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Stability Analysis of Stage Structure Prey-Predator Model with a Partially Dependent Predator and Prey Refuge

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Abstract: We propose a stage structure predator-prey model with a partially dependent predator and prey conservation. It is taken that the environment has been divided into two disjoint regions, namely, unreserved and reserved areas, where a predator is not allowed to enter the latter. The first model describes four species: prey refuge (prey in the reserved zone), prey in the unreserved zone, mature and immature predators. The predator is partially dependent on the prey in the unprotected area. The existence of ecological equilibria and their local and global stability is investigated. By using the Lyapunov theorem, sufficient conditions on the global stability of the equilibriums are obtained. Some numerical simulations show the viability of our results. The results show that the reserved area has a stabilizing impact on the stage structure predator-prey model.

Index Terms: Stage-structure, prey-predator, prey refuge, reserved zone.

1. Introduction

Many investigators have widely studied the interaction and co-existence of biological species via mathematical models [1]. A simple model of prey-predator interactions was proposed separately by Lotka and Volterra, but the model is now known as the Lotka – Volterra model [2,3]. In [4], the first simple mathematical model of two prey and one predator has been investigated and analysed to predict their dynamics. Subsequently, some researchers have studied numerous properties, such as co-existence, persistence, stability and extinction [4,5,6].

In an ecosystem, the reserved area plays a vital role in guaranteeing the stability of the populations [7]. The fundamental role of refuges/reserve areas in the predator-prey model has received significant attention from several researchers [8,9]. For instance, Collings [10] studied precisely the behaviour of a predator-prey system in the case of the existence of refuge to protect a constant amount of prey, with temperature-dependent parameters appropriately chosen for a mite interaction with a fruit species. His study showed that the existence of a temperature interval such that the quantity of the refuge increases dynamically destabilises the proposed system.

On the other hand, age factors are essential for the dynamics and evolution of many mammals. The growth rate and reproduction almost depend h on age or development stage[11]. It has been observed that the life cycle of many species is composed of at least two stages, immature and mature. The predator in the first stage often can neither attack prey nor reproduce, being raised by their adult parents. Most classical prey-predator models of two species assumed that all predators could attack prey and produce, ignoring the fact that the life cycle of most animals consists of at least two stages (immature and mature)[12]. Several prey-predator models with stage structures have been proposed and analysed in the last three decades [13,14,15,16].

This paper aims to consider the interaction among four populations: prey in the unreserved zone, prey refuge (prey in the reserved location), mature and immature predators. The mature predator can attack the first prey according to the type I functional response. Moreover, additional resources are suggested for the mature predator in the unprotected zone.

The rest of this paper is organised as follows: Section two investigates the equilibrium points for the proposed model. In section three, the stability of the possible equilibrium point has been analysed locally and globally. Finally, some numerical analyses have been provided in the last section to confirm our analytical result.

A. Mathematical Model

Suppose that $x(t), y(t)$ are the prey in the unreserved and reserved zone, respectively. $z_1(t)$, $z_2(t)$ are the immature and mature predators, respectively. Under the above assumptions, the model can be presented by the following system of differential equations:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\sigma_1 x+\sigma_2 y-p(x)z_2,\\ \frac{dy}{dt} &= sy\left(1-\frac{y}{l}\right)+\sigma_1 x-\sigma_2 y,\\ \frac{dz_1}{dt} &= \beta_2 x z_2+bz_2\left(1-\frac{z_2}{m_0}\right)-(D+d_1)z_1,\\ \frac{dz_2}{dt} &= Dz_1-d_2 z_2.\end{aligned} \tag{1}$$

Here, model (1) has been analysed with the initial conditions $x(0)\geq 0,\ y(0)\geq 0,\ z_1(0)\geq 0$ and $z_2(0)\geq 0$. $p(x)=\beta_1 x$ is the Lotka-Volterra type of functional response. All parameters of the system (1) are assumed to be positive and described as:

k, l and b are the carrying capacities of the first prey, second prey and mature predator, respectively, with intrinsic growth rates r, s and m_0 ; σ_1 is the migration rate coefficient of the prey species from the unreserved to the reserved area and σ_2 the migration rate coefficient of the prey species from the reserved to the unreserved zone. β_1 represents the attack rate of mature predator to the prey in the unreserved zone; D denoted the rate at which immature predator becomes mature predator; finally, the constants d_1 and d_2 represent the death rate of immature and mature predators, respectively.

The functions on the right-hand side of the system (1) are continuously differentiable functions on $R_+^4=\{(x,y,z_1,z_2),\ x\geq 0,\ y\geq 0,\ z_1\geq 0, z_2\geq 0\}$. Hence, there exists a unique solution for the system (1). And hence, they are Lipschitzian.

Remark: According to the system (1), it is easy to verify if there is no migration from reserved to unreserved zone (i.e., $\sigma_2=0$) and $r-\sigma_1<0$, then $\frac{dx}{dt}<0$. Similarly, if there is no migration from unreserved to reserved zone (i.e., $\sigma_1=0$) and $s-\sigma_2<0$, then $\frac{dy}{dt}<0$. Hence from now onward it is natural to assume that $r>\sigma_1$ and $s>\sigma_2$.

System (1) boundedness with non-negative initial conditions is shown in the following theorem.

Theorem 1: All the solutions of system (1), which initiate in R_+^4 are uniformly bounded.

Proof: let $(x(t),y(t),z_1(t),z_2(t))$ be any solution of the system (1) with a non-negative initial condition such that $R(t)=x(t)+y(t)+z_1(t)+z_2(t)$, then

$$\frac{dR}{dt}=rx-\frac{rx^2}{k}+sy-\frac{sy^2}{l}+bz_2-\frac{bz_2^2}{m_0}-(\beta_1-\beta_2)xz_2-d_1z_1-d_2z_2,$$

we obtain

$$\begin{aligned}\frac{dR}{dt} &\leq rx-\frac{rx^2}{k}+sy-\frac{sy^2}{l}+bz_2-\frac{bz_2^2}{m_0}-d_1z_1-d_2z_2\\ \frac{dR}{dt}+\xi_1 R &\leq 2rx-\frac{rx^2}{k}+2sy-\frac{sy^2}{l}+2bz_2-\frac{bz_2^2}{m_0}\end{aligned}$$

where $\xi_1=min.\{r,s,d_1,b+d_2\}$, hence

$$\frac{dR}{dt}+\xi_1 R\leq rk-\frac{r}{k}(x-k)^2+sl-\frac{s}{l}(y-l)^2+bm_0-\frac{b}{m_0}(z_2-m_0)^2\leq rk+sl+bm_0=\mu_0$$

By comparing the above differential inequality with the associated linear differential equation, we obtain

$$0<R(x(t),y(t),z_1(t),z_2(t))\leq\frac{\mu_0}{\xi_1}\left(1-e^{-t\xi_1}\right)+R(0)e^{-t\xi_1}.$$

Therefore, $0<R(t)\leq\frac{\mu_0}{\xi_1}$, as $t\to\infty$. Hence all the solutions of system (1) that initiate in R_+^4 are confined in the region $\Omega=\{(x,y,z_1,z_2)\in R_+^4 : R=x+y+z_1+z_2\leq\frac{\mu_0}{\xi_1}\}$ under the given condition. Thus, these solutions are uniformly bounded, and the proof is complete.

The existence and stability analysis of the above equilibrium point will discuss in the next section.

2. Existence and Stability Analysis of the Equilibrium Points:

The model given by system (1) has four non-negative equilibrium points, namely

1. $F_0 = (0,0,0,0)$.
2. $F_1 = (\hat{x}, \hat{y}, 0,0)$ coincides to E_1 of the system proposed in [1], and hence they have the same forms and existence conditions, i.e., $\hat{y} = \frac{1}{\sigma_2}\left[\frac{r\hat{x}^2}{k} - (r-\sigma_1)\hat{x}\right]$ and $\hat{x}$ is the positive root of

$$ax^3 + bx^2 + cx + d = 0. \tag{2}$$

where $a = \frac{sr^2}{l\sigma_2^2 k^2} > 0$, $b = \frac{-2rs(r-\sigma_1)}{kl\sigma_2^2} < 0$, $c = \frac{s(r-\sigma_1)^2}{l{\sigma_2}^2} - \frac{r(s-\sigma_2)}{k\sigma_2}$, $d = \frac{(r-\sigma_1)(s-\sigma_2)}{\sigma_2} - \sigma_1$.
So, by using Descartes rule of signs, Eq. (2) has a unique positive solution $x = \hat{x}$ if the following inequalities hold:

$$\frac{s(r-\sigma_1)^2}{l\sigma_2} < \frac{r(s-\sigma_2)}{k}, \tag{3}$$

$$(r-\sigma_1)(s-\sigma_2) < \sigma_1\sigma_2. \tag{4}$$

For $\hat{y}$ to be positive, we have to have $\hat{x} > \frac{k}{r}(r-\sigma_1)$.

3. $F_2 = (0,0,\bar{z}_1, \bar{z}_2)$ exists in the $Int. R_+^2$ of z_1z_2- plane, where

$$\bar{z}_1 = \frac{m_0 d_2}{D}\left(1 - \frac{(D+d_1)d_2}{bD}\right),$$
$$\bar{z}_2 = m_0\left(1 - \frac{(D+d_1)d_2}{bD}\right).$$

For $\bar{z}_1$ and $\bar{z}_2$to be positive, we must have

$$bD > d_2(D+d_1). \tag{5}$$

4. $F_3 = (x^\bullet, y^\bullet, z_1^\bullet, z_2^\bullet)$ exists in the $Int. R_+^4$, where

$$y^\bullet = \frac{l}{2s}\left((s-\sigma_2) + \sqrt{(s-\sigma_2)^2 + \frac{4s\sigma_1 x^\bullet}{l}}\right),$$
$$z_1^\bullet = \frac{d_2 m_0}{bD}\left((\beta_2 x^\bullet + b - \frac{d_2(D+d_1)}{D}\right),$$
$$z_2^\bullet = \frac{m_0}{b}\left((\beta_2 x^\bullet + b - \frac{d_2(D+d_1)}{D}\right),$$

and by Descartes rule of signs, $x^\bullet$ is the positive solution of $ax^3 + bx^2 + cx + d = 0$, where

$$a = \left(\frac{r}{k} + \frac{\beta_1\beta_2 m_0}{b}\right)^2 > 0,$$
$$b = 2\left(\frac{r}{k} + \frac{\beta_1\beta_2 m_0}{b}\right)\left((r-\sigma_1) - \beta_1 m_0 + \frac{\beta_1 m_0}{bD}(D+d_1)d_2\right),$$
$$c = 2\left(\frac{r}{k} + \frac{\beta_1\beta_2 m_0}{b}\right)\left(\frac{\sigma_2 l}{2s}(s-\sigma_2)\right) + \left((r-\sigma_1) - \beta_1 m_0 + \frac{\beta_1 m_0}{bD}(D+d_1)d_2\right)^2 > 0,$$
$$d = -\left(2\left(\frac{r}{k} + \frac{\beta_1\beta_2 m_0}{b}\right)y^\bullet\left(\frac{\sigma_2 l}{2s}(s-\sigma_2)\right) + \frac{\sigma_1\sigma_2^2 l}{s}\right) < 0,$$

if the following inequality holds:

$$(r-\sigma_1) + \frac{\beta_1 m_0}{bD}(D+d_1)d_2 > \beta_1 m_0 \tag{6}$$

For $z_1^\bullet$ and $z_2^\bullet$ to be positive, we must have

$$\beta_2 x^\bullet + b > \frac{d_2(D+d_1)}{D} \tag{7}$$

3. Local Stability

In this section, the conditions to guarantee the local behaviour of system (1) around each of the above equilibrium points are found. First, the Jacobian matrix of the system (1) at each point is determined, and then, the eigenvalues of the resulting matrix are computed.

1, The Jacobian matrix of the system (1) at $F_0 = (0,0,0,0)$ can be written as:

$$J(F_0) = \begin{bmatrix} r-\sigma_1 & \sigma_2 & 0 & 0 \\ \sigma_1 & s-\sigma_2 & 0 & 0 \\ 0 & 0 & -(D+d_1) & b \\ 0 & 0 & D & -d_2 \end{bmatrix}$$

Then, the eigenvalues of $J(F_0)$ are satisfying the following relations

$$\begin{gathered} \lambda_{01}+\lambda_{02}=(r-\sigma_1)+(s-\sigma_2)>0, \\ \lambda_{01}.\lambda_{02}=(r-\sigma_1)(s-\sigma_2)-\sigma_1\sigma_2, \\ \lambda_{03}+\lambda_{04}=-(D+d_1+d_2)<0, \\ \lambda_{03}.\lambda_{04}=d_2(D+d_1)-bD. \end{gathered}$$

Therefore F_0 is a saddle point with non-empty stable and unstable manifolds.

3. The Jacobian matrix of system (1) at the equilibrium point $F_1 = (\hat{x}, \hat{y}, 0,0)$ is given by

$$J(F_1) = \begin{bmatrix} r-\sigma_1-\dfrac{2r\hat{x}}{k} & \sigma_2 & 0 & -\beta_1\hat{x} \\ \sigma_1 & s-\sigma_2-\dfrac{2s\hat{y}}{l} & 0 & 0 \\ 0 & 0 & -(D+d_1) & \beta_2\hat{x}+b \\ 0 & 0 & D & -d_2 \end{bmatrix}$$

Then, the eigenvalues of $J(F_1)$ are satisfying the following relations

$$\begin{gathered} \lambda_{11}+\lambda_{12}=\left(r-\sigma_1-\tfrac{2r\hat{x}}{k}\right)+\left(s-\sigma_2-\tfrac{2s\hat{y}}{l}\right)<0, \\ \lambda_{11}.\lambda_{12}=\left(r-\sigma_1-\tfrac{2r\hat{x}}{k}\right)\left(s-\sigma_2-\tfrac{2s\hat{y}}{l}\right)-\sigma_1\sigma_2, \\ \lambda_{13}+\lambda_{14}=-(D+d_1+d_2)<0, \\ \lambda_{13}.\lambda_{14}=d_2(D+d_1)-(\beta_2\hat{x}+b)D. \end{gathered}$$

Hence F_1 is locally asymptotically stable if the following conditions hold:

$$d_2(D+d_1) > (\beta_2\hat{x}+b)D \tag{8}$$

$$\left(r-\sigma_1-\tfrac{2r\hat{x}}{k}\right)\left(s-\sigma_2-\tfrac{2s\hat{y}}{l}\right) > \sigma_1\sigma_2 \tag{9}$$

Otherwise, F_1 is a saddle point.

4. The Jacobian matrix of the system (1) at the equilibrium point $F_2 = (0,0,\bar{z}_1,\bar{z}_2)$ can be written as:

$$J(F_2) = \begin{bmatrix} r-\sigma_1-\beta_1\bar{z}_2 & \sigma_2 & 0 & 0 \\ \sigma_1 & s-\sigma_2 & 0 & 0 \\ \beta_2\bar{z}_2 & 0 & -(D+d_1) & b-\dfrac{2b\bar{z}_2}{m_0} \\ 0 & 0 & D & -d_2 \end{bmatrix}$$

Then, the eigenvalues of $J(F_2)$ are satisfying the following relations

$$\begin{gathered} \lambda_{21}+\lambda_{22}=(r-\sigma_1)+(s-\sigma_2)-\beta_1\bar{z}_2, \\ \lambda_{21}.\lambda_{22}=r(s-\sigma_2)-s\sigma_1-\beta_1\bar{z}_2(s-\sigma_2), \\ \lambda_{23}+\lambda_{24}=-(D+d_1+d_2)<0, \end{gathered}$$

$$\lambda_{23}.\lambda_{24} = d_2(D + d_1) + \left(\frac{2bD\bar{z}_2}{m_0}\right) - Db.$$

Hence F_2 is locally asymptotically stable if the following conditions hold:

$$d_2(D + d_1) + \left(\frac{2bD\bar{z}_2}{m_0}\right) > Db \quad (10)$$

$$(r - \sigma_1) + (s - \sigma_2) < \beta_1\bar{z}_2. \quad (11)$$

$$r(s - \sigma_2) > s\sigma_1 + \beta_1\bar{z}_2(s - \sigma_2) \quad (12)$$

Otherwise, F_2 is a saddle point.

5. The Jacobian matrix of the system (1) at the positive equilibrium point $F_3 = (x^{\bullet}, y^{\bullet}, z_1^{\bullet}, z_2^{\bullet})$ can be written as:

$$J(F_3) = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

where

$$b_{11} = \left(\frac{-rx^{\bullet}}{k} - \frac{\sigma_2 y^{\bullet}}{x^{\bullet}}\right) < 0;\ b_{12} = \sigma_2 > 0;\ b_{13} = 0;\ b_{14} = -\beta_1 x^{\bullet} < 0;$$
$$b_{21} = \sigma_1 > 0\ ;\ b_{22} = \left(\frac{-sy^{\bullet}}{l} - \frac{\sigma_1 x^{\bullet}}{y^{\bullet}}\right) < 0\ ;\ b_{23} = 0;\ b_{24} = 0;$$
$$b_{31} = \beta_2 z_2^{\bullet} > 0;\ b_{32} = 0; b_{33} = \left(\frac{-\beta_2 x^{\bullet} z_2^{\bullet}}{z_1^{\bullet}} - \frac{bz_2^{\bullet}}{z_1^{\bullet}} + \frac{b{z_2^{\bullet}}^2}{m_0 z_1^{\bullet}}\right) < 0;$$
$$b_{34} = \beta_2 x^{\bullet} > 0; b_{41} = 0;\ b_{42} = 0;\ b_{43} = D > 0;\ b_{44} = \left(\frac{-Dz_1^{\bullet}}{z_2^{\bullet}}\right) < 0.$$

Accordingly, the characteristic equation of $J(F_3)$ is given by:

$$\lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4 = 0$$

here

$$B_1 = -(N_1 + N_2),\ B_2 = I_1 + I_2 + N_1N_2,$$
$$B_3 = -(N_3 + I_1N_1 + I_2N_2),\ B_4 = (b_{22}N_3 + I_1I_2).$$

where

$$N_1 = b_{11} + b_{22};\ N_2 = b_{33} + b_{44};\ N_3 = b_{14}b_{31}b_{43};\ I_1 = b_{33}b_{44} - b_{34}b_{43};\ I_2 = b_{11}b_{22} - b_{12}b_{21}.$$

Now, according to the elements of $J(F_3)$, it is easy to verify that:

$$I_1 = bD\left(1 - \frac{z_2^{\bullet}}{m_0}\right) > 0;\quad \text{(since } z_2^{\bullet} \text{ doesn't exceed its carrying capacity)}$$
$$I_2 = \left(\frac{r{x^{\bullet}}^2({sy^{\bullet}}^2 + l\sigma_1 x^{\bullet}) + sk\sigma_2 {y^{\bullet}}^3}{klx^{\bullet}y^{\bullet}}\right) > 0;\ N_1 = -\left(\frac{r{x^{\bullet}}^2 + k\sigma_2 y^{\bullet}}{kx^{\bullet}} + \frac{s{y^{\bullet}}^2 + l\sigma_1 x^{\bullet}}{ly^{\bullet}}\right) < 0;$$
$$N_2 = -\left(\frac{\beta_2 x^{\bullet} z_2^{\bullet}}{z_1^{\bullet}} + \frac{bz_2^{\bullet}}{z_1^{\bullet}}\left(1 - \frac{z_2^{\bullet}}{m_0}\right) + \frac{Dz_1^{\bullet}}{z_2^{\bullet}}\right) < 0;\ N_3 = -(\beta_1\beta_2 Dx^{\bullet}z_2^{\bullet}) < 0.$$

Therefore,

$$B_1 = -(N_1 + N_2) > 0 \quad \text{(always)}$$
$$B_2 = I_1 + I_2 + N_1N_2 > 0 \quad \text{(always)}$$
$$B_3 = -(N_3 + I_1N_1 + I_2N_2) > 0 \text{ (always)}$$
$$B_4 = (b_{22}N_3 + I_1I_2) > 0. \text{ (always)}$$

Further,

$$\Delta = B_1B_2B_3 - B_3^2 - B_1^2B_4$$
$$= N_2^2N_3(N_1 - b_{22}) + N_1N_2((I_1 - I_2)^2 + I_2N_2^2) + N_3(N_1I_2 - N_3) + N_1N_3(N_1N_2 - I_1) + N_2I_2(N_1^2N_2 - N_3)$$
$$+ \quad N_2N_3(I_1 - b_{22}N_1) + N_1(N_1N_2I_1 - b_{22}N_3)(N_1 + N_2)$$

Now, according to the Routh-Hawirtiz criteria, all the eigenvalues of the $J(F_3)$ have roots with negative real parts if $B_i(i = 1,3,4) > 0$ and $\Delta > 0$ which are satisfied if the following conditions hold

$$\delta_1 < I_1 < N_1 N_2, \delta_1 = max\left\{b_{22}N_1, \frac{b_{22}N_3}{N_1 N_2}\right\} \quad (13)$$

$$\delta_2 < N_3 \text{ , } \delta_2 = max\{N_1^2 N_2, N_1 I_2\}. \quad (14)$$

Therefore F_3 is locally asymptotically stable if conditions (13) and (14) hold.

4. Global Stability

In this subsection, the global dynamics of the system (1) is investigated using the Lyapunov direct method as shown in the following theorems.

Theorem (2): Assume that the equilibrium point F_1 is locally asymptotically stable, then it is globally asymptotically stable in the R_+^4.

Proof: Consider the following positive definite function

$$R_1(x, y, z_1, z_2) = c_1\left(x - \hat{x} - \hat{x}\, ln\frac{x}{\hat{x}}\right) + c_2\left(y - \hat{y} - \hat{y}\, ln\frac{y}{\hat{y}}\right) + c_3 z_1 + c_4 z_2,$$

where c_1, c_2 , c_3 and c_4 are positive constants to be determined.

$$\frac{dR_1}{dt} = c_1\left(\frac{x - \hat{x}}{x}\right)\frac{dx}{dt} + c_2\left(\frac{y - \hat{y}}{y}\right)\frac{dy}{dt} + c_3\frac{dz_1}{dt} + c_4\frac{dz_2}{dt}$$
$$= c_1(x - \hat{x})\left(r - \frac{rx}{k} - \sigma_1 + \frac{\sigma_2 y}{x} - \beta_1 z_2\right) + c_2(y - \hat{y})\left(s - \frac{sy}{l} + \frac{\sigma_1 x}{y} - \sigma_2\right) +$$
$$c_3\left(\beta_2 x z_2 + b z_2 - \frac{b z_2^2}{m_0} - (D + d_1) z_1\right) + c_4(D z_1 - d_2 z_2),$$

therefore,

$$\frac{dR_1}{dt} = \frac{-c_1 r}{k}(x - \hat{x})^2 + c_1\sigma_2(x - \hat{x})\left(\frac{y\hat{x} - x\hat{y}}{x\hat{x}}\right) - c_1(x - \hat{x})\beta_1 z_2 - \frac{c_2 s}{l}(y - \hat{y})^2$$
$$+ c_2\sigma_1(y - \hat{y})\left(\frac{x\hat{y} - \hat{x}y}{y\hat{y}}\right) + c_3\left(\beta_2 x z_2 + b z_2 - \frac{b z_2^2}{m_0} - (D + d_1) z_1\right) + c_4(D z_1 + d_2 z_2).$$

By choosing the positive constants as:

$$c_1 = 1; c_2 = \frac{\sigma_2\hat{y}}{\sigma_1\hat{x}}; c_3 = \frac{\beta_1}{\beta_2}; c_4 = \frac{\beta_1}{d_2}\left(\hat{x} + \frac{b}{\beta_2}\right), \text{ we get}$$
$$\frac{dR_1}{dt} = -\left(\frac{r}{k}\right)(x - \hat{x})^2 - \left(\frac{\sigma_2 s\hat{y}}{\sigma_1 l\hat{x}}\right)(y - \hat{y})^2 - \left(\frac{\sigma_2}{x\hat{x}y}\right)(x\hat{y} - y\hat{x})^2 - \left(\frac{\beta_1}{d_2\beta_2}\right)\left(d_2(D + d_1) - D(\hat{x}\beta_2 + b)\right)z_1 - \left(\frac{b\beta_1}{\beta_2 m_0}\right)z_2^2.$$

Then, $\frac{dR_1}{dt} < 0$ under the local stability condition (8), hence R_1 is a Lyapunov function. Therefore, F_1 is globally asymptotically stable in the R_+^4.

Theorem (3): Assume that the equilibrium point F_2 exist, then it is globally asymptotically stable in $Int. R_+^2$ of $z_1 z_2$-plane.

Proof: let $H(z_1, z_2) = \frac{1}{z_1 z_2}$, $h_1(z_1, z_2) = -(D + d_1)z_1$ and $h_2(z_1, z_2) = D z_1 - d_2 z_2$. Then, $H(z_1, z_2) > 0$ for all $(z_1, z_2) \in Int. R_+^2$ and its C^1 function in $Int. R_+^2$ of $z_1 z_2$-plane.

Now, since

$$Hh_1(z_1, z_2) = \frac{-(D+d_1)}{z_2}; \text{ and } Hh_2(z_1, z_2) = \frac{D}{z_2} - \frac{d_2}{z_1}, \text{ hence } \Delta(z_1, z_2) = \frac{\partial(Hh_1)}{\partial z_1} + \frac{\partial(Hh_2)}{\partial z_2} = -\left(\frac{D}{z_2^2}\right) < 0.$$

Further, $\Delta(z_1, z_2)$ does not change sign and is not identically zero in the $Int. R_+^2$of $z_1 z_2$-plane. Then according to Bendixson-Dulic criteria, there is no periodic solution in the $Int. R_+^2$ of $z_1 z_2$-plane.

Now, since all the solutions of the system (1) are bounded and F_2 is a unique positive equilibrium point in $Int. R_+^2$ of $z_1 z_2$-plane, hence by using the Poincare-Bendixson theorem, F_2 is globally asymptotically stable, and hence the proof is complete.

Theorem (4): Assume that the positive equilibrium point $F_3 = (x^{\bullet}, y^{\bullet}, z_1^{\bullet}, z_2^{\bullet})$ is locally asymptotically stable with

$$\left(\beta_2 x - \frac{bz_2}{m_0}\right) < \frac{bz_2^{\bullet}}{m_0} < \beta_2 x^{\bullet}. \tag{15}$$

Then F_3 is globally asymptotically stable in the sub-region of R_+^4 which can be defined as $\Phi = \{(x, y, z_1, z_2): x > x^{\bullet}, y \geq 0, z_1 < z_1^{\bullet}, z_2 > z_2^{\bullet}\}$.

Proof: Consider the following positive definite function about

$$P(x^{\bullet}, y^{\bullet}, z_1^{\bullet}, z_2^{\bullet}) = c_1\left(x - x^{\bullet} - x^{\bullet}\, ln\frac{x}{x^{\bullet}}\right) + c_2\left(y - y^{\bullet} - y^{\bullet}\, ln\frac{y}{y^{\bullet}}\right) + c_3\left(z_1 - z_1^{\bullet} - z_1^{\bullet}\, ln\frac{z_1}{z_1^{\bullet}}\right)$$
$$+ c_4\left(z_2 - z_2^{\bullet} - z_2^{\bullet}\, ln\frac{z_2}{z_2^{\bullet}}\right),$$

where c_1, c_2 , c_3 and c_4 are positive constants to be determined.

Now the derivative of P along the trajectory of the system can be written as:

$$\frac{dP}{dt} = c_1\left(\frac{x - x^{\bullet}}{x}\right)\frac{dx}{dt} + c_2\left(\frac{y - y^{\bullet}}{y}\right)\frac{dy}{dt} + c_3\left(\frac{z_1 - z_1^{\bullet}}{z_1}\right)\frac{dz_1}{dt} + c_3\left(\frac{z_2 - z_2^{\bullet}}{z_2}\right)\frac{dz_2}{dt}$$
$$= c_1(x - x^{\bullet})\left(r\left(1 - \frac{x}{k}\right) - \sigma_1 + \frac{\sigma_2 y}{x} - \beta_1 z_2\right) + c_2(y - y^{\bullet})\left(s\left(1 - \frac{y}{l}\right) + \frac{\sigma_1 x}{y} - \sigma_2\right)$$
$$+ c_3(z_1 - z_1^{\bullet})\left(\frac{\beta_2 x z_2}{z_1} + \frac{bz_2}{z_1}\left(1 - \frac{z_2}{m_0}\right) - (D + d_1)\right) + c_4(z_2 - z_2^{\bullet})\left(\frac{Dz_1}{z_2} - d_2\right).$$

Therefore,

$$\frac{dP}{dt} = -c_1\left(\frac{r}{k}\right)(x - x^{\bullet})^2 - c_1\beta_1(x - x^{\bullet})(z_2 - z_2^{\bullet}) + c_1\sigma_2\left(\frac{yxx^{\bullet} - xyy^{\bullet}}{xx^{\bullet}}\right)(x - x^{\bullet}) - \left(\frac{c_2 s}{l}\right)(y - y^{\bullet})^2$$
$$+ c_2\sigma_1(y - y^{\bullet})\left(\frac{xy^{\bullet} - yx^{\bullet}}{yy^{\bullet}}\right) + \left(\frac{c_3\beta_2 z_2^{\bullet}}{z_1}\right)(x - x^{\bullet})(z_1 - z_1^{\bullet}) + \left(\frac{c_3}{z_1}\right)\left(\frac{b}{m_0}(z_2 + z_2^{\bullet}) - \beta_2 x\right)(z_1 - z_1^{\bullet})(z_2 - z_2^{\bullet})$$
$$+ c_3 b\left(\frac{z_2}{z_1} - \frac{z_2^{\bullet}}{z_1^{\bullet}}\right)(z_1 - z_1^{\bullet}) - \left(\frac{c_3 z_2^{\bullet}}{z_1 z_1^{\bullet}}\right)\left(\left(\beta_2 x^{\bullet} - \frac{bz_2^{\bullet}}{m_0}\right)\right)(z_1 - z_1^{\bullet})^2 + c_4 D\left(\frac{z_1}{z_2} - \frac{z_1^{\bullet}}{z_2^{\bullet}}\right)(z_2 - z_2^{\bullet}).$$

By choosing the positive constants as$c_1 = 1, c_2 = \frac{\sigma_2 y^{\bullet}}{\sigma_1 x^{\bullet}}, c_3 = 1, c_4 = \frac{bz_2^{\bullet}}{Dz_1^{\bullet}}$, then we obtain

$$\frac{dP}{dt} = -\left(\frac{r}{k}\right)(x - x^{\bullet})^2 - \left(\frac{\sigma_2}{xx^{\bullet}y}\right)(xy^{\bullet} - yx^{\bullet})^2 - \left(\frac{s\sigma_2 y^{\bullet}}{\sigma_1 x^{\bullet}}\right)(y - y^{\bullet})^2 - \left(\frac{b}{z_1 z_2 z_1^{\bullet}}\right)(z_1 z_2^{\bullet} - z_2 z_1^{\bullet})^2$$
$$+ \left(\frac{\beta_2 z_2^{\bullet}}{z_1}\right)(x - x^{\bullet})(z_1 - z_1^{\bullet}) - \left(\frac{z_2^{\bullet}}{z_1 z_1^{\bullet}}\right)\left((\beta_2 x^{\bullet} - \frac{bz_2^{\bullet}}{m_0})\right)(z_1 - z_1^{\bullet})^2$$
$$+ \left(\frac{1}{z_1}\right)\left(\frac{b}{m_0}(z_2 + z_2^{\bullet}) - \beta_2 x\right)(z_1 - z_1^{\bullet})(z_2 - z_2^{\bullet}),$$

hence $\frac{dP}{dt} < 0$ in Φ if and only if condition (15) holds, then P is a Lyapunov function. Therefore, F_3 is globally asymptotically stable in $\Phi \subset R_+^4$, which represents the basin of attraction for F_3.

5. Numerical Analysis

This section aims to find the system's critical parameters that affect the behaviour of the proposed system by using numerical simulations. The system dynamics is obtained by the solving system (1) numerically using the predictor-corrector method with the six order Range Kutta method. The time series of the system solution is drawn using MATLAB for different sets of parameters. Now, for the following set of parameters:

$$r = 4, k = 40, \sigma_1 = 2.5, \sigma_2 = 1.5, \beta_1 = 2, \beta_2 = 1.25, D = 2, s = 3.5, = 50, d_1 = 1, d_2 = 1, m_0 = 30, a = 30, \tag{16}$$

the condition (15) is satisfied. This shows that F_3 exists, and it is given by $(x^{\bullet}, y^{\bullet}, z_1^{\bullet}, z_2^{\bullet})$= (0.87, 29.63, 12.99, 25.98). (See Fig. 1)

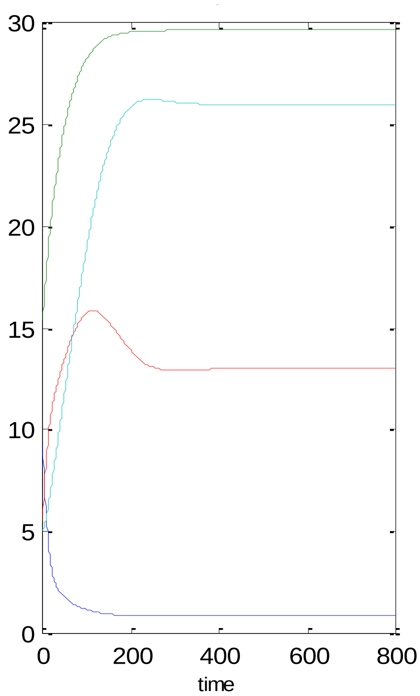

Fig. 1. Dynamics of the four species with the data given by Eq. (16) blue: x green: y red: z_1 light blue: z_2

Fig. 2 presents the dynamics of the four species with the data given by Eq. (16) with the intrinsic growth rate coefficient of the prey species in unreserved zone r. The trajectory of system (1) approaches asymptotically to $(0.87, 29.63, 12.99, 25.98)$ and (0.92, 29.68, 13.3, 26.61) for $r > 0$.

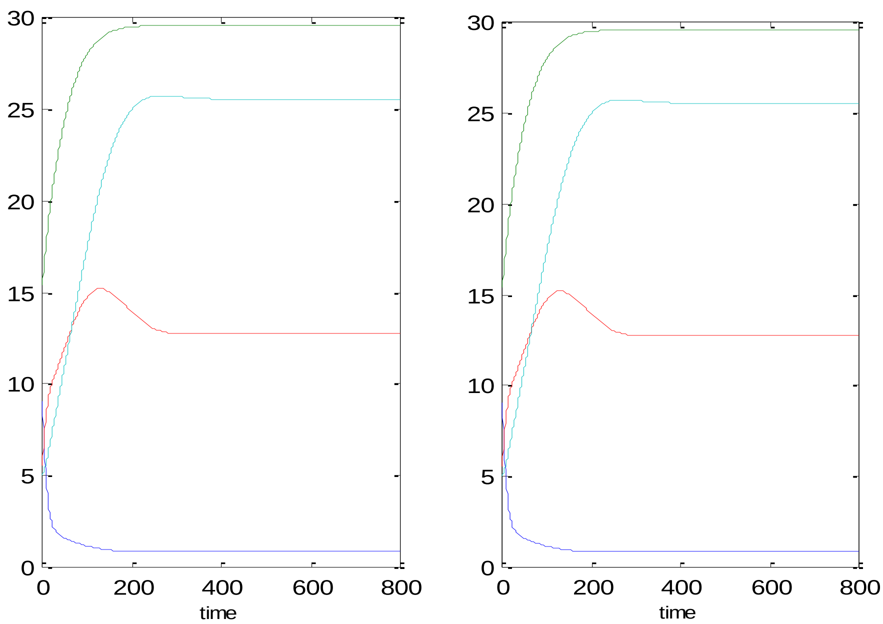

Fig. 2. Dynamics of the four species with the data given by Eq. (16) with r=1 and r=8, respectively blue: x green: y red: z_1 light blue: z_2

In order to explore the impact of σ_1, the time series of the trajectories of system (1) is drowned in Fig. 3. From this Figure, we note that the trajectory of system (1) approaches to the global stable point (0.88, 29.007, 13.02, 26.05) and (0.8, 31.66, 12.84, 25.69) for $\sigma_1 > 0$.

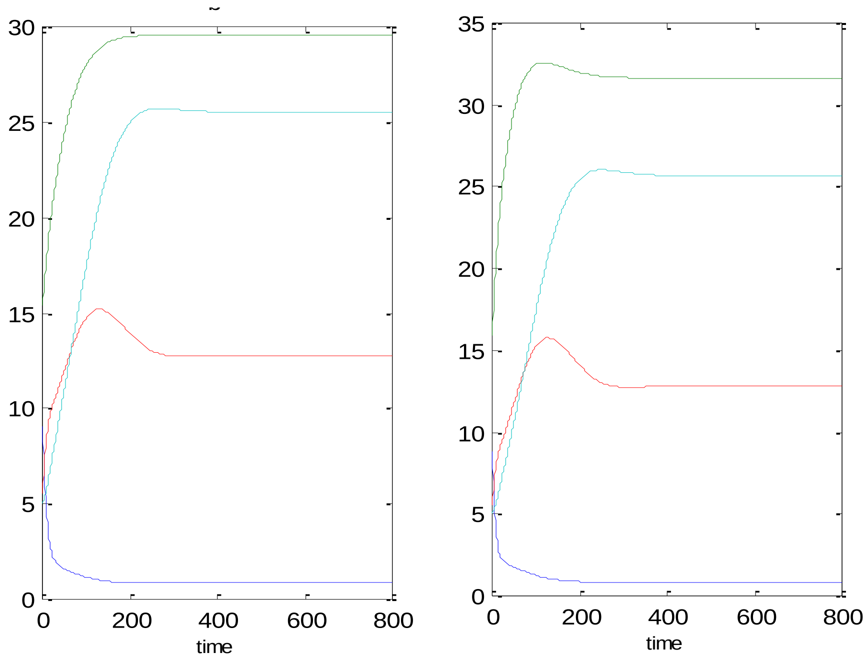

Fig. 3. Dynamics of the four species with the data given by Eq. (16) with $\sigma_1 = 1$ and $\sigma_1 = 8$ respectively blue: x green: y red: z_1 light blue: z_2

Fig. 4 shows the time series of system trajectories with varying of σ_2. It is observed that the trajectory of system (1) approaches asymptotically to $F_2 = (0,0,\bar{z}_1,\bar{z}_2) = (0,0,7.5,15)$ for $\sigma_2 \geq 3.84$, as shown in Fig. 4.

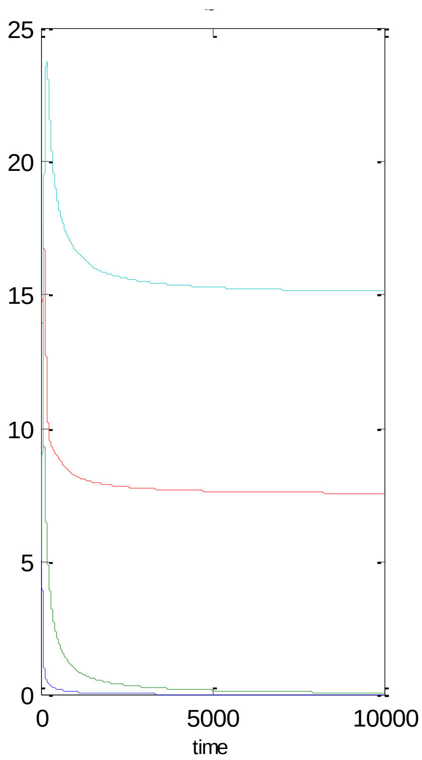

Fig. 4 Periodic attractor for the time series of the trajectories of system (1) with $\sigma_2 = 3.84$. blue: x green: y red:z_1 light blue: z_2

Finally, the trajectory of the system (1) approaches a globally asymptotically stable point for $D > 0$. See Fig. 5.

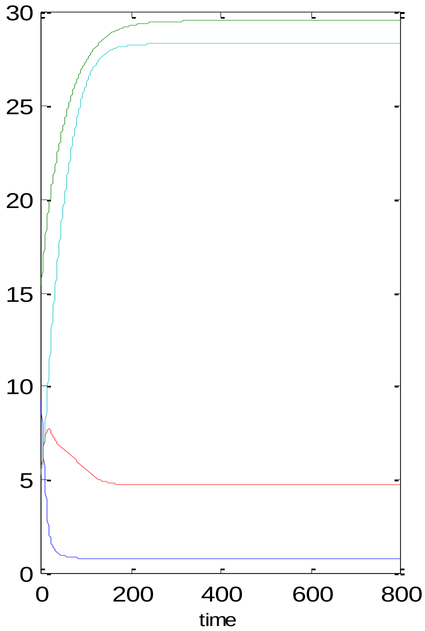

Fig.5 Global stable point (0.8, 29.54, 4.7, 28.35) for the time series of the trajectories of system (1) with D=6.m blue: x green: y red: z_1 light blue: z_2

6. Conclusion

In this section, a mathematical model has been proposed and analysed to study the role of stage structure on the dynamics of the prey-predator model with the reserved zone. The dynamical behaviour of the positive equilibrium points has been investigated locally and globally. According to the previous analysis, the following results are obtained.

1. The dynamical behaviours of the system (1) approaches a global asymptotically stable point for increases of σ_1 , r and D.
2. For $\sigma_2 < 3.84$, the trajectory of the system (1) approaches a global asymptotically stable point $Int.R_+^4$ while it approaches to F_2 for $\sigma_2 \geq 3.84$.
3. System (1) is persisting for all the values of parameters under consideration.

In conclusion, it has been shown that the alternative resource for the predator leads to a rise in the density of the prey and predator. Moreover, it has been found by using the stability theory of ordinary differential equations the positive equilibrium, whenever it exists, is always globally asymptotically stable. This shows that the reserve area has a stabilizing effect on the predator-prey system. This study suggests that the role of the reserved zone is an essential integrating idea in ecology and evolution. The prey species can raise without any external disturbances by producing reserved zones in the habitat where predators have no access or chance of settling. Hence, the prey species can be preserved at an appropriate level.

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