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The Influence of Stage Structure and Prey Refuge on the Stability of the Predator-Prey Model

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Abstract: This paper aims to study the prey refuge impact on the dynamic behaviour of a stage structure predator-prey model. The model consists of four ecological species: prey in the protected and unprotected area and immature and mature predators. It assumes the grown predator can feeds only on the prey in an unreserved area. The conditions that guarantee the existence of the possible fixed points are found. Further, the local stability around all of the equilibria is considered. Then, using the Lyapunov direct method, the essential conditions for the global stability of the equilibria are adopted. Numerical simulations are illustrated to confirm our results. It concluded that the protected area positively affects the system's co-existence.

Index Terms: Predator-prey model, Stage structure, Reserved area, Prey refuge.

1. Introduction

Many scholars have extensively considered the co-existence of ecological species [1,2]. Lotka and Volterra proposed the first predator-prey model, which is identified as the Lotka – Volterra system [3,4]. In [3], the system of one predator with two prey has been studied to predict their behaviour over time. Later, some academics studied some properties, such as persistence, extinction, co-existence and stability analysis [5,6].

External effects such as over-predation, overutilisation and over-harvesting lead to the extinction of some prey species. Prey may avoid becoming extinct or being attacked by predators either by living in a refuge area, where it will be out of sight of predators or by protecting themselves from the latter.

The impact of prey protection in the prey-predator model has drawn huge research attention [8,9]. Some studies have exposed the effects of prey refuges and believed that the shelters guarantee the steadiness of the biological populations [7]. For example, Collings [10] considered a predator-prey model in the case of the existence of shelter to protect some of the prey population. He discovered that a temperature interval increases the refuge, destabilising his system.

Modelling a predator-prey system is one of the significant targets in mathematical ecology, which has received huge attention from several scholars [11-15]. In the natural, many kinds of predator and prey species have at least two stages in their life cycle, namely immature and mature. Each step has different behavioural properties. Some works of stage structure predator-prey models have been provided in a wild number in the literature. In [16], we consider the interaction of four species when the adult predator partially relies on the unprotected prey.

This paper considers the interaction among the following species: prey in the unsheltered and sheltered zone and immature and mature predators. The role of refuge is used to protect prey from extinction. According to the functional response type 1, the adult predator can attack the prey in the unreserved region. The rest of this work is organised as follows: In section 2, the condition that grantee the existence of the equilibrium points has been investigated. Then, in section 3, the local and global behaviour of all of the balance points is provided. In the last section, some numerical simulations have been illustrated to confirm our logical result.

2. Mathematical Assumption

Assume that $x(t)$; $y(t)$ represent the prey in the unprotected and protected area at time t. $z_1(t)$, $z_2(t)$ denote the juvenile and adult predators at time t. Under the above expectations, the model can be offered by:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right) - \sigma_1 x + \sigma_2 y - p(x)z_2,$$

$$\frac{dy}{dt} = sy\left(1-\frac{y}{l}\right) + x - \sigma_2 y, \tag{1}$$
$$\frac{dz_1}{dt} = \beta_2 x z_2 - (d_1 + D)z_1, \frac{dz_2}{dt} = Dz_1 - d_2 z_2.$$

In this paper, model (1) has been studied with the initial conditions $(x, y, z_1, z_2) \in R_+^4$. All parameters of the system (1) belong to $(0, \infty)$ and are defined as in [2].

In system (1), all the interaction functions are continuous and continuous partial derivatives. Therefore they are Lipschitzain; the solution of system (1) exists and is unique. Moreover, the latter's solution is bounded, as exposed in the following theorem.

Theorem (1): All solutions of model (1), which initiate in R_+^4 are uniformly bounded.

Proof: suppose $(x(t), y(t), z_1(t), z_2(t))$ be any non-negative solution of the model (1) such that $U(t) = x(t) + y(t) + z_1(t) + z_2(t)$, then

$$\frac{dU}{dt} = rx - \frac{rx^2}{k} + sy - \frac{sy^2}{l} - (\beta_1 - \beta_2)xz_2 - d_1 z_1 - d_2 z_2.$$

Hence, we obtain

$$\frac{dU}{dt} \le rx - \frac{rx^2}{k} + sy - \frac{sy^2}{l} - d_1 z_1 - d_2 z_2$$
$$\frac{dU}{dt} + \xi U \le 2rx - \frac{rx^2}{k} + 2sy - \frac{sy^2}{l},$$

where $\xi = min.\{r, s, d_1, d_2\}$, then

$$\frac{dU}{dt} + \xi U \le rk - \frac{r}{k}(x-k)^2 + sl - \frac{s}{l}(y-l)^2 \le rk + sl = \mu$$

then, $0 < U(x(t), y(t), z_1(t), z_2(t)) \le \frac{\mu}{\xi}\left(1 - e^{-\xi t}\right) + U(0)e^{-\xi t}$. Therefore, $0 < U(t) \le \frac{\mu}{\xi}$, as $t \to \infty$.

Hence all the positive solutions are uniformly bounded.

3. Existence of the Equilibria:

System (1) has the following equilibrium points:

1. $e_0 = (0,0,0,0)$.
2. $e_1 = (\hat{x}, \hat{y}, 0, 0)$ accords to E_1 of the system presented in [3], so they have the same formulae and existence conditions, i.e., $\hat{y} = \frac{1}{\sigma_2}\left[\frac{r\hat{x}^2}{k} - (r - \sigma_1)\hat{x}\right]$ and $\hat{x}$ is a positive solution if

$$\frac{s(r-\sigma_1)^2}{l\sigma_2} < \frac{r(s-\sigma_2)}{k}, \tag{2}$$

$$(r - \sigma_1)(s - \sigma_2) < \sigma_1 \sigma_2. \tag{3}$$

While, $\hat{y} > 0$ provided that $\hat{x} > \frac{k}{r}(r - \sigma_1)$.

3. $e_2 = (x^*, y^*, z_1^*, z_2^*)$, where

$$x^* = \frac{d_2(D + d_1)}{\beta_2 D},$$
$$y^* = \frac{l}{2s}\left((s - \sigma_2) + \sqrt{(s-\sigma_2)^2 + \frac{4s\sigma_1 d_2(D + d_1)}{l\beta_2 D}}\right),$$
$$z_1^* = \frac{d_2}{D\beta_1}\left((r - \sigma_1) - \frac{rd_2(D + d_1)}{k\beta_2 D} + \frac{\sigma_2\beta_2 D y^*}{d_2(D + d_1)}\right),$$
$$z_2^* = \frac{1}{\beta_1}\left((r - \sigma_1) - \frac{rd_2(D + d_1)}{k\beta_2 D} + \frac{\sigma_2\beta_2 D y^*}{d_2(D + d_1)}\right).$$

Clearly, $z_1 > 0$ and $z_2 > 0$ if

$$(r-\sigma_1)+\frac{\sigma_2\beta_2 D y^*}{d_2(D+d_1)}>\frac{r d_2(D+d_1)}{k\beta_2 D}. \tag{4}$$

4. Local Stability:

In this section, the system (1) stability is determined using the nature of the eigenvalues of the corresponding Jacobian matrix around the equilibria.

1. The Jacobian matrix at the vanishing fixed point $e_0=(0,0,0,0)$ can be written as:

$$J(e_0)=\begin{bmatrix} r-\sigma_1 & \sigma_2 & 0 & 0 \\ \sigma_1 & s-\sigma_2 & 0 & 0 \\ 0 & 0 & -(D+d_1) & 0 \\ 0 & 0 & D & -d_2 \end{bmatrix}.$$

It is clear that the eigenvalues of $J(e_0)$ are satisfying the following relations

$$\begin{gathered} \lambda_{01}+\lambda_{02}=(r-\sigma_1)+(s-\sigma_2)>0, \\ \lambda_{01}.\lambda_{02}=(r-\sigma_1)(s-\sigma_2)-\sigma_1\sigma_2, \\ \lambda_{03}=-(D+d_1)<0, \\ \lambda_{04}=-d_2<0. \end{gathered}$$

Then e_0 is a locally asymptotical stable point fixed point when

$$(r-\sigma_1)(s-\sigma_2)>\sigma_1\sigma_2 \tag{5}$$

Otherwise e_0 is a saddle point.

2. The Jacobian matrix of system (1) at $e_1=(\hat{x},\hat{y},0,0)$ is given by

$$J(e_1)=\begin{bmatrix} r-\sigma_1-\frac{2r\hat{x}}{k} & \sigma_2 & 0 & -\beta_1\hat{x} \\ \sigma_1 & s-\sigma_2-\frac{2s\hat{y}}{l} & 0 & 0 \\ 0 & 0 & -(D+d_1) & \beta_2\hat{x} \\ 0 & 0 & D & -d_2 \end{bmatrix}.$$

The eigenvalues of $J(e_1)$ are satisfying the following relations

$$\begin{gathered} \lambda_{11}+\lambda_{12}=\left(r-\sigma_1-\frac{2r\hat{x}}{k}\right)+\left(s-\sigma_2-\frac{2s\hat{y}}{l}\right)<0, \\ \lambda_{11}.\lambda_{12}=\left(r-\sigma_1-\frac{2r\hat{x}}{k}\right)\left(s-\sigma_2-\frac{2s\hat{y}}{l}\right)-\sigma_1\sigma_2, \\ \lambda_{13}+\lambda_{14}=-(D+d_1+d_2)<0, \\ \lambda_{13}.\lambda_{14}=d_2(D+d_1)-\beta_2 D\hat{x}. \end{gathered}$$

Then e_1 is a locally asymptotical stable when

$$\begin{gathered} d_2(D+d_1)>\beta_2 D\hat{x} \\ \left(r-\sigma_1-\frac{2r\hat{x}}{k}\right)\left(s-\sigma_2-\frac{2s\hat{y}}{l}\right)>\sigma_1\sigma_2, \end{gathered} \tag{6}$$

hold, otherwise e_1 is a saddle point.

3. Finally, the $J(e_1)$can be written as:

$$J(e_2)=\left(a_{ij}\right)_{3\times 3},$$

where $a_{11}=\frac{-rx^*}{k}-\frac{\sigma_2 y^*}{x^*}<0$; $a_{12}=\sigma_2>0$; $a_{13}=0$; $a_{14}=-\beta_1 x^*<0$; $a_{21}=\sigma_1>0$; $a_{22}=\frac{-sy^*}{l}-\frac{\sigma_1 x^*}{y^*}<0$; $a_{23}=0$; $a_{24}=0$; $a_{31}=\beta_2 z_2^*>0$; $a_{32}=0$; $a_{33}=\frac{-\beta_2 x^* z_2^*}{z_1^*}<0$; $a_{34}=\beta_2 x^*>0$; $a_{41}=0$; $a_{42}=0$; $a_{43}=D>0$; $a_{44}=\frac{-Dz_1^*}{z_2^*}<0$.

The characteristic equation of $J(e_2)$ is given by:

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0, \tag{7}$$

here

$$A_1 = -(M_1 + M_2),\ A_2 = T_1 + T_2 + M_1M_2,\ A_3 = -(M_3 + T_1M_1 + T_2M_2),\ A_4 = (a_{22}M_3 + T_1T_2),$$

where

$$M_1 = a_{11} + a_{22};\ M_2 = a_{33} + a_{44};\ M_3 = a_{14}a_{31}a_{43};\ T_1 = a_{33}a_{44} - a_{34}a_{43};\ T_2 = a_{11}a_{22} - a_{12}a_{21}.$$

According to the elements of $J(e_2)$, it is easy to confirm that:

$$T_1 = 0; T_2 = \left(\frac{rx^{*2}(sy^{*2} + l\sigma_1x^*) + sk\sigma_2y^{*3}}{klx^*y^*}\right) > 0;$$
$$M_1 = -\left(\frac{rx^{*2} + k\sigma_2y^*}{kx^*} + \frac{sy^{*2} + l\sigma_1x^*}{ly^*}\right) < 0;$$
$$M_2 = -\left(\frac{\beta_2x^*z_2^*}{z_1^*} + \frac{Dz_1^*}{z_2^*}\right) < 0;\ M_3 = -(\beta_1\beta_2Dx^*z_2^*) < 0.$$

Consequently, the coefficients of the characteristic equation of $J(e_2)$ can be rewritten as:

$$A_1 = -(M_1 + M_2) > 0$$
$$A_2 = (T_2 + M_1M_2) > 0$$
$$A_3 = -(M_3 + T_2M_2) > 0$$
$$A_4 = (a_{22}M_3) > 0$$

Further,

$$\Delta = A_1A_2A_3 - A_3^2 - A_1^2A_4$$
$$= (M_1 + M_2)[M_2M_3(M_1 - a_{22}) + M_1(T_2M_2^2 - a_{22}M_3)] + (M_3 + T_2M_2)(M_1T_2 - M_3).$$

Now, according to the Routh-Hurwitz criteria, all the eigenvalues of $J(e_2)$ have roots with negative real parts, provided that $A_i(i = 1,3,4) > 0$ and $\Delta > 0$. Therefore, e_2 is locally asymptotically stable, if

$$T_2M_1 < M_3 < \left(\frac{T_2M_2^2}{a_{22}}\right) \tag{8}$$

hold. Therefore, e_2 is locally asymptotically stable in the $Int.R_+^4$ if and only if condition (8) holds.

5. Global Stability

In this section, the Lyapunov technique is used to study the global stability of system (1).

Theorem 2: Assume that the local stability condition (6) of e_1 is hold, then e_1 is globally asymptotically stable.

Proof: Define $U_1(x, y, z_1, z_2 = c_1\left(x - \hat{x} - \hat{x}\,ln\frac{x}{\hat{x}}\right) + c_2\left(y - \hat{y} - \hat{y}\,ln\frac{y}{\hat{y}}\right) + c_3z_1 + c_4z_2$, where c_1, c_2, c_3 and c_4 are positive constants to be determined.

Now,

$$\frac{dU_1}{dt} = c_1\left(\frac{x - \hat{x}}{x}\right)\frac{dx}{dt} + c_2\left(\frac{y - \hat{y}}{y}\right)\frac{dy}{dt} + c_3\frac{dz_1}{dt} + c_4\frac{dz_2}{dt}$$
$$= c_1(x - \hat{x})\left(r - \frac{rx}{k} - \sigma_1 + \frac{\sigma_2y}{x} - \beta_1z_2\right) + c_2(y - \hat{y})\left(s - \frac{sy}{l} + \frac{\sigma_1x}{y} - \sigma_2\right)$$
$$+ c_3(\beta_2xz_2 - (D + d_1)z_1) + c_4z_2(D - d_2)$$

$$\frac{dU_1}{dt} = c_1(x - \hat{x})\left(\frac{-r}{k}(x - \hat{x}) + \sigma_2\left(\frac{y}{x} - \frac{\hat{y}}{\hat{x}}\right) - \beta_1z_2\right) + c_2(-\hat{y})\left(\frac{-s}{l}(y - \hat{y}) + \sigma_1\left(\frac{x}{y} - \frac{\hat{x}}{\hat{y}}\right)\right) + c_3(\beta_2xz_2 - (D + d_1)z_1) + c_4(Dz_1 - d_2z_2).$$

Therefore,

$$\frac{dU_1}{dt} = \frac{-c_1 r}{k}(x-\hat{x})^2 + c_1\sigma_2(x-\hat{x})\left(\frac{y\hat{x}-x\hat{y}}{x\hat{x}}\right) - c_1(x-\hat{x})\beta_1 z_2 - \frac{c_2 s}{l}(y-\hat{y})^2 + c_2\sigma_1(y-\hat{y})\left(\frac{x\hat{y}-\hat{x}y}{y\hat{y}}\right) + c_3(\beta_2 x z_2 - (D+d_1)z_1) + c_4(Dz_1 + d_2 z_2)$$

By selecting the positive constants as: $c_1 = 1; c_2 = \frac{\sigma_2\hat{y}}{\sigma_1\hat{x}}; c_3 = \frac{\beta_1}{\beta_2}; c_4 = \frac{\beta_1\hat{x}}{d_2}$, we get

$$\frac{dU_1}{dt} = -\left(\frac{r}{k}\right)(x-\hat{x})^2 - \left(\frac{\sigma_2 s\hat{y}}{\sigma_1 l\hat{x}}\right)(y-\hat{y})^2 - \left(\frac{\sigma_2}{x\hat{x}y}\right)(x\hat{y}-y\hat{x})^2 + \left(\frac{D\hat{x}}{d_2} - \frac{(D+d_1)}{\beta_2}\right)\beta_1 z_1$$

Then, $\frac{dU_1}{dt} < 0$ under condition (6), hence U_1 is a Lyapunov function. Thus, e_1 is globally asymptotically stable.

Theorem (3): Suppose that $e_2 = (x^*, y^*, z_1^*, z_2^*)$ is exist; then the basin of attraction for e_2 is defined as $\Phi = \{(x, y, z_1, z_2): x > x^*, y \geq 0, z_1 < z_1^*, z_2 > z_2^*\}$.

Proof: Define

$$U_3(x, y, z_1, z_2) = c_1\left(x - x^* - x^* \ln\frac{x}{x^*}\right) + c_2\left(y - y^* - y^* \ln\frac{y}{y^*}\right) + c_3\left(z_1 - z_1^* - z_1^* \ln\frac{z_1}{z_1^*}\right) + c_4\left(z_2 - z_2^* - z_2^* \ln\frac{z_2}{z_2^*}\right),$$

where c_1, c_2 , c_3 and c_4 are positive constants to be determined.

Now,

$$\frac{dU_3}{dt} = c_1\left(\frac{x-x^*}{x}\right)\frac{dx}{dt} + c_2\left(\frac{y-y^*}{y}\right)\frac{dy}{dt} + c_3\left(\frac{z_1-z_1^*}{z_1}\right)\frac{dz_1}{dt} + c_4\left(\frac{z_2-z_2^*}{z_2}\right)\frac{dz_2}{dt}$$
$$= c_1(x-x^*)\left(r\left(1-\frac{x}{k}\right) - \sigma_1 + \frac{\sigma_2 y}{x} - \beta_1 z_2\right) + c_2(y-y^*)\left(s\left(1-\frac{y}{l}\right) + \frac{\sigma_1 x}{y} - \sigma_2\right) + c_3(z_1-z_1^*)\left(\frac{\beta_2 x z_2}{z_1} - (D+d_1)\right) + c_4(z_2-z_2^*)\left(\frac{Dz_1}{z_2} - d_2\right).$$

Therefore,

$$\frac{dU_3}{dt} = -c_1\left(\frac{r}{k}\right)(x-x^*)^2 - c_1\beta_1(x-x^*)(z_2-z_2^*) + c_1\sigma_2\left(\frac{yxx^*-xyy^*}{xx^*}\right)(x-x^*) - \left(\frac{c_2 s}{l}\right)(y-y^*)^2 + c_2\sigma_1(y-y^*)\left(\frac{xy^*-yx^*}{yy^*}\right) + \left(\frac{c_3\beta_2 z_2}{z_1}\right)(x-x^*)(z_1-z_1^*) - \left(\frac{c_3\beta_2 x^*}{z_1}\right)(z_1-z_1^*)(z_2-z_2^*) - \left(\frac{c_3\beta_2 x^* z_2^*}{z_1 z_1^*}\right)(z_1-z_1^*)^2 + \left(\frac{c_4 D}{z_2}\right)(z_1-z_1^*)(z_2-z_2^*) - \left(\frac{c_4 D z_1^*}{z_2 z_2^*}\right)(z_2-z_2^*)^2$$

By indicating the positive constants as $c_1 = 1, c_2 = \frac{\sigma_2 y^*}{\sigma_1 x^*}, c_3 = \frac{1}{\beta_2}, c_4 = \frac{1}{D}$, then we obtain

$$\frac{dU_3}{dt} = -\left(\frac{r}{k}\right)(x-x^*)^2 - \left(\frac{\sigma_2}{xx^*y}\right)(xy^*-yx^*)^2 - \left(\frac{s\sigma_2 y^*}{\sigma_1 x^*}\right)(y-y^*)^2 - \left(\frac{x^* z_2^*}{z_1 z_1^*}\right)(z_1-z_1^*)^2 - \left(\frac{z_1^*}{z_2 z_2^*}\right)(z_2-z_2^*)^2 - \beta_1(x-x^*)(z_2-z_2^*) + \left(\frac{z_2}{z_1}\right)(x-x^*)(z_1-z_1^*) + \left(\frac{x^* z_2+z_1}{z_1 z_2}\right)(z_1-z_1^*)(z_2-z_2^*).$$

Hence $\frac{dU_3}{dt} < 0$ in Φ, and then U_3 is a Lyapunov function. Consequently, e_2 is globally asymptotically stable in $\Phi \subset R_+^4$.

6. Numerical simulation

This section objects to finding the system's (1) key parameters that disturb the system's (1) dynamics by using numerical simulations. System (1) is solved numerically with the help of MATLAB for diverse sets of parameters. For the following collection of parameters:

$$r = 4, k = 40, \sigma_1 = 2.5, \sigma_2 = 1.5, \beta_1 = 2, \beta_2 = 1.25, D = 2, s = 3.5, = 50, d_1 = 1, d_2, \tag{9}$$

the condition (8) is satisfied. This confirms that e_2 exists, and it is given by (x^*, y^*, z_1^*, z_2^*) = (1.2, 30, 9.72, 19.44). (See Figure 1)

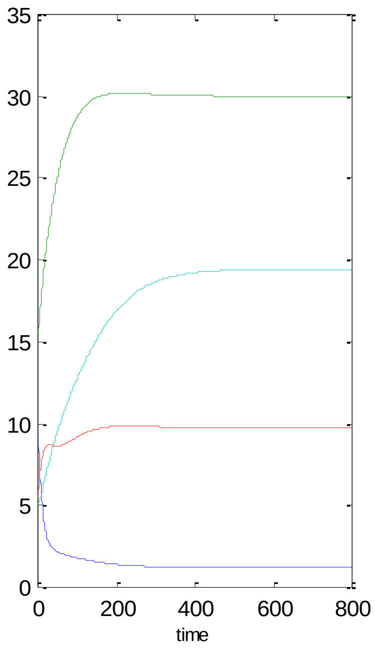

Fig. 1. Dynamics of system (1) with the data given by Eq. (9) blue: x green: y red: z_1 light blue: z_2

Figure 2 shows the behaviour of system (1) with the data given by Eq. (9) with r. The system (1) trajectory asymptotically approaches to $(x^*, y^*, z_1^*, z_2^*) = (1.2, 30, 8.99, 17.98)$ and (1.2, 30, 10.69, 21.38) for $r > 0$.

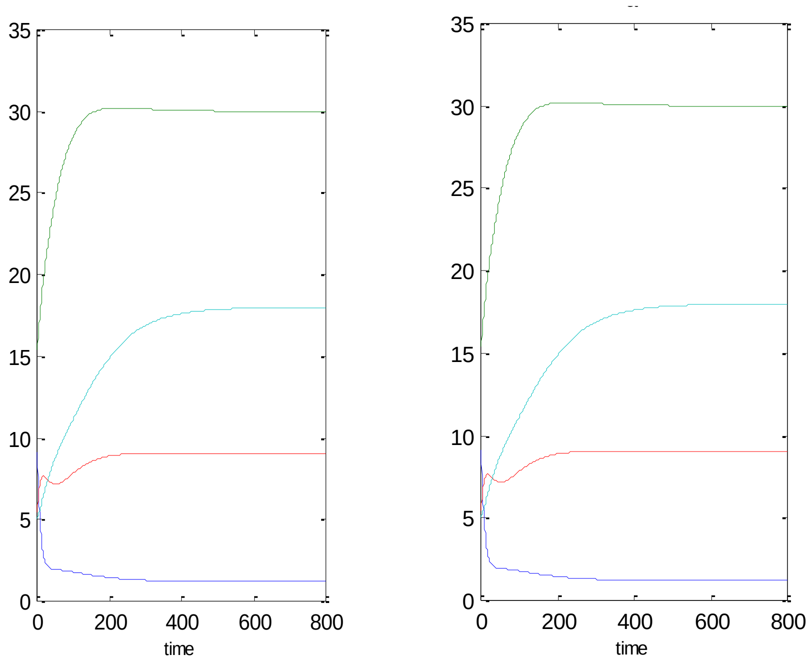

Fig. 2. Numerical simulation of system (1) with r=1 and r=8. (blue: x green: y red: z_1 light blue: z_2)

To study the impact of σ_1, the time series of system (1) trajectories (1) is presented in Fig. (3). We note from Fig. (3) that the trajectory of system (1) converges to (1.2, 29.15, 9.83, 19.66) and (1.2, 32.75, 9.2, 18.41) for $\sigma_1 > 0$.

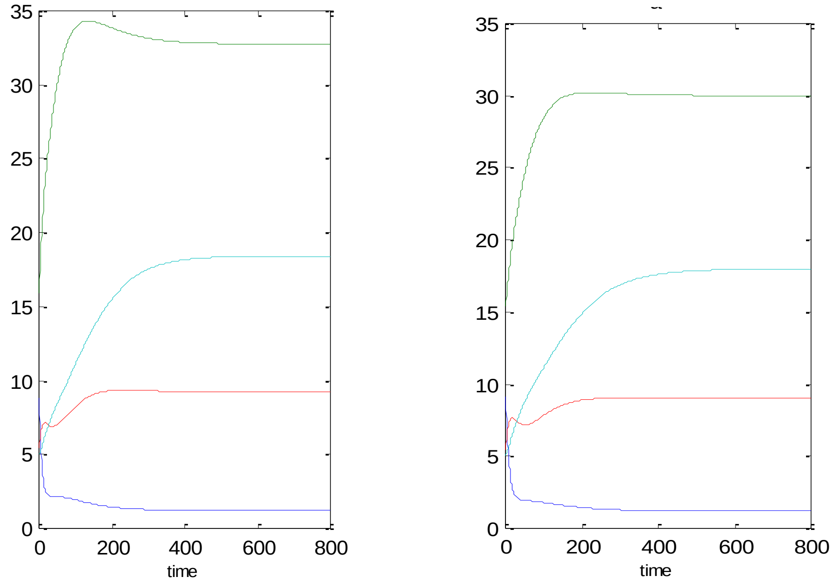

Fig. 3. Numerical simulation of system (1) with $\sigma_1 = 1$ and $\sigma_1 = 8$. (blue: x green: y red: z_1 light blue: z_2)

Fig. 4 demonstrates the system (1) trajectories with changing of σ_2. It is detected that the trajectory approaches the interior fixed point $e_2 = (x^*, y^*, z_1^*, z_2^*)$ for $\sigma_2 < 6.38$, as shown in Fig. (1). In contrast, it is approached periodic dynamics for $\sigma_2 \geq 6.38$, see Fig. (4).

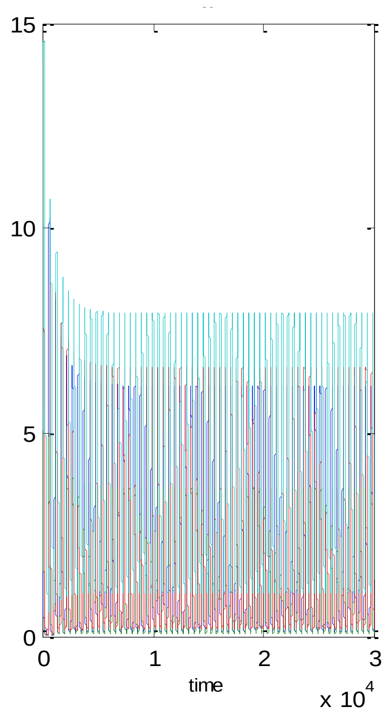

Fig. 4. The periodic attractor of system (1) with $\sigma_2 = 7$. blue: x green: y red: z_1 light blue: z_2

Lastly, the system (1) trajectory approaches $e_2 = (x^*, y^*, z_1^*, z_2^*) = (0.93, 29.69, 4.09, 24.56)$ for $D > 0$. See Fig. (5).

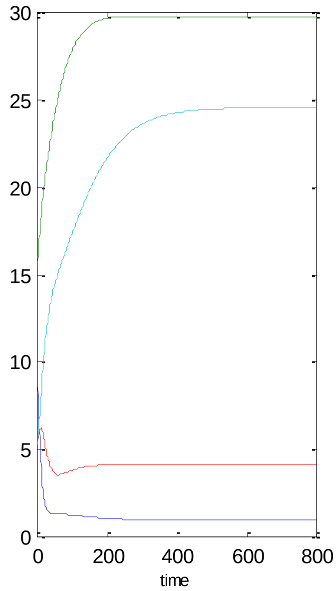

Fig. 5. Numerical simulation of system (1) with D=6. (blue: x green: y red: z_1 light blue: z_2)

7. Conclusion

This paper presents mathematical modelling for studying the role of the stage structure and reserved zone on the dynamics of the predator-prey model. In the proposed model, different equilibrium points at which the system is stable are observed. The system's (1) stability is specified based on the conditions at which the steady states of the model equations exist. The conditions that guarantee the equilibrium points' local and global behaviour have been investigated. Since there is a refuge effect for the prey, system stability (1) can stay for a long time by controlling the parameter σ_2. Moreover, the following numerical result is found:

1. The trajectory of system (1) approaches $e_2 = (x^*, y^*, z_1^*, z_2^*)$ for increases of $r, \sigma_1$and D.
2. For $\sigma_2 < 6.38$ the system (1) trajectory converges to e_2 in R_+^4 while its transfers to periodic dynamics for $\sigma_2 \geq 6.38$
3. All species persist for all the values of parameters.

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