therefore to consider how to achieve a quantum wavelet transform [39], [43]. The author in [44] calculated the admissibility constant and the demonstrated the covariant of Morlet wavelets, and hence eliminated two of the barriers to full use of this powerful technology for the analysis of foundational questions in quantum mechanics. He took the four-dimensional mother Morlet wavelet as the direct product of four one-dimensional mother Morlet wavelets, one for each coordinate.

III. SCHRÖDINGER LIKE EQUATION FOR WAVELETS

Computation of high frequency waves is a necessity in many scientific applications [72]. Fields requiring such computations include the semi-classical limit of the Schrödinger equation, communication networks, radio antenna engineering, laser optics, under water acoustics, seismic wave propagation, and reflection seismology [32]. Christopher L. Mueller [49] and H. Kogelinik and T. Li [50], used the concept of electromagnetic wave beams to describe mathematical techniques for Resonant Interferometer and Laser beams resonators. Gaussian beams are approximate high frequency solutions to PDEs which are concentrated on a single ray through space-time. The quantum amplitude Ψ linked to a quantum object satisfies the differential wave equation:

$$\nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial t^2} \quad (2)$$

where ∇ is the Laplacian operator and Ψ represents a space filling and time-dependent physical field [51]. It is known that in most cases the laser beam is similar to plane waves; except that their intensity distribution is not uniform [50]. In this paper we will attempt to use the optical differential wave equation and the sine polynomial windowed by a Gaussian envelope to model some optical, microwave and radar system elements and some quantum phenomena based on the matrix of vertices of the tetrahedral graph to easy the computation complexity of PDEs.

In this paper, we consider the modulated wave form given by Eqn. (3) whose plane representation is shown in Fig.1.

$$\Psi(t) = 100 \cos(2\sigma^2 \pi t) \sin(\pi t) \cos(t) e^{\frac{-t^2}{2\sigma^2}} \quad (3)$$

Applying Euler's formulae on Eqn. (3) and simplifying expressions transforms it into a family of sine polynomials given in Eqn. (4).

$$\Psi_m(t) = \left[25 \sum_{m=1}^4 \sin(\omega_m t) \right] e^{\frac{-t^2}{2\sigma^2}} \quad (4)$$

where

$$\begin{aligned} \omega_1 &= 2\sigma^2 \pi + \pi + 1 \\ \omega_2 &= 2\sigma^2 \pi - \pi + 1 \\ \omega_3 &= 2\sigma^2 \pi + \pi - 1 \\ \omega_4 &= 2\sigma^2 \pi - \pi - 1 \end{aligned} \quad (5)$$

Given the Schrödinger wave equation:

$$-\frac{\hbar}{2m} \frac{d^2}{dt^2} \sin kt = \frac{\hbar}{2m} k^2 \sin kt \quad (6)$$

\hbar is the Plank's constant divided by 2π and m is the mass of the particle that is traveling through space, $\sin kt$ is an eigenfunction of the Schrödinger equation, the eigenvalue k^2 is the energy. The compact form of the Schrödinger equation is:

$$H \Psi = E \Psi \quad (7)$$

where

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dt^2} \quad (8)$$

is the energy operator,

$$E = -\frac{\hbar^2}{2m} k^2 \quad (9)$$

is the energy eigenvalue or energy whereas $\Psi = \sin kt$ is the wave function. The complete solution with boundary conditions is the linear combination of partial solutions; then we have:

$$\Psi(t) = \sqrt{\frac{2}{L}} \sum_{-\infty}^{+\infty} \sin \frac{n\pi}{L} t \quad (10)$$

Note that n is a quantum number [70]. The mathematical model describing the higher order light beams is a product of Hermite function and Gaussian functions [50].

$$\Psi_{mm}(r, t) = \frac{1}{\sqrt{\sigma 2^n n! \sqrt{\pi}}} H_{mm} \left(\frac{r, t}{\sigma} \right) \quad (11)$$

From Eqn. (4) we define the phase-space transformation matrix given by:

$$P = \begin{pmatrix} 2\pi\sigma^2 & 2\pi\sigma^2 & 2\pi\sigma^2 & 2\pi\sigma^2 \\ \pi & -\pi & \pi & -\pi \\ 1 & 1 & -1 & -1 \end{pmatrix} \quad (12)$$

After performing some operations on the transformation P then we get the matrix representing the vertices of the tetrahedral graph given by:

$$R = \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \end{pmatrix} \quad (13)$$

The column vectors are the vertices of a family of tetrahedrons defined in a 3-dimension real space R^3 . The coordinates of the tetrahedron at the origin and those of the dual tetrahedral graph refer to [52]. Therefore the tetrahedron set of vertices V has the cardinality of 4 ($\#V = 4$) i.e. four vertices and the set of edges E has the cardinality ($\#E = 6$) i.e. six edges. Note that the tetrahedron, the octahedron, and the cube are the only "Platonic" solids that exist in any dimension [53]. For the properties of tetrahedral graph refer to [54]. Each vertex of the tetrahedral graph corresponds the quantum number.

IV. RESULTS ANALYSIS

A. Spectral Analysis

In this section, we are interested in computing the product of the Matrix P and its transpose as follow:

$$F = P^T \times P \quad (14a)$$

we get the following matrices:

$$F = 4\pi^2 \sigma^4 (uu^T) + \pi^2 (vv^T) + (ww^T) \quad (14b)$$

$$F = 4\pi^2 \sigma^4 U + \pi^2 V + W \quad (14c)$$

The product of the transformation matrix R and its transpose $P^T \times P$ indicates that the transformation representing the wavelet transform on a bipartite regular undirected graph. We realize that the product F is a linear combination of three Gram matrices U , V and W , where $U = uu^T$, $V = vv^T$ and $W = ww^T$

$$u = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; v = \begin{pmatrix} +1 \\ -1 \\ +1 \\ -1 \end{pmatrix}; w = \begin{pmatrix} +1 \\ +1 \\ -1 \\ -1 \end{pmatrix}; U = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix};$$

$$V = \begin{pmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \end{pmatrix}, W = \begin{pmatrix} +1 & +1 & -1 & -1 \\ +1 & +1 & -1 & -1 \\ -1 & -1 & +1 & +1 \\ -1 & -1 & +1 & +1 \end{pmatrix};$$

Multilevel block partitioning is of interest only if the blocks of the levels exhibit some structure [55]. U , V and W exhibits block structures. U and W have the following block structures:

$$U = \sqrt{2} \begin{pmatrix} C & C \\ C & C \end{pmatrix}; W = \sqrt{2} \begin{pmatrix} +C & -C \\ -C & +C \end{pmatrix} \quad (15)$$

where C is the Temperley-Leib generator given by:

$$C = \begin{pmatrix} 1/d & 1/d \\ 1/d & 1/d \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (16)$$

For the Ising anyons model, the Kauffman variable is $\zeta = ie^{i\frac{\pi}{8}}$, and the quantum dimension of the spin $1/2$ is $d = \sqrt{2}$

Alternatively, matrices U , V and W can be expressed as follows:

$$W = D - A + N - M \quad (17a)$$

$$V = D - A + M - N \quad (17b)$$

$$V + W = 2D - 2A = 2L \quad (17c)$$

Where

$$L = \begin{pmatrix} +1 & 0 & 0 & -1 \\ 0 & +1 & -1 & 0 \\ 0 & -1 & +1 & 0 \\ -1 & 0 & 0 & +1 \end{pmatrix}; D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (18)$$

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; Q = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

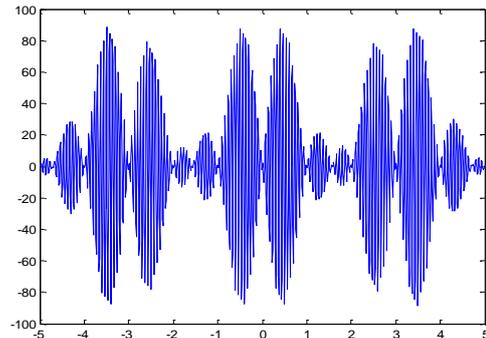


Fig.1. Plot of the wavelet transform in Eqn.(3) for $2\sigma^2 = 230$

with $L = D - A$ being the Laplacian matrix, A , the adjacency matrix, D the degree matrix and $Q = D + A$, the signless matrix of the tetrahedral graph. The eigenvectors corresponding to different eigenvalues are orthogonal and therefore the adjacency A matrix is orthogonally diagnosable. The diagonalisation means that there exists a diagonal matrix Λ such that:

$$A = V_A \Lambda_A V_A^{-1} = V_A \Lambda_A V_A^T \quad (19)$$

where

$$V_A^{-1} = V_A^T$$

$$U, V, W, D, L \in Spect_4 = \{ X \in S_+^4 \mid Tr(X) = 4 \}$$

where S_+^4 is the set of 4×4 positive semi-definite matrices and $Tr(X)$ is the trace of the matrix X . This means that the matrices U, V, W, D, L and Q having the same trace equal to one and belong to the same spectrahedron of the tetrahedron which is the subject of investigation. The matrices U, V and W , are circulant matrices and we know that circulants are a special type of Toeplitz matrix and have unique properties. Looking at the tetrahedral graph we see that its adjacency matrix is symmetric and we can find its eigenvalues. They are obtained by solving the characteristic polynomial to zero.

$$P_A(\lambda) = (\lambda I - A) = (\lambda - 1)^2(\lambda + 1)^2 \quad (20a)$$

The roots which are eigenvalues having the following values:

$$\lambda_1 = \lambda_2 = -1, \lambda_3 = \lambda_4 = 1 \quad (20b)$$

The eigenvalues matrix associated with the adjacency matrix A of the tetrahedral graph is:

$$\Lambda_A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (21a)$$

The square of the matrix V_A diagonalizing the adjacency matrix is described by:

$$V_A^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (21b)$$

The vectors of V_A form an orthonormal basis of R^4 .
The Laplacian characteristic polynomial

$$P_L(\mu) = P_Q(\mu) = (\mu I - A) = \mu^2(\mu - 2)^2 \quad (22a)$$

The eigenvalues of the Laplacian matrix

$$\mu_1 = \mu_2 = 2; \mu_3 = \mu_4 = 0 \quad (22b)$$

and we realize that the square of the matrix V_L diagonalizing the Laplacian matrix is described by:

$$V_L^2 = V_Q^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (22c)$$

We have the following matrix equations from the tetrahedral graph:

$$\begin{aligned} A^2 &= M^2 = N^2 = D = I \\ A \times M &= M \times A = N \\ A \times N &= N \times A = M \\ M \times N &= N \times M = A \\ Q + M + N &= D + A + M + N = U \\ R^T \times R &= 3D - A + M + N \\ M + N &= R^T \times R - 3D + A \\ (M + N)^2 &= 2D + 2A \end{aligned} \quad (23)$$

$$A = A_{K_{4,4}} + A_{C_4} - A_{K_4} \quad (24)$$

where $A_{K_{4,4}}$, A_{C_4} and A_{K_4} are the adjacency matrices of the complete bipartite graph, the cyclic graph and the complete graph respectively.

(xi) The degree matrix D is also a product of a Haar transform matrix of order 4 and its transpose, that is,

$$D = H_4^T H_4 \quad (25)$$

B. Generalize Gell-Mann Matrix Basis

In this paper, we will discuss three different bases that can be used to decompose the matrices derived from the tetrahedral graph. We begin with the generalized Gell-Mann matrices which are higher-dimensional extensions of the Pauli matrices (for qubits) and the Gell-Mann matrices (for qutrits), they are the standard $SU(N)$ generators[84]. The six symmetric Gell-Mann matrices of $SU(4)$ are given below:

$$\Lambda_s^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Lambda_s^{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\Lambda_s^{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \Lambda_s^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (25a)$$

$$\Lambda_s^{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Lambda_s^{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let us express the adjacency matrix A of the tetrahedral graph as a function of the generalized Gell-Mann matrices:

$$A = \Lambda_s^{14} + \Lambda_s^{23} \quad (25b)$$

From Eqn. (23a) and (25a) then the degree matrix D is given by:

$$D = (\Lambda_s^{12})^2 = (\Lambda_s^{13})^2 = (\Lambda_s^{14})^2 + (\Lambda_s^{23})^2 \quad (25c)$$

with

$$\Lambda_s^{14} \Lambda_s^{23} = \Lambda_s^{23} \Lambda_s^{14} = 0 \quad (25d)$$

Since the two matrices are linearly independent. We can see that at the center of the matrix Λ_s^{23} there is the submatrix of adjacency matrix Hamming cube of order one. The consequence of the above statement is that the Laplacian matrix L is given by:

$$L = D - A = (\Lambda_s^{23})^2 + (\Lambda_s^{14})^2 - (\Lambda_s^{23} + \Lambda_s^{14}) \quad (25e)$$

and therefore, the signless matrix Q is equal to:

$$Q = D + A = (\Lambda_s^{23})^2 + (\Lambda_s^{14})^2 + \Lambda_s^{23} + \Lambda_s^{14} \quad (25f)$$

$$(A_1^2 + A_2)^2 = (A_2^2 + A_1)^2 = A_1^2 + A_2^2 \quad (25g)$$

$$M = \Lambda_s^{13} + \Lambda_s^{24} \quad (25h)$$

$$N = \Lambda_s^{12} + \Lambda_s^{34} \quad (25i)$$

$$(\Lambda_s^{24})^2 + (\Lambda_s^{13})^2 = (\Lambda_s^{14})^2 + (\Lambda_s^{23})^2 \quad (25j)$$

$$\Lambda_s^{24} \Lambda_s^{13} = 0 \quad (25k)$$

C. Frames and Hadamard-Walsh Transform

Frames are generalizations of bases which lead to redundant signal expansions. In recent years, frames have been used in signal processing for sampling techniques, detection problems, error correction codes, analysis and design of packet-based communication systems [74], [75], [76], [77] & [78].

In this section we consider the frame $F_2 = V_1, V_2, V_3, V_4$ defined by the following vectors:

$$V_1 = \begin{pmatrix} +1 \\ +1 \\ +1 \\ +1 \end{pmatrix}, V_2 = \begin{pmatrix} +1 \\ -1 \\ +1 \\ -1 \end{pmatrix}, V_3 = \begin{pmatrix} +1 \\ +1 \\ -1 \\ -1 \end{pmatrix} \text{ and } V_4 = \begin{pmatrix} +1 \\ -1 \\ -1 \\ +1 \end{pmatrix} \quad (26)$$

is a group under component wise multiplication. Any group of order p or p^2 , p a prime is necessarily abelian [57]. The matrix W_{F_2} corresponds to the character table of the point group C_{2h} symmetry where following equations:

$$\begin{aligned} E &= V_1 = u \\ C_2 &= V_2 = v \\ i &= V_3 = w \\ \sigma_h &= V_4 \end{aligned} \quad (27)$$

$$\begin{aligned} V_1 &= V_2 \times V_3 \times V_4 \\ V_2 &= V_3 \times V_4 \\ V_3 &= V_2 \times V_4 \\ V_4 &= V_2 \times V_3 \end{aligned} \quad (28)$$

Let us consider the matrix W_{F_2} made of the column vectors of the frame F_2 ,

$$W_{F_2} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad (29a)$$

$$W_{F_2} = W_T P_4 \quad (29b)$$

where

$$W_T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

are the Walsh transform matrix for a 4x4 image and the permutation of coordinate axes respectively. It is easy to show that:

$$K^2 = N, K^3 = M, K^4 = D, K^5 = A \quad (29c)$$

Using Eqn. (29c) then Eqns. (17a), (17b) and (17c) become:

$$V = K^3 + K^4 - K^2 - K^5 \quad (29d)$$

$$W = K^2 + K^4 - K^3 - K^5 \quad (29e)$$

$$V + W = 2K^4 - 2K^5 = 2L \quad (29f)$$

From Eqn. (23d) and (29c) one can obtain

$$A = M K^2 \quad (29g)$$

From the result in [71] and Eqn. (25c) we realize that the integration of the cross-product of two Walsh function vectors is the degree matrix of the tetrahedral graph:

$$\int_0^1 W(t)W^T(t) = (\Lambda_s^{14})^2 + (\Lambda_s^{23})^2 \quad (30)$$

The matrix W_{F_2} defined by the frame F_2 contains four Dirac matrices. Reference to Table IV [58] given by M. Karlsson. we can express the matrix W_{B_2} in terms of four matrices as follows:

$$W_{F_2} = \sum_{k=1}^2 \sum_{j=1}^2 D_{jk} \quad (31a)$$

Equivalently we have:

$$W_{F_2} = D_{11} + D_{12} + D_{21} + D_{22} \quad (31b)$$

$$W_{F_2}^3 = 4D_{11} \quad (31c)$$

Therefore, the square of the sum of the Dirac matrices D_{12} and D_{21} becomes

$$(D_{12} + D_{21})^2 = \left(W_{F_2} - \frac{1}{4} W_{F_2}^3 - A \right)^2 = 2D + 2D_{33} \quad (31d)$$

Now we consider the tight frame characterization of the multiwavelet vector [59] in terms of the polyphaser matrices:

$$P(\omega) = \left[H_\gamma^l(\omega) \right]_{l=0, \gamma \in \Gamma}^{m-1} \quad (32a)$$

with

$$H_\gamma^l(\omega) = \frac{1}{|\det(A)|^{\frac{l}{2}}} \sum_{\gamma \in \Gamma} e^{-i\omega\gamma} H_\gamma^l(\omega A) \quad (33b)$$

where $A \in Z^{d \times d}$ is an expanding dilation matrix. Each polyphaser matrix contains the degree matrix D of the tetrahedral graph and it is expressed in term of matrices in Eqn. (33f) as follows:

$$D = B_0 + B_1 + B_4 + \tilde{B}_0 + \tilde{B}_1 + \tilde{B}_4 \quad (33c)$$

The matrix of eigenvalues of the adjacency matrix is given by:

$$\Lambda_A = 4B_2^2 - 4B_3^2 \quad (33d)$$

The permutation matrix N is the sum of two matrices $2B_2$ and $2B_3$ i.e.

$$N = 2B_2 + 2B_3 \quad (33e)$$

Where [59]

$$\begin{aligned} B_0 &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tilde{B}_0 = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ B_1 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tilde{B}_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ B_2 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, B_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ B_4 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \tilde{B}_4 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{aligned} \quad (33f)$$

Therefore, the square of the matrix diagonalizing the Laplacian matrix $V_L^2 = V_Q^2$ can be written as:

$$V_L^2 = B_0 + \tilde{B}_0 + A_1^2 A + B_1 + \tilde{B}_1 - 4B_2^2 \quad (34a)$$

The following equations describe the connection among frames, wavelets, quantum measurement and the tetrahedral graph [58], 59 & [83]:

Using Eqn. (25c) and (33c) one can obtain

$$D = (\Lambda_s^{14})^2 + (\Lambda_s^{23})^2 = B_0 + B_1 + B_4 + \tilde{B}_0 + \tilde{B}_1 + \tilde{B}_4 \quad (34b)$$

Using Eqn.(25i) and (33e) one can obtain

$$N = \Lambda_s^{12} + \Lambda_s^{34} = 2B_2 + 2B_3 \quad (34c)$$

Using Eqn. (23d), (25b) and (34c) above, one can obtain

$$M = AN^{-1} = \Lambda_s^{13} + \Lambda_s^{24} = (\Lambda_s^{14} + \Lambda_s^{23})(2B_2 + 2B_3)^{-1} \quad (34d)$$

Such that

$$(\Lambda_s^{13})^2 = B_0 + \tilde{B}_0 + B_1 + \tilde{B}_1 \quad (34e)$$

$$(\Lambda_s^{24})^2 = B_4 + \tilde{B}_4 \quad (34f)$$

$$(\Lambda_s^{14})^2 = B_0 + \tilde{B}_0 - B_1 - \tilde{B}_1 + 4B_2^2 \quad (34g)$$

$$(\Lambda_s^{23})^2 = 2B_1 + 2\tilde{B}_1 + B_4 + \tilde{B}_4 - 4B_2^2 \quad (34h)$$

The matrix W_{B_2} of the column vectors of the frame F_2 is the Hadamard matrix H_4 of the order 4. The quaternion units $D \equiv D_{00}$, iD_{13} , iD_{30} and $D_{01}A$ link Williamson type and related Hadamard matrices [81]. The Hadamard matrix H_4 can be expressed in terms of the Hadamard matrix second order H_2 as follows:

$$H_4 = \begin{pmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{pmatrix} \quad (35a)$$

The characteristic equation of the Hadamard matrix H_4

$$P_{H_4}(\lambda) = (\lambda I - A) = (\lambda - 2)^2(\lambda + 2)^2 \quad (35b)$$

and

$$\Lambda_{H_4} = 2\Lambda_A \quad (35c)$$

where Λ_A is the eigenvalue matrix as defined in Eqn.(21a) The matrix W_{F_2} of the frame contains the matrix R of the vertices of the tetrahedral graph. In the Hadamard matrix of order 4 the first three columns constitute the matrix R defined by the vertices of the tetrahedron.

Hadamard transform finds practical application in many areas such as data encryption, signal processing, quantum information processing and data compression algorithms. The Fast Walsh Hadamard transform (FWHT) is used to obtain local structure of images. FWHT can be considered as a sparse factorization of the transform matrix, and refer to each factor as a stage. The property that relates matrix H with its inverse is given by:

$$HR^n = R^n(HR^n)^{-1} \quad (35d)$$

where $HR^n =$ Radix-R Walsh Hadamard transform; $R^n =$ radix-R factorizations $n =$ input element. The special aspects of WHT are the saving of time (48% compared with DF) [60]. Sasikala and Neelaveni [61] approached the problem of multimodality medical image with a fundamental concept of correlation coefficient as a matching measure and showed that the FWHT is faster than the Walsh transform (WT) reducing time consumption for medical image registration.

D. Qubits Solution to Yang-Baxter

Yang-Baxter equation has been studied as the master equation in integrable models in statistical mechanics and quantum field theory [54]. Many scientists have found solutions for the Yang-Baxter equation; however, the full classification of its solutions remains an open problem [55]. The problem of finding solutions to the Yang-Baxter equation that are unitary turns out to be surprisingly difficult [56]. Dye H. [57] described all unitary solutions to the Yang-Baxter equation in dimension four.

In this section, we consider the following Yang-Baxter equation:

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R) \quad (36a)$$

where the matrix R is the unitary solution of the Yang-Baxter equation and I denotes the identity operator of order two. The matrix solution R is the universal quantum gate known as the familiar change-of-basis matrix from the standard basis to the Bell basis of entangled states. The above unitary braid matrix can be expressed in terms of the degree matrix and adjacency matrix of the tetrahedral graph as follows.

$$R = \frac{1}{\sqrt{2}}(D + D_{01}A) \quad (56b)$$

where D and A are the degree matrix and the adjacency matrix of the tetrahedral graph respectively. We have the following identities:

$$(D_{01}A)A = -A(D_{01}A) \quad (57a)$$

$$-(V_A^2 V_L^2)A = -A(-V_A^2 V_L^2) \quad (57b)$$

$$(D_{01}A)(-V_A^2 V_L^2) = (-V_A^2 V_L^2)(D_{01}A) \quad (57c)$$

$$-(D_{01}A)^2 = -(-V_A^2 V_L^2)^2 = A^2 = D \quad (57d)$$

$$-V_A^2 V_L^2 = D_{01}A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (57e)$$

$$D_{01} = V_A^2 + V_L^2 - A \quad (57f)$$

is the gamma matrix as defined in [58]. The matrix representation of the SWAP gate S :

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (58a)$$

The SWAP gate

$$S = (\Lambda_s^{14})^2 + \Lambda_s^{23} \quad (58b)$$

The quantum logic gate appears in the reduction of the four-dimension Lorentz transformation to spinor representation form in the electric field [61]. J.M De Freitas and MA. Player used the matrix $B = (\Lambda_s^{14})^2 + \Lambda_s^{23}$, in the polarization effect in heterodyne interferometry [73]. It is not difficult to show that

$$S^2 = D = (\Lambda_s^{12})^2 = (\Lambda_s^{13})^2 = (\Lambda_s^{14})^2 + (\Lambda_s^{23})^2 \quad (58c)$$

The adjacency matrix A and the opposite of the degree matrix D of the tetrahedral graph are the simplified unipolar forms of Bipolar NOT Gate and Bipolar Negation Gate respectively [60]. Circulant matrices are applicable to many areas of math and science such as physics and differential equations. They are also useful in digital image processing [62]. There is no circulant Hadamard matrix of size $N \times N$, for any $N \neq 4$ [63]. The only known circulant Hadamard matrices are described by the following matrices expressions:

$$\begin{aligned} &\pm(A - D + M + N); \pm(A + D - M + N) \quad (59a) \\ &\pm(D_{01}A + D + M - iD_{03}); \pm(D - D_{01}A + M + iD_{03}) \end{aligned}$$

come from the vectors:

$$\begin{aligned} \alpha = &\pm(+1, -1, -1, -1), \pm(+1, -i, +1, +i), \\ &\pm(+1, +1, -1, +1), \pm(+1, +i, +1, -i) \end{aligned} \quad (59b)$$

From Eqn. (29c), circulant Hadamard matrices in Eqn. (59a) can be written as:

$$\begin{aligned} &\pm(K^5 - K^4 + K^3 + K^2) \\ &\pm(K^5 + K^4 - K^3 - K^2) \quad (59c) \\ &\pm(D_{01}K^5 + K^4 + K^3 - iD_{03}) \\ &\pm(K^4 - D_{01}K^5 + K^3 + iD_{03}) \end{aligned}$$

where D_{01} and D_{03} are Dirac matrices given in [58].

At $N = 4$ the conjugate of the Fourier matrix is given by:

$$F^* = \frac{1}{2} \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & -i & -1 & +i \\ +1 & -1 & +1 & -1 \\ +1 & +i & -1 & -i \end{pmatrix} \quad (60)$$

Thus the vector $\alpha = F^* \gamma$ are indeed those in the statement.

$$F^{*2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (61a)$$

We therefore have the above matrix F^{*2} expressed in terms of Gell-Mann matrices as follows:

$$F^{*2} = (\Lambda_s^{13})^2 + \Lambda_s^{24} \quad (61b)$$

From Eqn. (60) then the conjugate of the Fourier matrix of order 4 ($N = 4$) to the power 4 gives the degree matrix of the tetrahedral graph, that is

$$F^{*4} = D \quad (61c)$$

The quantum controlled $-Z$ (CZ) gate is given by:

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (62a)$$

$$CZ = V_L^2 \quad (62b)$$

Therefore, using Eqn. (33f) then the quantum controlled $-Z$ (CZ) gate is written as:

$$CZ = B_0 + \tilde{B}_0 + A_1^2 A + B_1 + \tilde{B}_1 - 4B_2^2 \quad (62c)$$

The square of the matrix diagonalizing the adjacency matrix V_A^2 and the square of the matrix diagonalizing the

Laplacian matrix $V_L^2 = V_Q^2$ are solutions to the Yang-Baxter equation and hence represent universal gates, see criteria in [54].

We now represent the Berkeley B Gate with the Hamiltonian in terms of the degree matrix and adjacency matrix of the tetrahedral graph as follows:

$$H = \frac{\pi}{8} (2X \otimes X + Y \otimes Y) \quad (63)$$

where the gate is given by $U = e^{iH}$ and

$$B = e^{\frac{\pi}{8}(2X \otimes X + Y \otimes Y)}$$

Therefore,

$$B = D \cos\left(\frac{3\pi}{8}\right) + iA \sin\left(\frac{3\pi}{8}\right) \quad (64)$$

This means that the Berkeley B Gate is a linear combination of the degree matrix D and adjacency matrix A of the tetrahedral graph. The special 2-Qubit Gates, iSWAP arises naturally in superconducting quantum computing via Hamiltonians implementing the so-called XY model [72].

$$iSWAP = (\Lambda_s^{14})^2 A + i\Lambda_s^{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (65)$$

$$SWAP^\alpha = (\Lambda_s^{14})^2 + \frac{1}{2}(1 + e^{i\pi\alpha})(\Lambda_s^{23})^2 + \frac{1}{2}(1 - e^{i\pi\alpha})(\Lambda_s^{23})^2 \quad (66)$$

Controlled-SIGN Gate (CSIGN)

$$CSIGN = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (67)$$

$$CSIGN = 4B_3^2 + 2B_1 + 2\tilde{B}_1 - 4B_2^2 \quad (68)$$

CSIGN arises naturally in Quantum Linear Optical Computing (LOQC) [64].

We distinguish two versions of the CNOT gate, topCNOT and botCNOT conditioned on the top and bottom lines respectively: (i) botCNOT exchanges $|01\rangle \leftrightarrow |11\rangle$ i.e. CNOT controlled by the top line, and (ii) topCNOT exchanges $|10\rangle \leftrightarrow |11\rangle$ [65]. Those gates can be represented by matrices equations:

$$topCNOT = AK^{-1} = 4B_3^2 + 2B_2 \quad (69a)$$

$$botCNOT = F^{*2} \text{ (the square of } F^* \text{)} \quad (69b)$$

it is a universal gate of 2-level for the Bell basis.

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (70a)$$

From Eqn. (29g) one can compute CNOT as follows:

$$CNOT = MK = AK^{-1} \quad (70b)$$

Using Eqn. (29a), (70a), (70b) then the universal gate F^{*2} is described by:

$$\begin{aligned} F^{*2} &= \frac{1}{4}W_{F_2} CNOT W_{F_2} = \frac{1}{4}W_{F_2} MK W_{F_2} = \frac{1}{4}W_{F_2} AK^{-1} W_{F_2} \quad (70c) \\ &= \frac{1}{4}H_4 CNOT H_4 = \frac{1}{4}H_4 MK H_4 \frac{1}{4}H_4 AK^{-1} H_4 \end{aligned}$$

Using Eqn. (33f), the phase gate CSIGN is related to controlled NOT (CNOT) gate as follows:

$$CSIGN = CNOT + 2B_1 + 2\tilde{B}_1 - 2B_2 - 4B_2^2 \quad (71a)$$

$$CSIGN = 2B_1 + 2\tilde{B}_1 + 4B_3^2 - 4B_2^2 \quad (71b)$$

The controlled phase shift gate is given by the following matrix equation:

$$B(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\varphi} \end{pmatrix} \quad (72a)$$

$$B(\varphi) = 4B_3^2 + B_1 + \tilde{B}_1 + e^{i\varphi}(4B_2^2 - B_1 - \tilde{B}_1) \quad (72b)$$

Therefore, the NOT operation on the second qubit of a two-qubit system is represented by the unitary matrix, and we realize that $U_{NOT} = N$. The Toffoli or simply the three-bit Toffoli gate, T3 applies a NOT to the third bit if the first two bits are in $|11\rangle$, but otherwise having no effect [66]. It can be represented as follows:

$$\begin{pmatrix} D & O \\ O & CNOT \end{pmatrix} = \begin{pmatrix} D & O \\ O & MK \end{pmatrix} = \begin{pmatrix} D & O \\ O & AK^{-1} \end{pmatrix} \quad (73)$$

The Fredkin (controlled-SWAP) is universal for classical reversible computing and has the following representation:

$$\begin{pmatrix} D & O \\ O & S \end{pmatrix} \quad (74)$$

where D is the degree matrix of the tetrahedral graph. The adjacency matrix A is the Hermitian P part of the unitary polar decomposition of $E^*U_f E$, where

$$E = \frac{\sqrt{2}}{2} \begin{pmatrix} +1 & +i & 0 & 0 \\ 0 & 0 & +i & +1 \\ 0 & 0 & +i & -1 \\ +1 & -i & 0 & 0 \end{pmatrix} \quad (75)$$

E^* and U_f are the entangler gate, the disentangler gate and black-box gate respectively. the entangler gate is useful for breaking down arbitrary two-qubit computations into elementary gates [65]. U_f swaps $|00\rangle \leftrightarrow |01\rangle$.

$$E^*U_f EE^t U_f^t \bar{E} = PP^t = P^2 = (CNOT)K = MK^2 = A \quad (76)$$

Consider the elementary braid matrices for 6 quasi-particles defined in [67], Eqn.(33e), (33f), (23a) and (25a), one can obtain

$$\rho(\sigma_1) = 4B_3^2 + i4B_2^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \quad (77a)$$

$$\rho(\sigma_2) = (\Lambda_s^{13})^2 - i\Lambda_s^{12} = \begin{pmatrix} +1 & 0 & -i & 0 \\ 0 & +1 & 0 & -i \\ -i & 0 & +1 & 0 \\ 0 & -i & 0 & +1 \end{pmatrix} \quad (77b)$$

$$\rho(\sigma_3) = (\Lambda_s^{14})^2 + i(\Lambda_s^{23})^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (77c)$$

$$\rho(\sigma_4) = (\Lambda_s^{12})^2 - i\Lambda_s^{12} = \begin{pmatrix} +1 & -i & 0 & 0 \\ -i & +1 & 0 & 0 \\ 0 & 0 & +1 & -i \\ 0 & 0 & -i & +1 \end{pmatrix} \quad (77d)$$

$$\rho(\sigma_5) = (\Lambda_s^{13})^2 + i(\Lambda_s^{24})^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \quad (77e)$$

Therefore, the Hadamard gate H_1 acting on acting on qubit 1 and the Hadamard gate H_2 acting on the qubit 2 are given by:

$$H_1 \approx \Lambda_s^{12} - (4B_2^2 - 4B_3^2) = \rho(\sigma_1^{-1})\rho(\sigma_2^{-1})\rho(\sigma_1^{-1}) \quad (78)$$

See Eqn.(21a) and (33d)

$$H_2 \approx \Lambda_s^{12} - D_{01} = \rho(\sigma_4)\rho(\sigma_5)\rho(\sigma_4)$$

respectively.

$$H_2 \approx 2B_0 + 2B_1 + 2\tilde{B}_4 \quad (79)$$

See Eqn. (25a) and(33f)

The Controlled-Z gate is given by the square of the matrix diagonalizing the Laplacian matrix.

$$CZ \approx V_L^2 = \rho(\sigma_1)\rho(\sigma_3^{-1})\rho(\sigma_5) \quad (80)$$

V. CONCLUSION

This research has aimed at investigating the product of the sine polynomial consisting of a sum of four sine waves equivalent to the superposition of four plane waves. First, we have transformed it into a phase-space yielding the matrix of the vertices of the tetrahedral graph. Next we have derived the adjacency and the Laplacian matrices of the tetrahedral graph. Then we have presented different matrix bases which are useful to the decomposition of the matrices derived from the tetrahedral graph. Finally, the spectral domain analysis of the tetrahedral graph under study manifested huge potential attracting properties of the spectrahedron in image processing, quantum information processing and in the design of quantum logic gates. The adjacency matrix and the opposite of the degree matrix of the tetrahedral graph are the simplified unipolar forms of Bipolar NOT Gate and Bipolar Negation Gate respectively. It is also the Hermitian part of the unitary polar decomposition in terms of elementary gates for quantum computation. In this paper, we have also showed that the square of the matrices diagonalizing both the adjacency matrix and Laplacian matrix V_A^2 and V_L^2 respectively, are solutions to the Yang-Baxter equation and hence represent universal gates. The Berkley B gate is a function of adjacency matrix and degree matrix. The powers of the permutation matrix K describe circulant Hadamard matrices and hence the tetrahedral. The results presented in this paper permit to extract the essential signal information by simply knowing the tetrahedral graph vertices. This paper obviously serves as a foundation for a variety of useful applications of graph theory to quantum image processing and quantum data compression and related areas. The link among the tetrahedral graph, wavelets, frames and quantum logic gates opens up new directions

for future [69] research. The findings of this work deserve more deep investigations in the areas of signal processing techniques, optical communication systems, coding theory and orbital structures applications.

ACKNOWLEDGEMENT

The author would like to thank the referees for their valuable comments and insights.

REFERENCES

- [1] Nora Leonardi, Dimitri Van De Ville, "Wavelet Frames on Graphs Defined by FMRI Functional Connectivity", *IEEE, ISBI*, 2011.
- [2] Akie Sakiyama and Yuichi Tanaka, "Spectral Graph Wavelets and Filter Banks With Low Approximation Error", *IEEE Transactions on Signal and Information Processing Over Networks*, Vol. 2, No 3, June 2016., "doi:10.1109/TSIPN.2016.2581303
- [3] Nicolas Tremblay and Pierre Borgnat, "Subgraph Base Filter banks for Graphs Signals", Dec 2015, HAL Id: hal-0124388.
- [4] D. Shuman, S. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, "The emerging field of signal processing on graphs: Extending highdimensional data analysis to networks and other irregular domains," *Signal Processing Magazine, IEEE*, Vol. 30, no. 3, pp. 83–98, May 2013.
- [5] R. Vilela Mendes, Hugo C. Mendes, Tanya Ara'ujo," Signal Processing on Graphs: Transforms and Tomograms", June 2014.
- [6] D. I. Shuman, S. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, "The Emerging Field of Signal Processing on Graphs: Extending Highdimensional Data Analysis to Networks and Other Irregular Domains," *IEEE Signal Process. Mag.*, vol. 30, No. 3, pp. 83–98, 2013.
- [7] A. Sandryhaila and J. M. F. Moura, "Discrete Signal Processing on graphs," *IEEE Trans. Signal Process.*, Vol. 61, No. 7, pp. 1644–1656, 2013.
- [8] David K. Hammond, Pierre Vandergheynst and Remi Gribonval, Wavelets on Graphs via Spectral Graph Theory, Applied and Computational Harmonics Analysis, Vol. 30, pp. 129-150, 2011
- [9] C. Godsil and G. F. Royle, Algebraic Graph Theory. Springer, 2001.
- [10] F. K. Chung, Spectral Graph Theory. Vol. 92 of the CBMS Regional Conference Series in Mathematics, AMS Bokstore, 1997.
- [11] D. Spielman, "Spectral graph theory," in Combinatorial Scientific Computing. Chapman and Hall / CRC Press, 2012.
- [12] Sunkyuu Yu, Xianji Piao, Jiho Hong and Namkyoo Park Inter-dimensional Optical Isospectrality Inspired by Graph Networks, 2016, <https://arxiv.org/ftp/arxiv/papers/1601/1601.05635.pdf>
- [13] Chensheng Wu, Jonathan Ko, Christopher C. Davis, Using Graph Theory and a Plenoptic Sensor to Recognize Phase Distortions of a Laser Beam, <https://arxiv.org/ftp/arxiv/papers/1506/1506.00310.pdf>
- [14] Kan Wu, Javier Garcia de Abajo, Cesare Soci, Perry Ping Shum, and Nikolay I Zheludev, An Optical Fiber Network Oracle for NP-Complete Problems, Light: Science & Applications (2014) 3,e147; doi:10.1038/lsa.2014.28
- [15] Mario Krenn, Xuemei Gu and Anton Zeilinger, Quantum Experiments and Graphs: Multiparty States as Coherent Superpositions of Perfect Matchings, May 2017.
- [16] M. Krenn, M. Malik, R. Fickler, R. Lapkiewicz and A. Zeilinger, Automated Search for New Quantum Experiments. Physical review letters 116, 09040, 5 2016.
- [17] M. Hein, J. Eisert and H.J. Briegel, Multiparty Entanglement in Graph States. Physical Review A 69,062311, 2004.
- [18] C.Y. Lu, X.Q. Zhou, O. Gühne, W.B. Gao, J. Zhang, Z.S. Yuan, A. Goebel, T. Yang and J.W. Pan, Experimental Entanglement of Six Photons in Graph States. Nature Physics 3, 91–95, 2007.
- [19] S.L. Braunstein, S. Ghosh and S. Severini, The Laplacian of a Graph as a Density Matrix: A Basic Combinatorial Approach to Separability of Mixed States. Annals of Combinatorics 10, 291–317, 2006.
- [20] Dragan Stevanovic, Applications of Graph Spectra in Quantum Mechanics, Mathematics Subject Classification, 2010.
- [21] R. Raussendorf and H.J. Briegel, A one-way quantum computer. Physical Review Letters 86, 5188, 2001.
- [22] Ashok Muthukrishnan, Classical and Quantum Logic Gates: An Introduction to Quantum Computing, Quantum Information Seminar, September 1999.
- [23] L. Pauling, J. Chem. Phys. 4, 673 (1936).
- [24] H. Kuhn, Helv. Chim. Acta, 31, 1441 (1948).
- [25] C. Flesia, R. Johnston, and H. Kunz, Europhys. Lett. 3, 497 (1987).
- [26] R. Mitra and S. W. Lee, Analytical techniques in the Theory of Guided Waves (Macmillan, New York, 1971).
- [27] Y. Imry, Introduction to Mesoscopic Systems (Oxford, New York, 1996).
- [28] D. Kowal, U. Sivan, O. Entin-Wohlman, Y. Imry, Phys. Rev. B 42, 9009 (1990).
- [29] E. L. Ivchenko, A. A. Kiselev, JETP Lett. 67, 43 (1998).
- [30] J.A. Sanchez-Gil, V. Freilikher, I. Yurkevich, and A. A. Maradudin, Phys. Rev. Lett. 80, 948 (1998).
- [31] Y. Avishai and J.M. Luck, Phys. Rev. B 45, 1074 (1992).
- [32] T. Nakayama, K. Yakubo, and R. L. Orbach, Rev. Mod. Phys. 66, 381 (1994).
- [33] T. Kottos and H. Schanz, Physica E 9, 523 (2003).
- [34] T. Kottos and U. Smilansky, J. Phys. A 36, 3501 (2003).
- [35] Oleh Hul, Oleg Tymoshchuk, Szymon Bauch, Peter M. Koch and Leszek Sirko, Experimental Investigation of Wigner's Reaction Matrix for Irregular Graphs with Absorption, March 2009.
- [36] Trystan Koch, Ziyuan FU, Electrical Networks and the Graph Model, 2016
- [37] Darwin Gosal and Wayne Lawton, Quantum Haar Wavelet Transforms and Their Applications, November 2001.
- [38] C.P. Williams, Explorations in Quantum Computing, Texts in Computer Science, Chapter 2: Quantum Gates, DOI 10.1007/978-1-84628-887-6_2, Springer-Verlag London Limited, 2011.
- [39] Amir Fijany and Colin P. Williams, Quantum Wavelet Transforms: Fast Algorithms and Complete Circuits, 1st NASA Int. Conf. on Quantum Computing and Communication, September 1998.
- [40] P. Shor, Algorithms for Quantum Computation: Discrete Logarithms and Factoring, Proc. 35th Annual Symposium on Foundations of Computer Science, p. 124, 1994.
- [41] L. K. Grover, A Fast Quantum Mechanical Algorithm for Database Search, Proc. 28th Annual ACM Symposium on the Theory of Computing, Philadelphia, p. 212, 1996.
- [42] R. Jozsa, Quantum Algorithms and the Fourier Transform, Los Alamos preprint archive, <http://xxx.lanl.gov/archive/quant-ph/9707033>, 1997.
- [43] C. Van Loan, Computational Frameworks for the Fast Fourier Transform, SIAM Publications, Philadelphia, 1992.

- [44] John Ashmead, Morlet Wavelets in Quantum Mechanics, *Quanta*, Vol.1, Issue 1, pp:58-78, November 2012, doi: 10.12743/quanta.v1i1.5
- [45] Georgiev L. S., Topologically Protected Quantum Gates for Computation with NonAbelian Anyons in the Pfaffian Quantum Hall State, *Phys. Rev. B* 74, 235112 [arXiv:condmat/0607125 [cond-mat.mes-hall],2006.
- [46] Georgiev L. S., Towards a Universal Set of Topologically Protected Gates for Quantum Computation with Pfaffian qubits, *Nucl. Phys. B* 789, 552 [arXiv:hep-th/0611340], 2008
- [47] Georgiev L. S., Ultimate Braid-Group Generators for Coordinate Exchanges of Ising Anyons from the Multi-Anyon Pfaffian wave functions, *J. Phys. A: Math. Theor.* 42, 225203 [arXiv:0812.2334 [math-ph], 2009.
- [48] Ovidiu Racorean, Quantum Gates and Quantum Circuits of Stock Portfoliofile:///F:/STOCK%20QUANTUM.pdf
- [49] Christopher L. Mueller, Techniques for Resonant Optical Interferometry with Applications to the Advanced Ligo Gravitational Wave Detectors, PhD Thesis, 2014
- [50] H.Kogelnik and T.Li, Laser Beams and Resonators, 1150 *Applied Physics*, Vol.5, No 10, October 1966.
- [51] Greyson Gilson, Unified Theory of Wave-Particle Duality, the Schrödinger Equations and Quantum Diffraction, September 2014.
- [52] Martin Aigner, On the Tetrahedral Graph, *Pacific Journal of Mathematics*, Vol.25, No 2, October 1968.
- [53] H.S.M. Coxeter, *Regular Polytopes*, Dover, New York, 1973.
- [54] Lois H. Kauffman and Samuel J Lomonaco Jr, Braiding Operators are Universal Quantum Gates, *New Journal of Physics* 6 (2004) 134, October 2004. doi:10.1088/1367-2630/6/1/134
- [55] Florin Nichita, Introduction to the Yang-Baxter Equation with Open Problems, *Axioms* 2012, 1, 33-37; doi:10.3390/axioms1010033, April 2012.
- [56] Michio Jimbo, Introduction to Yang-bater Equation, *Internation Journal of Modern Physics A*, Vol.4, No.15 (1989), pp.3759-3777, March 1989.
- [57] Dye H, *Quantum Inform Process.* 2 117 (Preprint quant-ph/0211050v3), August 2003.
- [58] Magnus Karlsson, The Connection between Polarization Calculus and Four-Dimensional Rotations, March 2013.
- [59] Jean Krommweh, Tight Frame Characterization of Multiwavelet Vector Function in Terms of the Polyphase Matrix, January 2008. file:///F:/QUANTUM%20QUINCUNX%203.pdf
- [60] Wen-Ran Zhang, Bipolar Quantum Logic Gates and Quantum Cellular Combinatorics – A Logical Extension to Quantum Entanglement, June 2013. file:///F:/Drive/Bipolar%20Quantum%20Logic%20Gates%20and%20Quantum%20Cellular%20Combinatorics%20%20.pdf
- [61] Strang, G., *Linear Algebra and Its Applications*, 4th Edn (Thomson/Brooks/Cole), 2006.
- [62] Theodor Banica, Ion Netchita and Karol Zyczkowski, *Almost Hadamard Matrices: General Theory and Examples*, 2000.
- [63] Theodor Banica, Ion Netchita and Karol Zyczkowski, *Almost Hadamard Matrices: General Theory and Examples*, 2000.
- [64] C.P. Williams, *Explorations in Quantum Computing*, Texts in Computer Science, Chapter2: Quantum Gates, DOI 10.1007/978-1-84628-887-6_2, Springer-Verlag London Limited, 2011.
- [65] Stephen S. Bullock and Igor L. Markov, An Arbitrary Two-qubit Computation In 23 Elementary Gates, Feb. 2008.
- [66] Ashok Muthukrishnan, *Classical and Quantum Logic Gates: An Introduction to Quantum Computing*, Quantum Information Seminar, September 1999.
- [67] Ovidiu Racorean, *Quantum Gates and Quantum Circuits of Stock Portfolio*file:///F:/STOCK%20QUANTUM.pdf
- [68] Chapter 2: Matrix Description of Wave Propagation and Polarization,file:///F:/QUANTUM%20OPTIC%20BEAM%20SPLITTER.pdf
- [69] L.D. Baumert and Marshall Hall, Jr., Hadamard Matrices of the Williamson Type, http://mathscinet.ru/files/Williamson_1965.pdf
- [70] http://www4.ncsu.edu/~franzen/public_html/CH736/lecture/Particle_in_a_box.pdf
- [71] Y. Ordokhani, An Application of Walsh Functions for of Fredholm-Hammerstein Integro-Differential Equation, *Int.J. Contemp.Math Sciences*, Vol.5, No22, pp.1055-1063, 2010.
- [72] Ashish M. Kothari and Ved Vyas Dwivedi, Video Watermaking- Combination of Discrete Wavelet and Cosine Transform to Achieve Extra Robustness, *IJIGSP*, Vol.5, No. 3, pp. 36-41, March 2013, DOI: 10.5815/ijigsp.2013.03.05
- [73] J.M De Freitas and M.A Player, Polarization Effect in Heterodyne Interferometry, *Journal of Modern Optics*, Vol.42, No. 9, pp. 1875-1899, 1995.
- [74] R. J. Duffin and A. C. Schaeffer, "A class of nonharmonic Fourier series," *Trans. Amer. Math. Soc.*, vol. 72, pp. 314–366, 1952.
- [75] R. M. Young, *An introduction to nonharmonic Fourier series*, New York: Academic Press, 1980.
- [76] A. Aldroubi and K. Gröchenig, "Non-uniform sampling and reconstruction in shift-invariant spaces," *Siam Rev.*, to appear.
- [77] Y. C. Eldar and A. V. Oppenheim, "Orthogonal and projected orthogonal matched-filter detection," submitted to *IEEE Trans. Signal Processing*, Jan. 2001.
- [78] Y. C. Eldar and A. M. Chan, "Orthogonal and projected orthogonal multiuser detection," submitted to *IEEE Trans. Inform. Theory*, May 2001.
- [79] V. K. Goyal, J. Kovačević, and M. Vetterli, "Multiple description transform coding: Robustness to erasures using tight frame expansions," in *Proc. IEEE Int. Symp. Inform. Theory, Cambridge, MA*, Aug. 1998, p. 408.
- [80] H. Bölcskei, *Oversampled filter banks and predictive subband coders*, Ph.D. thesis, Vienna University of Technology, Nov. 1997.
- [81] H. Bölcskei, F. Hlawatsch, and H. G. Feichtinger, "Frame-theoretic analysis of oversampled filter banks," *IEEE Trans. on Signal Processing*, vol. 46, no. 12, pp. 3256–3268, Dec. 1998.
- [82] P. J. S. G. Ferreira, "Mathematics for multimedia signal processing II — discrete finite frames and signal reconstruction," in *Signal processing for multimedia*, J. S. Byrnes, Ed., pp. 35–54. IOC Press, 1999.
- [83] Yonina C. Eldar and G. David Forney, Jr., *Optimal Tight Frames and Quantum Measurement*, March 2018.
- [84] Reinhold A. Bertlmann, and Philipp Krammer, *Bloch Vectors for Qudits*, June 2008.

Author's Profile

Eng. Jean Bosco Mugiraneza is a PhD student at the University of Rwanda (UR), African Centre of Excellence in Energy for Sustainable Development (ACE-ESD). He serves at the Board of Directors of the “Société Internationale d’Electricité des Pays des Grands Lacs”. He is the former Chief Executive Officer of Rwanda Energy Group Limited and

former Board Member of the Executive Committee of the Association of Power Utilities of Africa. Eng. Jean Bosco Mugiraneza is a Professional Electrical Engineer, Fellow of the

Association of Engineers Rwanda (FIER) and Member of International Association of Engineers (MIAENG). He graduated from the City University of New York & Graduate Centre in USA with the Master of Engineering Degree in Electrical Engineering in 2006. His research interest includes the applications of Julia Sets and the dynamics of Rational Maps in wavelets analysis as well as Quantum wavelet transforms in electrical power systems. He has published some papers on Continuous Wavelets Transforms in the Dynamics of Rational Maps. He is the co-author of the book on Principles of Engineering Analysis.

How to cite this paper: Jean Bosco Mugiraneza, "Quantum Wavelet Transforms Generated by the Product of the Sine Polynomial and the Gaussian Envelope on the Tetrahedral Graph", *International Journal of Image, Graphics and Signal Processing(IJIGSP)*, Vol.10, No.7, pp. 11-24, 2018.DOI: 10.5815/ijigsp.2018.07.02