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Four-dimensional Vector Matrix Determinant and Inverse

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Abstract

The theory of two-dimensional matrix has been popularized in multi-dimensional matrix. However applications of multi-dimensional matrix also bring space redundancy and time redundancy, we put forward a multi-dimensional vector matrix model. This is new series of study to define multidimensional vector matrix mathematics, including four-dimensional vector matrix determinant, four-dimensional vector matrix inverse and related properties. There is innovative concept of multi-dimensional vector matrix mathematics created by author with numerous applications in engineering, math, video conferencing, 3D TV, and other fields.

Index Terms: Multidimensional vector matrix, four-dimensional vector matrix determinant, four-dimensional vector matrix inverse

1. Introduction

This paper brings a new branch of mathematics called multidimensional vector matrix mathematics and its new subsets, four-dimensional vector matrix determinant and four-dimensional vector matrix inverse. The traditional matrix mathematics [1] that engineering, science, and math students are usually introduced to in college handles matrices of one or two dimensions. Ashu M. G. Solo [2] also defined some multidimensional matrix algebra operations. Based on these theories and papers, multidimensional vector matrix extends traditional matrix math to any figure of dimensions. Therefore, traditional matrix math is a subset of multidimensional vector matrix mathematics.

This paper mainly brings forward the definition of four-dimensional vector matrix determinant and the four-dimensional vector matrix inverse. We adopt the form that is different from the definition of two-dimensional matrix. But the properties of two-dimensional matrix determinant and inverse can be extended to the four-dimensional vector matrix. The extension of classical matrix mathematics to any figure of dimensions has various applications in many branches of engineering, math, image compression, coding and other fields. We

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should promote the other applications of multidimensional vector matrix math that could not be done without his multidimensional vector matrix math.

Our group has proposed the definition of multidimensional vector matrix, multiplication of multi-dimensional vector matrices, multidimensional Walsh orthogonal transform and traditional discrete cosine transform [3]. Their application in color image compression and coding is more and more common and widespread. For one thing, it conquers the restriction of traditional two-dimensional matrix multiplication. For another thing, it carries on high efficiency of traditional matrix transform in the aspect of removing redundancy of color space. By means of multi-dimensional vector matrix model, color image data can be expressed and processed in a unified mathematical model, and better compression results are received.

In Section 2, a multi-dimensional vector matrix model will be introduced, and the related properties will be discussed. In Section 3, we will propose the definitions of four-dimensional vector matrix determinant and inverse. Verification the truth of formula with regard to the four-dimensional vector matrix determinant and inverse will be also given in the same Section. Section 4 concludes this paper.

2. Proposed Theory

Based on the multidimensional vector matrix definition proposed by our group, we will further study four-dimensional vector matrix adjoin matrix, determinant, inverse matrix and related properties.

A. The definition of multi-dimensional vector matrix:

An array of numbers $\left(a_{i_1 i_2}\right)$ in two directions (one direction has M entries and the other direction has N entries) is called two-dimensional matrix, and the set of all such matrices is represented as $M_{M\times N}$ .An array of numbers $\left(a_{i_1 i_2 \cdots i_n}\right)$ in n directions (each direction has $I\ i$ entries, $1 \le i \le n$. $I\ i$ can be called the order in this direction) is called multi-dimensional matrix, and the set of all such matrices is denoted as $M_{I_1\times I_2\times\cdots\times I_n}$ [4].

If the dimensions of multi-dimensional matrix $M_{K_1\times K_2\times\cdots\times K_r}$ are separated into two sets and the matrix is denoted as $M_{\overrightarrow{I_1\times\cdots I_m}\times\overrightarrow{J_1\times\cdots\times J_n}}$, where $m+n=r$. $M_{\overrightarrow{I_1\times\cdots I_m}\times\overrightarrow{J_1\times\cdots\times J_n}}$ can be denoted as M_{IJ}, where I and J are for the vectors, $I=\left(I_1,I_2,\dots,I_m\right)$, $J=\left(J_1,\ J_2,\dots,\ J_n\right)$. $M_{\overrightarrow{I_1\times\cdots I_m}\times\overrightarrow{J_1\times\cdots\times J_n}}$ can be called multi-dimensional vector matrix separated according to the vector I, and J, multidimensional vector matrix in short[4].

A multi-dimensional matrix has various relevant multi-dimensional vector matrices, whereas a multi-dimensional vector matrix has unique relevant multi-dimensional matrix.

B. Multi-dimensional vector identity matrix:

Let A_{IJ} be a multidimensional vector matrix, where $I=\left(I_1,I_2,\dots,I_m\right)$, $J=\left(J_1,\ J_2,\dots,\ J_n\right)$. If vector $I=J$, then A_{IJ} is called multidimensional vector square matrix [5].

Let $\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$, where $i=j$ represents vector $I=\left(I_1,I_2,\dots,I_m\right)$, $J=\left(J_1,\ J_2,\dots,\ J_n\right)$, if it has the same dimension, the meanings of $i=j$ is that $m=n$, and $i_1=j_1,\cdots,i_m=j_n$.

If $A_{IJ}=\left(\delta_{ij}\right)$, A_{IJ} is said to be multi-dimensional vector identity matrix , denoted as E_{II} , or E simply.

C. Multiplication of multi-dimensional vector matrices:

Let A_{IJ} and B_{UV} be two multi-dimensional vector matrices, in which $I=(I_1,I_2,...,I_m)$, $J=(J_1,J_2,...,J_n)$, $U=(U_1,U_2,...,U_s)$, $V=(V_1,V_2,...,V_t)$, If $J=U$, then A_{IJ} and B_{UV} are multiplicative.

Let A_{IL} be I × L matrix and B_{LJ} be L × J matrix. The result of multiplication of A_{IL} and B_{LJ} is defined as a I × J matrix [4] $C=\left(c_{\overrightarrow{i_1\cdots i_m}\,\overrightarrow{j_1\cdots j_n}}\right)$,

$$c_{\overrightarrow{i_1\cdots i_m}\,\overrightarrow{j_1\cdots j_n}}=\sum_{\mathbf{L}} a_{\mathbf{il}}b_{\mathbf{lj}}=\sum_{l_1}^{L_1}\cdots\sum_{l_k}^{L_k} a_{\overrightarrow{i_1\cdots i_m}\,\overrightarrow{l_1\cdots l_k}}\, b_{\overrightarrow{l_1\cdots l_k}\,\overrightarrow{j_1\cdots j_n}} \tag{1}$$

Which is denoted as $C_{IJ}=A_{IL}\;B_{LJ}$.

For simplicity, the signal $\sum_{l_1}^{L_1}\cdots\sum_{l_k}^{L_k}(\cdots)$ is rewritten as $\sum_{L}(\cdots)$ and the signal $a_{\overrightarrow{i_1\cdots i_m}\,\overrightarrow{l_1\cdots l_k}}$ is rewritten as a_{il}.If no specified, this kind of form is default.

D. Multi-dimensional vector matrix transpose

The definition of multi-dimensional vector matrix transpose

$$A_{IJ}^{\;T}=A_{JI}$$

3. Four-dimensional Vector Matrix Determinant And Inverse

The multidimensional vector matrix determinant for a one-dimensional matrix is undefined. The multidimensional vector matrix determinant for a two-dimensional square matrix is calculated using the traditional methods. The multidimensional vector matrix determinant of a two-dimensional non-square matrix is undefined.

Hence, at first, a four-dimensional vector matrix which can be calculated determinant should be a four-dimensional vector square matrix. Secondly, commutative matrices must be square matrices with the same orders.

For instants, a four-dimensional vector square matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}=\left(a_{\overrightarrow{i_1i_2}\,\overrightarrow{j_1j_2}}\right)_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$, including $1\le i_1\le m$, $1\le i_2\le n$, $1\le j_1\le m$ and $1\le j_2\le n$.

For a four-dimensional square vector matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$, all the elements of four vector directions where the element $a_{\overrightarrow{i_1i_2}\,\overrightarrow{j_1j_2}}$ in the matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$ is located can be cancelled. The other elements are regularly collected in a matrix with the orders of $(m\times n-1)$ and then its determinant can be calculated. The matrix determinant can be called the cofactor of the element $a_{\overrightarrow{i_1i_2}\,\overrightarrow{j_1j_2}}$, denoted as $M_{\overrightarrow{i_1i_2}\,\overrightarrow{j_1j_2}}$, then

$$A_{\overrightarrow{i_1i_2}\,\overrightarrow{j_1j_2}}=(-1)^{\left[(i_1-1)n+i_2\right]+\left[(j_1-1)n+j_2\right]}M_{\overrightarrow{i_1i_2}\,\overrightarrow{j_1j_2}}$$

$A_{\overrightarrow{i_1i_2}\,\overrightarrow{j_1j_2}}$ can be said the vector cofactor of the element $a_{\overrightarrow{i_1i_2}\,\overrightarrow{j_1j_2}}$.

A. The definition of four-dimensional vector square matrix determinant

For a four-dimensional vector square matrix, each element of any vector direction is multiplied by its vector cofactor and then all the products are added. The product can be called the four-dimensional vector square matrix determinant.

$$\left|A_{\vec{m}\times n\times\vec{m}\times n}\right| = \sum_{i_1=1}^{m}\sum_{i_2=1}^{n} a_{i_1 i_2 \vec{j_1} \vec{j_2}} A_{i_1 i_2 \vec{j_1} \vec{j_2}} \quad \begin{matrix} j_1=1,2,\ldots,m \\ j_2=1,2,\ldots,n \end{matrix}$$

$$\text{or } \left|A_{\vec{m}\times n\times\vec{m}\times n}\right| = \sum_{j_1=1}^{m}\sum_{j_2=1}^{n} a_{i_1 i_2 \vec{j_1} \vec{j_2}} A_{i_1 i_2 \vec{j_1} \vec{j_2}} \quad \begin{matrix} i_1=1,2,\ldots,m \\ i_2=1,2,\ldots,n \end{matrix}$$

If n=m,

$$\left|A_{\vec{m}\times m\times\vec{m}\times m}\right| = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m} a_{i_1 i_2 \vec{j_1} \vec{j_2}} A_{i_1 i_2 \vec{j_1} \vec{j_2}} \quad \begin{matrix} j_1=1,2,\ldots,m \\ j_2=1,2,\ldots,m \end{matrix}$$

$$\text{or} \left|A_{\vec{m}\times m\times\vec{m}\times m}\right| = \sum_{j_1=1}^{m}\sum_{j_2=1}^{m} a_{i_1 i_2 \vec{j_1} \vec{j_2}} A_{i_1 i_2 \vec{j_1} \vec{j_2}} \quad \begin{matrix} i_1=1,2,\ldots,m \\ i_2=1,2,\ldots,m \end{matrix}$$

Similarly, all elements of any vector direction in the four-dimensional vector square matrix are multiplied by the vector cofactor of corresponding elements in another vector direction and then all the products are added. The result is zero.

$$a_{\vec{i_1}\vec{i_2}11} A_{\vec{j_1}\vec{j_2}11} + \cdots + a_{\vec{i_1}\vec{i_2}1n} A_{\vec{j_1}\vec{j_2}1n} + \cdots + a_{\vec{i_1}\vec{i_2}mn} A_{\vec{j_1}\vec{j_2}mn} = 0$$

$$a_{11\vec{i_1}\vec{i_2}} A_{11\vec{j_1}\vec{j_2}} + \cdots + a_{1n\vec{i_1}\vec{i_2}} A_{1n\vec{j_1}\vec{j_2}} + \cdots + a_{mn\vec{i_1}\vec{i_2}} A_{mn\vec{j_1}\vec{j_2}} = 0$$

If n=m,

$$a_{\vec{i_1}\vec{i_2}11} A_{\vec{j_1}\vec{j_2}11} + \cdots + a_{\vec{i_1}\vec{i_2}1m} A_{\vec{j_1}\vec{j_2}1m} + \cdots + a_{\vec{i_1}\vec{i_2}mm} A_{\vec{j_1}\vec{j_2}mm} = 0$$

$$a_{11\vec{i_1}\vec{i_2}} A_{11\vec{j_1}\vec{j_2}} + \cdots + a_{1m\vec{i_1}\vec{i_2}} A_{1m\vec{j_1}\vec{j_2}} + \cdots + a_{mm\vec{i_1}\vec{i_2}} A_{mm\vec{j_1}\vec{j_2}} = 0$$

In conclusion,

$$a_{\vec{i_1}\vec{i_2}11} A_{\vec{j_1}\vec{j_2}11} + \cdots + a_{\vec{i_1}\vec{i_2}mn} A_{\vec{j_1}\vec{j_2}mn} = \begin{cases} |A| & i_1=j_1, i_2=j_2 \\ 0 & i_1\neq j_1, i_2\neq j_2 \end{cases} \tag{2}$$

$$a_{11\vec{i_1}\vec{i_2}} A_{11\vec{j_1}\vec{j_2}} + \cdots + a_{mn\vec{i_1}\vec{i_2}} A_{mn\vec{j_1}\vec{j_2}} = \begin{cases} |A| & i_1=j_1, i_2=j_2 \\ 0 & i_1\neq j_1, i_2\neq j_2 \end{cases} \tag{3}$$

B. The definition of four-dimensional vector square matrix inverse

The multidimensional vector matrix inverse for a one-dimensional matrix is undefined. The multidimensional vector matrix inverse of a two-dimensional matrix exists if it is a square matrix and has a nonzero determinant, and is calculated using the standard means in traditional matrix math. For the four-dimensional vector matrix, each four-dimensional vector square matrix with a nonzero determinant is necessary. Firstly, we define the four-dimensional vector adjoin matrix.

The definition of four-dimensional vector adjoin matrix,

$$A_{\vec{m}\times n\times\vec{m}\times n}^{*} = \left(A_{i_1 i_2 \vec{j_1}\vec{j_2}}\right)_{\vec{m}\times n\times\vec{m}\times n}^{T}$$

If a four-dimensional vector square matrix $A_{\vec{m}\times n\times\vec{m}\times n}$ is invertible, and $\left|A_{\vec{m}\times n\times\vec{m}\times n}\right|$, then

$$A_{\vec{m}\times n\times\vec{m}\times n}^{-1} = \frac{1}{\left|A_{\vec{m}\times n\times\vec{m}\times n}\right|} A_{\vec{m}\times n\times\vec{m}\times n}^{*} \tag{4}$$

If n=m, $$A_{\vec{m}\times m\times\vec{m}\times m}^{-1} = \frac{1}{\left|A_{\vec{m}\times m\times\vec{m}\times m}\right|} A_{\vec{m}\times m\times\vec{m}\times m}^{*}$$

C. The properties of four-dimensional vector square matrix determinant and inverse

In traditional matrix mathematics, if a matrix possesses an inverse and that matrix is multiplied by its inverse, the product is an identity matrix with the same dimensions.

Because multidimensional vector matrices are a concatenation of two-dimensional matrices, if a four-dimensional vector matrix has an inverse and that four-dimensional vector matrix is multiplied by its inverse, then the product will be a four-dimensional vector identity matrix with the same dimensions.

Due to the definition of four-dimensional vector matrix inverse（4）and $\left|A_{\overline{m\times n}\times\overline{m\times n}}\right|\neq 0$, if this matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{*}$ is the four-dimensional vector adjoin matrix, we can conclude

$$\left(A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{-1}\right)A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}=\left(\frac{1}{\left|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}\right|}A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{*}\right)A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$$

$$=\frac{1}{\left|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}\right|}\begin{bmatrix}\begin{bmatrix}A_{\overrightarrow{11}\overrightarrow{11}} & \cdots & A_{\overrightarrow{11}\overrightarrow{1n}}\\ \vdots & \ddots & \vdots\\ A_{\overrightarrow{11}\overrightarrow{m1}} & \cdots & A_{\overrightarrow{11}\overrightarrow{mn}}\end{bmatrix} & \iddots & \begin{bmatrix}A_{\overrightarrow{1n}\overrightarrow{11}} & \cdots & A_{\overrightarrow{1n}\overrightarrow{1n}}\\ \vdots & \ddots & \vdots\\ A_{\overrightarrow{1n}\overrightarrow{m1}} & \cdots & A_{\overrightarrow{1n}\overrightarrow{mn}}\end{bmatrix}\\ & & \\ \begin{bmatrix}A_{\overrightarrow{m1}\overrightarrow{11}} & \cdots & A_{\overrightarrow{m1}\overrightarrow{1n}}\\ \vdots & \ddots & \vdots\\ A_{\overrightarrow{m1}\overrightarrow{m1}} & \cdots & A_{\overrightarrow{m1}\overrightarrow{mn}}\end{bmatrix} & \iddots & \begin{bmatrix}A_{\overrightarrow{mn}\overrightarrow{11}} & \cdots & A_{\overrightarrow{mn}\overrightarrow{1n}}\\ \vdots & \ddots & \vdots\\ A_{\overrightarrow{mn}\overrightarrow{m1}} & \cdots & A_{\overrightarrow{mn}\overrightarrow{mn}}\end{bmatrix}\end{bmatrix}_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$$

$$\times\begin{bmatrix}\begin{bmatrix}a_{\overrightarrow{11}\overrightarrow{11}} & \cdots & a_{\overrightarrow{1n}\overrightarrow{11}}\\ \vdots & \ddots & \vdots\\ a_{\overrightarrow{m1}\overrightarrow{11}} & \cdots & a_{\overrightarrow{mn}\overrightarrow{11}}\end{bmatrix} & \iddots & \begin{bmatrix}a_{\overrightarrow{11}\overrightarrow{1n}} & \cdots & a_{\overrightarrow{1n}\overrightarrow{1n}}\\ \vdots & \ddots & \vdots\\ a_{\overrightarrow{m1}\overrightarrow{1n}} & \cdots & a_{\overrightarrow{mn}\overrightarrow{1n}}\end{bmatrix}\\ & & \\ \begin{bmatrix}a_{\overrightarrow{11}\overrightarrow{m1}} & \cdots & a_{\overrightarrow{1n}\overrightarrow{m1}}\\ \vdots & \ddots & \vdots\\ a_{\overrightarrow{m1}\overrightarrow{m1}} & \cdots & a_{\overrightarrow{mn}\overrightarrow{m1}}\end{bmatrix} & \iddots & \begin{bmatrix}a_{\overrightarrow{11}\overrightarrow{mn}} & \cdots & a_{\overrightarrow{1n}\overrightarrow{mn}}\\ \vdots & \ddots & \vdots\\ a_{\overrightarrow{m1}\overrightarrow{mn}} & \cdots & a_{\overrightarrow{mn}\overrightarrow{mn}}\end{bmatrix}\end{bmatrix}_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$$

$$=\frac{1}{\left|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}\right|}\begin{bmatrix}\begin{bmatrix}\left|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}\right| & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\end{bmatrix} & \iddots & \begin{bmatrix}0 & \cdots & \left|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}\right|\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\end{bmatrix}\\ & & \\ \begin{bmatrix}0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ \left|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}\right| & \cdots & 0\end{bmatrix} & \iddots & \begin{bmatrix}0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \left|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}\right|\end{bmatrix}\end{bmatrix}_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}=UNIT$$

Due to the multiplication of multi-dimensional vector matrices (1) and the formula of four-dimensional vector matrix determinant (2) and (3), that is $\left(A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{-1}\right)A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}=UNIT$

For example, the four-dimensional vector matrix $A_{\overrightarrow{2\times 2}\times\overrightarrow{2\times 2}}$ with two orders is given. By means of the program's operation, we can calculate the four-dimensional vector inverse matrix.

$$A_{\overrightarrow{2\times 2}\times\overrightarrow{2\times 2}}A_{\overrightarrow{2\times 2}\times\overrightarrow{2\times 2}}^{-1}=\begin{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}2&3\\4&1\end{bmatrix} & \\ & \begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}2&3\\4&1\end{bmatrix}\end{bmatrix}_{\overrightarrow{2\times 2}\times\overrightarrow{2\times 2}}\begin{bmatrix}\begin{bmatrix}-\frac{9}{40}&\frac{1}{40}\\\frac{1}{40}&\frac{11}{40}\end{bmatrix}\begin{bmatrix}\frac{1}{40}&\frac{1}{40}\\\frac{11}{40}&-\frac{9}{40}\end{bmatrix} & \\ & \begin{bmatrix}\frac{1}{40}&\frac{11}{40}\\-\frac{9}{40}&\frac{1}{40}\end{bmatrix}\begin{bmatrix}\frac{11}{40}&-\frac{9}{40}\\\frac{1}{40}&\frac{1}{40}\end{bmatrix}\end{bmatrix}_{\overrightarrow{2\times 2}\times\overrightarrow{2\times 2}}$$

$$=\begin{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} & \\ & \begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}\end{bmatrix}_{\overrightarrow{2\times 2}\times\overrightarrow{2\times 2}}=UNIT$$

In traditional matrix mathematics, if a matrix is an identity matrix, the determinant of two-dimensional matrix is 1.

Similarly, multidimensional vector matrices are a concatenation of two-dimensional matrices, if a four-dimensional vector matrix is a four-dimensional vector identity matrix, the result of four-dimensional vector identity matrix determinant is 1. That is $\left|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}\right|=1$. For example,

$$\left|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}\right|=\begin{vmatrix}\begin{bmatrix}a_{\vec{1}\vec{1}\vec{1}\vec{1}}&\cdots&a_{\vec{1}\vec{n}\vec{1}\vec{1}}\\\vdots&\ddots&\vdots\\a_{\vec{m}\vec{1}\vec{1}\vec{1}}&\cdots&a_{\vec{m}\vec{n}\vec{1}\vec{1}}\end{bmatrix}\cdots\begin{bmatrix}a_{\vec{1}\vec{1}\vec{1}\vec{n}}&\cdots&a_{\vec{1}\vec{n}\vec{1}\vec{n}}\\\vdots&\ddots&\vdots\\a_{\vec{m}\vec{1}\vec{1}\vec{n}}&\cdots&a_{\vec{m}\vec{n}\vec{1}\vec{n}}\end{bmatrix} & \\ & \begin{bmatrix}a_{\vec{1}\vec{1}\vec{m}\vec{1}}&\cdots&a_{\vec{1}\vec{n}\vec{m}\vec{1}}\\\vdots&\ddots&\vdots\\a_{\vec{m}\vec{1}\vec{m}\vec{1}}&\cdots&a_{\vec{m}\vec{n}\vec{m}\vec{1}}\end{bmatrix}\cdots\begin{bmatrix}a_{\vec{1}\vec{1}\vec{m}\vec{n}}&\cdots&a_{\vec{1}\vec{n}\vec{m}\vec{n}}\\\vdots&\ddots&\vdots\\a_{\vec{m}\vec{1}\vec{m}\vec{n}}&\cdots&a_{\vec{m}\vec{n}\vec{m}\vec{n}}\end{bmatrix}\end{vmatrix}_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$$

$$=\begin{vmatrix}\begin{bmatrix}1&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&0\end{bmatrix}\cdots\begin{bmatrix}0&\cdots&1\\\vdots&\ddots&\vdots\\0&\cdots&0\end{bmatrix} & \\ & \begin{bmatrix}0&\cdots&0\\\vdots&\ddots&\vdots\\1&\cdots&0\end{bmatrix}\cdots\begin{bmatrix}0&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&1\end{bmatrix}\end{vmatrix}_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}=1$$

For a four-dimensional vector square matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$,

$$\left|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}\right|=\sum_{i_1=1}^{m}\sum_{i_2=1}^{n}a_{\vec{i_1}\vec{i_2}\vec{j_1}\vec{j_2}}A_{\vec{i_1}\vec{i_2}\vec{j_1}\vec{j_2}}$$

$$=a_{\vec{1}\vec{1}\vec{1}\vec{1}}A_{\vec{1}\vec{1}\vec{1}\vec{1}}+\ldots+a_{\vec{1}\vec{n}\vec{1}\vec{1}}A_{\vec{1}\vec{n}\vec{1}\vec{1}}+\ldots+a_{\vec{m}\vec{n}\vec{1}\vec{1}}A_{\vec{m}\vec{n}\vec{1}\vec{1}}\qquad=\sum_{j_1=1}^{m}\sum_{j=1}^{n}a_{\vec{i_1}\vec{i_2}\vec{j_1}\vec{j_2}}A_{\vec{i_1}\vec{i_2}\vec{j_1}\vec{j_2}}$$

$$=a_{\vec{1}\vec{1}\vec{1}\vec{1}}A_{\vec{1}\vec{1}\vec{1}\vec{1}}+\ldots+a_{\vec{1}\vec{1}\vec{1}\vec{n}}A_{\vec{1}\vec{1}\vec{1}\vec{n}}+\ldots+a_{\vec{1}\vec{1}\vec{m}\vec{n}}A_{\vec{1}\vec{1}\vec{m}\vec{n}}\quad\left|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{T}\right|=\sum_{j_1=1}^{m}\sum_{j=1}^{n}a_{\vec{j_1}\vec{j_2}\vec{i_1}\vec{i_2}}A_{\vec{j_1}\vec{j_2}\vec{i_1}\vec{i_2}}$$

$$=a_{\vec{1}\vec{1}\vec{1}\vec{1}}A_{\vec{1}\vec{1}\vec{1}\vec{1}}+\dots+a_{\vec{1}\vec{n}\vec{1}\vec{1}}A_{\vec{1}\vec{n}\vec{1}\vec{1}}+\dots+a_{\vec{m}\vec{n}\vec{1}\vec{1}}A_{\vec{m}\vec{n}\vec{1}\vec{1}} \quad = \sum_{i_1=1}^{m}\sum_{i_2=1}^{n} a_{\vec{j_1}\vec{j_2}\vec{i_1}\vec{i_2}}A_{\vec{j_1}\vec{j_2}\vec{i_1}\vec{i_2}}$$

$$=a_{\vec{1}\vec{1}\vec{1}\vec{1}}A_{\vec{1}\vec{1}\vec{1}\vec{1}}+\dots+a_{\vec{1}\vec{1}\vec{1}\vec{n}}A_{\vec{1}\vec{1}\vec{1}\vec{n}}+\dots+a_{\vec{1}\vec{1}\vec{m}\vec{n}}A_{\vec{1}\vec{1}\vec{m}\vec{n}}$$

So $|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}| = |A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{T}|$.

There are still many properties of two-dimensional matrix that can be extend to the four-dimensional vector matrix.

If all the elements of any vector direction are zero in a four-dimensional vector square matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$, then $|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}| = 0$.

If one vector direction is proportional to another vector direction of a four-dimensional vector square matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$, then $|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}| = 0$.

If one vector direction is a linear combination of one or more other vector directions of a four-dimensional vector square matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$, then $|A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}| = 0$.

If two vector directions of a four-dimensional vector square matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$ are interchanged, the sign of the determinant of the matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$ is changed.

A four-dimensional vector square matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$ inverse which it is an invertible matrix can be unique.

If four-dimensional vector square matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$ is invertible, $\left(A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{-1}\right)^{-1} = A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$.

If four-dimensional vector square matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$ is invertible, $\lambda \neq 0$, and $\lambda A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$ is also invertible, then

$$\left(\lambda A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}\right)^{-1} = \frac{1}{\lambda} A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{-1}$$

If four-dimensional vector square matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$ and $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{T}$ are both invertible, then

$$\left(A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{T}\right)^{-1} = \left(A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{-1}\right)^{T}$$

If four-dimensional vector square matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$ and $B_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$ are both invertible and four-dimensional vector square matrix $A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}B_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}$ is also invertible, then

$$\left(A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}B_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}\right)^{-1} = B_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{-1}A_{\overrightarrow{m\times n}\times\overrightarrow{m\times n}}^{-1}$$

There are various properties of four-dimensional vector matrix determinant and inverse to prove the correctness of four-dimensional vector matrix determinant and inverse definition in this paper. Meanwhile, we run successfully the corresponding program to verify the definition of the four-dimensional vector matrix determinant and inverse.

4. Conclusion

On the basis of newly operation laws of multidimensional vector matrix, we define the four-dimensional vector matrix determinant, inverse and related properties. We also prove the correctness of these formulas by mathematics and program.

In this paper, we have introduced mainly the theory of multi-dimensional vector matrix, the four-dimensional vector matrix determinant and inverse. The future work is to extend the four-dimensional vector matrix inverse to multidimensional vector matrix inverse. We will apply these theories and definitions on multidimensional vector matrix.

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